On the Gaussian MAC with Imperfect Feedback

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Abstract

New achievable rate regions are derived for the two-user additive white Gaussian multiple-access channel with noisy feedback. The regions exhibit the following two properties. Irrespective of the (finite) Gaussian feedback-noise variances, our regions include rate points which lie strictly outside the no-feedback capacity region, and when the feedback-noise variances tend to 0 our regions converge to the perfect feedback capacity region.

The new achievable regions also apply to the partial-feedback setting where one of the transmitters has a noisy feedback link and the other transmitter has no feedback at all. Again, irrespective of the (finite) noise variance on the only feedback link, the regions include rate points which lie strictly outside the no-feedback capacity region. Moreover, in the case of perfect partial feedback, i.e., where the only feedback link is noise-free, for certain channel parameters the new regions even include rate points which lie strictly outside the Cover-Leung region. This answers in the negative the question posed by van der Meulen as to whether the Cover-Leung region equals the capacity region of the Gaussian multiple-access channel with perfect partial feedback.

Finally, we propose new achievable regions also for a setting where the receiver is cognizant of the realizations of the noise sequences on the feedback links.

1 Introduction

In [5] Gaarder and Wolf showed that for some memoryless multiple-access channels (MAC) the capacity region is strictly increased compared to the classical setting if both transmitters have perfect feedback from the channel output. That this also holds for the two-user additive white Gaussian noise (AWGN) MAC was shown by Ozarow in [9], where he also determined the capacity region of this channel with perfect feedback. Here, we study the capacity region of the two-user AWGN MAC with imperfect feedback. We consider the following settings:

- \textit{noisy feedback} where the feedback links are corrupted by additive white Gaussian noise;

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- **noisy partial feedback** where one of the two transmitters has a noisy feedback link whereas the other transmitter has no feedback at all;

- **perfect partial feedback** where one of the two transmitters has a perfect (noise-free) feedback link whereas the other transmitter has no feedback at all; and

- **noisy feedback with receiver side-information** where both transmitters have noisy feedback links and the receiver (but not the transmitters) is perfectly cognizant of the feedback-noise sequences.

The last situation arises for example when the receiver actively feeds back a quantized version of the output over perfect feedback links, and thus the feedback noises model the quantization noises known at the receiver. The MAC with quantized feedback has also been considered in [12] but under the assumption of a rate limitation on the feedback links and for the discrete memoryless case. We show that in all these settings the capacity region is strictly larger than the no-feedback capacity region. Moreover, we show that for noisy feedback the capacity region tends to Ozarow’s perfect-feedback capacity region [9] as the feedback-noise variances tend to zero. Finally, in the case of perfect partial feedback we show that for certain channel parameters the capacity region strictly includes the Cover-Leung region [4], a region that was originally derived for the perfect-feedback setting and that was later shown by Carleial [2] and (for the discrete memoryless case) by Willems and van der Meulen [16] to be achievable also in the perfect partial-feedback setting. This answers in the negative the question posed by van der Meulen in [13] as to whether the Cover-Leung region equals the capacity region of the Gaussian MAC with perfect partial feedback.

To derive these results we propose coding schemes for the described settings and analyze the rates that they achieve. The idea of our schemes is to generalize Ozarow’s capacity-achieving perfect-feedback scheme to imperfect feedback. Ozarow’s scheme is based on the following strategy. The transmitters first map their messages onto message points in the interval $[-\frac{1}{2}, \frac{1}{2}]$. They then successively refine the receiver’s estimates of these message points by sending scaled versions of the receiver’s linear minimum mean-squared errors (LMMSE) of the message points. Besides achieving capacity, Ozarow’s scheme has the advantage of a double-exponential decay of the probability of error. However, a drawback of the scheme is that it is extremely sensitive to noise on the feedback links: it does not achieve any positive rate if the feedback links are not noise-free. To overcome this weakness, we propose to apply an outer code around a modified version of Ozarow’s scheme where the transmitters instead of refining the message points they successively refine the input symbols from the outer code. We further modify Ozarow’s scheme in that we allow the transmitters to refine the input symbols by sending arbitrary linear updates (i.e., not necessarily LMMSE-updates) and in that the number of refinements of each input symbol is a constant (that can be optimized) and does not grow with the blocklength. These modifications yield a scheme which achieves high rates also for channels with imperfect feedback. In particular, for noisy feedback and for noisy partial feedback our scheme exhibits the following key properties:

- for all finite feedback-noise variances, our scheme achieves rate points that lie strictly outside the capacity region without feedback, and

for noisy feedback
the scheme achieves rate regions that converge to Ozarow’s perfect-feedback capacity region when the feedback-noise variances tend to zero.

Previous achievable regions for the AWGN MAC with imperfect feedback were given by Carleial [2], by Gastpar [6], and by Willems et al. [18]. Carleial and Willems et al. generalized the Cover-Leung coding scheme [4]. Gastpar’s result is also based on Ozarow’s scheme and on the idea of modifying it to use only a finite number of refinements which does not grow with the blocklength. However, these regions collapse to the no-feedback capacity region when the feedback-noise variances exceed a certain threshold. Moreover, as the feedback-noise variances tend to zero the first two mentioned regions converge to the Cover-Leung region which is a strict subset of Ozarow’s region.

An outer bound on the capacity region of the MAC with imperfect feedback was presented by Gastpar and Kramer [7]. For Gaussian channels this outer bound meets a known achievable region only for the case with perfect feedback [9] and for the case without feedback [3, 20].

We conclude this section with notation. In the following $A^\ell$ denotes the $\ell$-dimensional random column vector with components $A_1, \ldots, A_\ell$, i.e., $A^\ell = (A_1, A_2, \ldots, A_\ell)^T$; $\text{diag}(a_1, \ldots, a_\ell)$ denotes the diagonal matrix with diagonal entries $a_1, \ldots, a_\ell$; $I_\ell$ is the $\ell \times \ell$ identity matrix; $A^T$ denotes the transpose of a matrix $A$, $|A|$ its determinant, and $\text{tr}(A)$ its trace. Also, for zero-mean random vectors $S$ and $T$ we define the covariance matrices $K_{S,T} \triangleq \mathbb{E}[ST^T]$ and $K_S \triangleq \mathbb{E}[SS^T]$.

2 Channel Model

This paper focuses on the additive white Gaussian multiple-access channel with two transmitters wishing to transmit messages $M_1$ and $M_2$ to a single receiver. The two messages are assumed to be independent of each other and uniformly distributed over the discrete finite sets $\mathcal{M}_1$ and $\mathcal{M}_2$.

1The result in [18] is for the discrete memoryless case, but it easily extends to the Gaussian case.
2The idea of using a finite number of refinements was already mentioned in [11]. However, only in combination with zero rate or nonvanishing probability of error.
3It can be shown that the achievable rate region in [6] converges to Ozarow’s region when the feedback-noise variances tend to 0.
To describe the channel model (see also Fig 1) we introduce the sequence \( \{Z_t\} \) of independent and identically distributed (IID) zero-mean variance-\( N \) Gaussian random variables that will be used to model the additive noise at the receiver. Using this sequence we can describe the time-\( t \) channel output \( Y_t \) corresponding to the time-\( t \) channel inputs \( x_{1,t} \) and \( x_{2,t} \) by

\[
Y_t = x_{1,t} + x_{2,t} + Z_t.
\]

The sequence \( \{Z_t\} \) is assumed to be independent of the messages \( (M_1, M_2) \). Also, we introduce the IID sequence of bi-variate zero-mean Gaussians \( \{(W_{1,t}, W_{2,t})\} \) of covariance matrix

\[
K_{W_1 W_2} \triangleq \begin{pmatrix} \mathbb{E}[W_{1,t}^2] & \mathbb{E}[W_{1,t} W_{2,t}] \\ \mathbb{E}[W_{1,t} W_{2,t}] & \mathbb{E}[W_{2,t}^2] \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{pmatrix}
\]

where \( \sigma_1, \sigma_2 \geq 0 \) and \( \rho \in [-1, 1] \). The sequence \( \{(W_{1,t}, W_{2,t})\} \) is used to model the additive noise corrupting the feedback links. The time-\( t \) feedback output \( V_{\nu,t} \) at Transmitter \( \nu \) can then be modeled as

\[
V_{\nu,t} = Y_t + W_{\nu,t}, \quad \nu \in \{1, 2\}.
\]

The sequence \( \{(W_{1,t}, W_{2,t})\} \) is assumed to be independent of \( (M_1, M_2, \{Z_t\}) \).

The transmitters observe the feedback outputs in a causal fashion, i.e., before they have to compute their time-\( t \) channel inputs \( X_{1,t} \) and \( X_{2,t} \), they have observed all previous feedback outputs \( V_{1,1}, \ldots, V_{1,t-1} \) and \( V_{2,1}, \ldots, V_{2,t-1} \). Thus, Transmitter \( \nu \) computes its channel inputs by mapping the Message \( M_\nu \) and the previous feedback outputs \( V_{\nu,1}, \ldots, V_{\nu,t-1} \) into the time-\( t \) channel input \( X_{\nu,t} \),

\[
X_{\nu,t} = \varphi^{(n)}_{\nu,t}(M_\nu, V_{\nu,1}, \ldots, V_{\nu,t-1}), \quad t \in \{1, \ldots, n\}, \quad \nu \in \{1, 2\}, \tag{1}
\]

for some sequences of encoding functions

\[
\varphi^{(n)}_{\nu,t} : \mathcal{M}_\nu \times \mathbb{R}^{t-1} \to \mathbb{R}, \quad t \in \{1, \ldots, n\}, \quad \nu \in \{1, 2\}, \tag{2}
\]

where \( n \) denotes the blocklength of the scheme. We shall only allow encoding functions which fulfill the power constraints

\[
\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \left( \varphi^{(n)}_{\nu,t}(M_\nu, V_{\nu,1}, \ldots, V_{\nu,t-1}) \right)^2 \right] \leq P_\nu, \quad \nu \in \{1, 2\}, \tag{3}
\]

where the expectation is over the messages and the realizations of the channel, i.e., the noise sequences \( \{Z_t\}, \{W_{1,t}\}, \) and \( \{W_{2,t}\} \).\footnote{The achievability results in this paper remain valid also when the expected average block-power constraints \( (3) \) are replaced by average block-power constraints that hold with probability 1.}

A blocklength \( n \), powers \( (P_1, P_2) \), feedback-code of rate pair \( (\frac{1}{n} \log(|\mathcal{M}_1|), \frac{1}{n} \log(|\mathcal{M}_2|)) \) is a triple

\[
\left( \left\{ \varphi^{(n)}_{1,t} \right\}_{t=1}^{n}, \left\{ \varphi^{(n)}_{2,t} \right\}_{t=1}^{n}, \phi^{(n)} \right),
\]

where

\[
\phi^{(n)} : \mathbb{R}^n \to \mathcal{M}_1 \times \mathcal{M}_2.
\]
and where \( \{\varphi_{1,t}^{(n)}\} \) and \( \{\varphi_{2,t}^{(n)}\} \) are of the form (2) and satisfy (3). In the following we say that a rate pair \((R_1, R_2)\) is achievable if for every \( \delta > 0 \) and every sufficiently large \( n \) there exists a blocklength \( n \), powers \((P_1, P_2)\), feedback code of rates exceeding \( R_1 - \delta \) and \( R_2 - \delta \) such that the average probability of a decoding error,

\[
\Pr[\phi^{(n)}(Y_1, \ldots, Y_n) \neq (M_1, M_2)],
\]

tends to 0 as the blocklength \( n \to \infty \). The set of all achievable rate pairs for this setting is called the capacity region and will be denoted by \( C_{\text{NoisyFB}}(P_1, P_2, N, W_1, W_2) \).

The case \( \sigma_1^2 = \sigma_2^2 = 0 \) corresponds to the special case when the feedback links are completely noise-free. We refer to this setting as the “perfect-feedback” setting and denote the capacity region by \( C_{\text{PerfectFB}}(P_1, P_2, N) \), i.e.,

\[
C_{\text{PerfectFB}}(P_1, P_2, N) \triangleq C_{\text{NoisyFB}}(P_1, P_2, N, 0),
\]

where 0 is the \( 2 \times 2 \) all-zero matrix.

In addition to the noisy-feedback setting we also consider the “partial-feedback” setting (see also Figure 2) where only one of the two transmitters has feedback. We denote the transmitter with feedback Transmitter 2. For the partial-feedback setting (1) and (2) are modified such that the sequence \( \{X_{1,1}, \ldots, X_{1,n}\} \) is computed as a function of Message \( M_1 \) only. Since the only feedback link can be noisy we shall refer to this setting also as “noisy partial feedback” and denote its capacity region by \( C_{\text{NoisyPartialFB}}(P_1, P_2, N, \sigma_2^2) \), where \( \sigma_2^2 \geq 0 \) denotes the noise variance on the feedback link to Transmitter 2. In the special case of \( \sigma_2^2 = 0 \), i.e., when the only feedback link is noise-free, we refer to the setting as “perfect partial feedback” (see also Figure 3) and denote the capacity region by \( C_{\text{PerfectPartialFB}}(P_1, P_2, N) \).

By the “no-feedback” setting we refer to the classical MAC where neither transmitter has a feedback link. In this case (1) and (2) have to be modified so that both sequences
Figure 4: AWGN MAC with noisy feedback and receiver side-information.

\{X_1, \ldots, X_{1,n}\} and \{X_2, \ldots, X_{2,n}\} are functions of the respective messages only. We denote the capacity region of the classical MAC by \(C_{\text{MAC}}(P_1, P_2, N)\).

Finally, we also consider a noisy-feedback setting where the receiver perfectly knows the realizations of the Gaussian noise sequences \(\{W_{1,k}\}\) and \(\{W_{2,k}\}\) corrupting the feedback signals (see also Figure 4). We refer to this setting as the “noisy feedback with receiver side-information” setting. For this setting the formal description of the communication scenario is the same as in the noisy-feedback setting, except for the notion of the decoder \(\phi_{\text{SI}}^{(n)}\) which is given by a mapping of the form

\[
\phi_{\text{SI}}^{(n)} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{M}_1 \times \mathcal{M}_2, \\
(Y^n_1, W^n_{1,1} W^n_{2,1}) \mapsto (\hat{M}_1, \hat{M}_2).
\]

We denote the capacity region of the MAC with noisy feedback and perfect receiver side-information by \(C_{\text{NoisyFB\ SI}}(P_1, P_2, N, K_{W_1 W_2})\).

### 3 Previous Results

We survey previous results that will be needed in subsequent sections.

The capacity region of the classical Gaussian MAC without feedback \(C_{\text{MAC}}(P_1, P_2, N)\) was independently determined by Cover [3] and Wyner [20] and is given by the set of all rate pairs \((R_1, R_2)\) satisfying

\[
R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{N} \right), \tag{4}
\]

\[
R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_2}{N} \right), \tag{5}
\]

\[
R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{N} \right). \tag{6}
\]

The capacity region of the Gaussian MAC with perfect feedback \(C_{\text{PerfectFB}}(P_1, P_2, N)\)

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5Since we do not consider any delay constraints and the receiver cannot actively feed back a signal, it does not matter whether the receiver learns the feedback-noise sequences \(\{W_{1,k}\}\) and \(\{W_{2,k}\}\) causally or acausally.
was determined by Ozarow [9]:

$$\mathcal{C}_{\text{PerfectFB}}(P_1, P_2, N) = \bigcup_{\rho \in [0, 1]} \mathcal{R}_\text{Oz}^\rho(P_1, P_2, N),$$  \hspace{1cm} (7)

where $\mathcal{R}_\text{Oz}^\rho(P_1, P_2, N)$ is the set of all rate pairs $(R_1, R_2)$ satisfying

$$R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 (1 - \rho^2)}{N} \right),$$  \hspace{1cm} (8)

$$R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_2 (1 - \rho^2)}{N} \right),$$  \hspace{1cm} (9)

$$R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2 \rho}}{N} \right).$$  \hspace{1cm} (10)

We next describe some properties of the regions $\mathcal{R}_\text{Oz}^\rho(P_1, P_2, N)$ and $\mathcal{C}_{\text{PerfectFB}}(P_1, P_2, N)$ that will be needed in subsequent sections. Some of the properties, Remarks 2–4 and Remark 8, have been reported in [9].

**Definition 1.** The parameter $\rho^*(P_1, P_2, N)$ (for short $\rho^*$) is defined as the unique solution in the interval $[0, 1]$ of the following quartic equation in $\rho$

$$N(N + P_1 + P_2 + 2\sqrt{P_1 P_2 \rho}) = (N + P_1 (1 - \rho^2))(N + P_2 (1 - \rho^2)).$$  \hspace{1cm} (11)

**Remark 2.** Equation (11) is equivalent to the right-hand side of (10) being equal to the sum of the right-hand sides of (8) and (9).

That (11) has a unique solution in the interval $[0, 1]$ can be seen as follows. At $\rho = 0$ the left-hand side of (11) is smaller than its right-hand side, whereas for $\rho = 1$ the left-hand side is larger. Since the expressions on both sides of (11) are continuous, by the Intermediate Value Theorem there must exist at least one solution to (11) in $[0, 1]$. The uniqueness of the solution follows by noting that the left-hand side of (11) is strictly increasing in $\rho$ whereas the right-hand side is strictly decreasing in $\rho \in [0, 1]$.

Next, we discuss the region $\mathcal{R}_\text{Oz}^\rho(P_1, P_2, N)$ and have a closer look at the rate constraints (8)–(10) defining the region. The two single-rate constraints (8) and (9) are strictly decreasing in $\rho \in [0, 1]$, whereas the sum-rate constraint (10) is strictly increasing in $\rho$. By these properties, by Definition 1, and by Remark 2 the following two remarks follow.

**Remark 3.** For $\rho = \rho^*$ the sum of the two single-rate constraints (8) and (9) equals the sum-rate constraint (10); for $\rho \in [0, \rho^*)$ the sum of the two single-rate constraints is strictly larger than the sum-rate constraint; and for $\rho \in (\rho^*, 1]$ the sum-rate constraint is strictly larger than the sum of the two single-rate constraints.

**Remark 4.** For every $\rho \in [0, \rho^*)$ the rate region $\mathcal{R}_\text{Oz}^\rho(P_1, P_2, N)$ has the shape of a pentagon and for every $\rho \in [\rho^*, 1]$ the rate region $\mathcal{R}_\text{Oz}^\rho(P_1, P_2, N)$ has the shape of a rectangle. Furthermore, all rectangles $\mathcal{R}_\text{Oz}^\rho(P_1, P_2, N)$ for $\rho \in (\rho^*, 1]$ are strictly contained in the rectangle $\mathcal{R}_\text{Oz}^{\rho^*}(P_1, P_2, N)$, and thus in (7) it is enough to take the union over all $\rho \in [0, \rho^*)$. 

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For the next two observation we introduce the notation of a dominant corner point as in [10]. A corner point of a given rate region is called dominant if it is of maximum sum-rate in the considered region.

**Remark 5.** The boundary points of $C_{\text{PerfectFB}}(P_1, P_2, N)$ that have sum-rate larger or equal to $\frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{N} \right)$ (see Fig.5) are given by the dominant corner points of the regions $R^\rho_{\text{Oz}}(P_1, P_2, N)$ for $\rho \in [0, \rho^*]$.

Remark 5 follows by Remark 4, by continuity considerations, and by the monotonicities of the constraints (8)–(10), see Remark 3. To state the next observation we define:

**Definition 6.** Let $P_1, P_2, N > 0$ and $\rho \in [0, \rho^*]$ be given. Then, define $R^\rho_1,\text{Oz}(P_1, P_2, N)$ as the set of all rate pairs $(R_1, R_2)$ satisfying

$$R_1 \leq \frac{1}{2} \log \left( 1 + \frac{P_1(1 - \rho^2)}{N} \right),$$

$$R_2 \leq \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2\sqrt{P_1P_2\rho} + N}{P_1(1 - \rho^2) + N} \right).$$

Similarly, define $R^\rho_2,\text{Oz}(P_1, P_2, N)$ as the set of all rate pairs $(R_1, R_2)$ satisfying

$$R_1 \leq \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2\sqrt{P_1P_2\rho} + N}{P_2(1 - \rho^2) + N} \right),$$

$$R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_2(1 - \rho^2)}{N} \right).$$

Notice that by Remark 4, $R^\rho_1,\text{Oz}(P_1, P_2, N) = R^\rho_2,\text{Oz}(P_1, P_2, N) = R^\rho_{\text{Oz}}(P_1, P_2, N)$. Further, notice that for fixed $\rho \in [0, \rho^*]$ the regions $R^\rho_1,\text{Oz}(P_1, P_2, N)$ and $R^\rho_2,\text{Oz}(P_1, P_2, N)$ correspond to the rectangles with dominant corner point equal to one of the dominant corner points of $R^\rho_{\text{Oz}}(P_1, P_2, N)$, see Figure 5. By these observations and by Remark 5 the following remark is obtained.

**Remark 7.** The perfect-feedback capacity region can be expressed as

$$C_{\text{PerfectFB}}(P_1, P_2, N) = \bigcup_{\rho \in [0, \rho^*]} (R^\rho_1,\text{Oz}(P_1, P_2, N) \cup R^\rho_2,\text{Oz}(P_1, P_2, N)).$$

Figure 5: Perfect-feedback capacity region with an example of $R^\rho_{\text{Oz}}, R^\rho_1,\text{Oz}$, and $R^\rho_2,\text{Oz}$ for $0 < \rho < \rho^*$. 
Finally, last remark follows by Remark 5 and because the sum-rate constraint (10) is strictly increasing in \( \rho \).

**Remark 8.** The dominant corner point of the rectangle \( \mathcal{R}_{Ox}^*(P_1, P_2, N) \) is the only rate point of maximum sum-rate in \( \mathcal{C}_{\text{PerfectFB}}(P_1, P_2, N) \).

This concludes our discussion of the perfect-feedback capacity region. Next, we present an achievability result for general discrete memoryless MACs and Gaussian MACs with perfect feedback due to Cover and Leung [4]. The scheme is known to achieve capacity for a specific class of discrete memoryless MACs with perfect feedback [15]. However, for general channels it can be suboptimal, e.g., for Gaussian channels. For Gaussian channels the optimization problem defining the Cover-Leung region is solved by jointly Gaussian inputs, see [14, 1], and therefore the Cover-Leung region is given by

\[
\mathcal{R}_{\text{CL}}(P_1, P_2, N) = \bigcup_{\rho_1, \rho_2 \in [0, 1]} \mathcal{R}_{\text{CL}}^{(\rho_1, \rho_2)}(P_1, P_2, N),
\]

where \( \mathcal{R}_{\text{CL}}^{(\rho_1, \rho_2)}(P_1, P_2, N) \) includes all rate pairs \((R_1, R_2)\) satisfying

\[
\begin{align*}
R_1 &\leq \frac{1}{2} \log \left(1 + \frac{P_1 (1 - \rho_1^2)}{N}\right), \\
R_2 &\leq \frac{1}{2} \log \left(1 + \frac{P_2 (1 - \rho_2^2)}{N}\right), \\
R_1 + R_2 &\leq \frac{1}{2} \log \left(1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2 \rho_1 \rho_2}}{N}\right).
\end{align*}
\]

Carleial [2] and Willems [16] independently proved that to achieve the Cover-Leung region \( \mathcal{R}_{\text{CL}}(P_1, P_2, N) \) it suffices that only one of the two transmitters has a perfect feedback link, i.e., they proved that the Cover-Leung region is achievable also in a perfect partial-feedback setting. Thereupon, van der Meulen in a survey paper on multiple-access channels with feedback [13] posed the question whether the Cover-Leung region equals the capacity region for discrete memoryless MACs or Gaussian MACs with perfect partial feedback. We will answer this question in the negative for Gaussian channels by proving that for certain channel parameters \((P_1, P_2, N)\) there exist rate pairs that lie strictly outside the Cover-Leung region \( \mathcal{R}_{\text{CL}}(P_1, P_2, N) \) but that are achievable in the perfect partial-feedback setting.

For the Gaussian MAC with perfect partial feedback also Willems, van der Meulen, and Schalkwijk proposed an encoding scheme [17] which is based on the scheme by Schalkwijk and Kailath [11]. Unfortunately, the achievable rate region can only be stated in an implicit form and is difficult to evaluate analytically and to compare to the Cover-Leung region.

In [2] Carleial proposed a coding scheme for the discrete memoryless MAC and the Gaussian MAC with “generalized” feedback. In the Gaussian case, “generalized” feedback includes as special cases noisy feedback, noisy partial feedback, and perfect partial feedback. We present Carleial’s region for the Gaussian MAC with noisy feedback in Appendix A, where we also prove that if the feedback noise variances \( \sigma_1^2 \) and \( \sigma_2^2 \) exceed a certain threshold depending on the channel parameters \( P_1, P_2, \) and \( N \), then Carleial’s region collapses to the no-feedback capacity region in (4)–(6) (Proposition 44 in Appendix A). For perfect partial feedback and for perfect feedback Carleial’s scheme equals the Cover-Leung region.
4 Concatenated Schemes

We present our coding schemes with general parameters for the settings with noisy feedback, with noisy partial feedback, with perfect partial feedback, and with noisy feedback with receiver side-information. We refer to these schemes as the “concatenated schemes.”

In the first subsection we present the concatenated scheme for noisy feedback. In the next-following subsections, we explain how to modify the scheme to apply also for noisy or perfect partial feedback and for noisy feedback with receiver side-information.

4.1 Noisy Feedback

We propose an encoding scheme with a concatenated structure where each of the encoders and the decoder consists of an outer part and an inner part. (Here the inner parts are the parts that are closer to the physical channel, see also Figure 6.) In our scheme the various parts fulfill the following tasks. The outer encoders map the messages into codewords (without using the feedback) and feed these codewords to their corresponding inner encoders. The inner encoders produce for every fed symbol a sequence of $\eta$ channel inputs to the MAC with feedback, for some positive integer $\eta$. In particular, when fed by the symbol $i$. Hence, in the case of perfect feedback Carleial’s scheme is known to be strictly suboptimal for the two-user Gaussian MAC.

Another coding scheme for the MAC with imperfect feedback has been proposed by Willems et al. in [18]. Even though they considered only discrete memoryless channels the modifications to treat the Gaussian case are straightforward, and we state their achievable rate region for the Gaussian MAC with noisy feedback in Appendix B. Similar to Carleial’s scheme also Willems et al.’s scheme collapses to the no-feedback capacity region when the feedback-noise variances $\sigma_1^2$ and $\sigma_2^2$ exceed a certain threshold (Proposition 46 in Section B), and for perfect feedback or perfect partial feedback the region equals the Cover-Leung region. Thus, for very noisy feedback, for perfect feedback, and for perfect partial feedback Carleial’s region and Willems et al.’s region coincide.
ξ₁ ∈ ℝ, Inner Encoder 1 produces η inputs which depend on ξ₁ and on the observed feedback outputs; all symbols fed to the inner encoder are treated in the same way. Inner Encoder 2 is analogously defined. The η symbols which the MAC outputs for every pair of input symbols (ξ₁, ξ₂) are then linearly mapped by the inner decoder to a pair of estimates (Ξ₁, Ξ₂), and the estimates are fed to the outer decoder. Thus, the outer decoder is fed with a vector in ℝ² every η channel uses. Based on the sequence of vectors produced by the inner decoder, the outer decoder then decodes the transmitted messages.

Consequently, the inner encoders and the inner decoder transform each subblock of η channel uses of the original MAC into a single channel use of a “new” time-invariant and memoryless MAC which for given inputs ξ₁ ∈ ℝ and ξ₂ ∈ ℝ produces the channel output (Ξ₁, Ξ₂)ᵀ ∈ ℝ². We denote the new MAC by ξ₁, ξ₂ ↦→ (Ξ₁, Ξ₂). We can then think of the overall scheme as a no-feedback scheme over the new MAC ξ₁, ξ₂ ↦→ (Ξ₁, Ξ₂). As a consequence, the capacity of the original MAC with feedback, which we denote by x₁, x₂ ↦→ Y, is inner bounded by the capacity of the new MAC ξ₁, ξ₂ ↦→ (Ξ₁, Ξ₂) without feedback but scaled by η⁻¹ to account for the fact that to send the symbols ξ₁, ξ₂ over the new MAC the original channel is used η times.

In the remaining of this subsection we describe concrete choices for the different parts of the scheme. We first sketch some of the properties of the inner encoders and the inner decoder and postpone their detailed description to after the description of the outer encoders and decoder. As we shall see, we choose the inner encoders and the inner decoder so that the MAC ξ₁, ξ₂ ↦→ (Ξ₁, Ξ₂) can be described by

\[
\begin{pmatrix}
\hat{Ξ}_1 \\
\hat{Ξ}_2
\end{pmatrix} = A \begin{pmatrix}
ξ_1 \\
ξ_2
\end{pmatrix} + T,
\]

(16)

where A is a deterministic 2 × 2 matrix and where T is a bi-variate Gaussian whose law does not depend on the pair of inputs (ξ₁, ξ₂). Also, the inner encoders are designed so that if both outer encoders satisfy a unit average block-power constraint (over time and messages) and if at every epoch the symbols produced by the outer encoders are zero-mean (when averaged over the messages), then the channel inputs to the original MAC x₁, x₂ ↦→ Y satisfy the average power constraints (3).

For the outer code (encoders and decoder) we choose a capacity achieving zero-mean code for the MAC ξ₁, ξ₂ ↦→ (Ξ₁, Ξ₂) under an average block-power constraint of 1. Note that there is no loss in optimality, in restricting ourselves to zero-mean codes because subtracting the mean of the code can only reduce its average power (averaged over time and messages) and does not change the performance on an additive noise MAC such as (16). We shall need the property that the outer encoders produce zero-mean symbols in the power-analysis of the input sequences to the original channel x₁, x₂ ↦→ Y.

For the inner encoders and the inner decoder we choose linear mappings. To obtain a compact description of the linear mappings we stack the η channel inputs, X₁, X₂, . . . , X₇, produced by Inner Encoder ν in an η-dimensional column vector

\[
X_ν \triangleq (X_{ν,1}, \ldots, X_{ν,η})ᵀ, \quad ν ∈ \{1, 2\},
\]

and similarly we stack the η feedback outputs, V₁, V₂, . . . , V₇, observed by Inner Encoder ν in the η-dimensional vector

\[
V_ν = (V_{ν,1}, \ldots, V_{ν,η})ᵀ, \quad ν ∈ \{1, 2\}.
\]
We can then describe our choice of the inner encoders as follows. When fed by input symbol \( \xi_\nu \in \mathbb{R} \), Inner Encoder \( \nu \) produces
\[
X_\nu = a_\nu \xi_\nu + B_\nu V_\nu, \quad \nu \in \{1, 2\},
\]
where \( a_\nu \) are \( \eta \)-dimensional column vectors and \( B_\nu \) are \( \eta \times \eta \) matrices which are strictly lower-triangular (so as not to violate the causality of the feedback). Also, as previously mentioned, we restrict the set of possible inner encoders to encoders which produce sequences of inputs to the original MAC \( x_1, x_2 \mapsto Y \) satisfying the average block-power constraints (3) when the outer encoders feed them with zero-mean sequences of unit average block-power. In the following we derive explicit conditions on the linear mappings \( a_1, a_2, B_1, \) and \( B_2 \) which ensure that the inner encoders satisfy these restrictions. To this end, we define the \( \eta \times 2 \) matrix
\[
A_b \triangleq \begin{pmatrix} a_1 & a_2 \end{pmatrix},
\]
the \( 2\eta \times 2 \) matrix
\[
A_d \triangleq \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},
\]
the \( \eta \times 2\eta \) matrix
\[
B_b \triangleq \begin{pmatrix} B_1 & B_2 \end{pmatrix},
\]
the \( 2\eta \times \eta \) matrix
\[
B_t \triangleq \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},
\]
the \( 2\eta \times 2\eta \) block-diagonal matrix
\[
B_d \triangleq \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},
\]
the \( 2\eta \times 2\eta \) matrix
\[
B_B \triangleq \begin{pmatrix} B_1 & B_1 \\ B_2 & B_2 \end{pmatrix},
\]
and the \( \eta \times \eta \) matrix
\[
B_\Sigma \triangleq B_1 + B_2.
\]
Then, by (17) the inner encoders satisfy the power restrictions whenever the vectors \( a_1 \) and \( a_2 \) and the matrices \( B_1 \) and \( B_2 \) satisfy the following two trace constraints
\[
\text{tr}\left( A_d^T (I_{2\eta} - B_B)^{-T} \begin{pmatrix} I_\eta & 0 \\ 0 & 0 \end{pmatrix} (I_{2\eta} - B_B)^{-1} A_d + (K_{W_1W_2} \otimes I_\eta) B_d^T (I_{2\eta} - B_B)^{-T} \begin{pmatrix} I_\eta & 0 \\ 0 & 0 \end{pmatrix} (I_{2\eta} - B_B)^{-1} B_d + NB_t^T (I_{2\eta} - B_B)^{-T} \begin{pmatrix} I_\eta & 0 \\ 0 & 0 \end{pmatrix} (I_{2\eta} - B_B)^{-1} B_t \right) \leq \eta P_1
\]
\[ \operatorname{tr} \left( A_d^\top (I_{2\eta} - B_B)^{-1} \left( \begin{array}{cc} 0 & 0 \\ 0 & I_{\eta} \end{array} \right) (I_{2\eta} - B_B)^{-1} A_d \right) + (K_{W_1 W_2} \otimes I_{\eta}) B_d^\top (I_{2\eta} - B_B)^{-1} \cdot \left( \begin{array}{cc} 0 & 0 \\ 0 & I_{\eta} \end{array} \right) (I_{2\eta} - B_B)^{-1} B_d \] 
\[ + NB_t^\top (I_{2\eta} - B_B)^{-1} \left( \begin{array}{cc} 0 & 0 \\ 0 & I_{\eta} \end{array} \right) (I_{2\eta} - B_B)^{-1} B_t \right) \leq \eta P_2, \]

(22)

where \( \otimes \) denotes the Kronecker product. Thus, in the following we only allow for vectors \( a_1 \) and \( a_2 \) and for strictly lower-triangular matrices \( B_1 \) and \( B_2 \) satisfying (21) and (22). Note that, since \( B_1 \) and \( B_2 \) are strictly lower-triangular, the matrix \( (I_{2\eta} - B_B) \) is nonsingular and its inverse exists.

To describe our linear choice of the inner decoder, we stack the \( \eta \) outputs \( Y_1, \ldots, Y_\eta \), which the original MAC produces for the pairs of inputs \((X_{1,1}, X_{2,1}), \ldots, (X_{1,\eta}, X_{2,\eta})\), into the \( \eta \)-dimensional column vector

\[ Y \triangleq (Y_1, \ldots, Y_\eta)^\top. \]

We can then express the estimates produced by the outer decoder by

\[ \begin{pmatrix} \hat{\Xi}_1 \\ \hat{\Xi}_2 \end{pmatrix} = D Y, \]

(23)

for some matrix of our choice \( D \in \mathbb{R}^{2 \times \eta} \).

In the following we describe the MAC \( \xi_1, \xi_2 \mapsto (\hat{\Xi}_1, \hat{\Xi}_2) \) as induced by \( \eta, a_1, a_2, B_1, B_2, \) and \( D \). Given inputs \( \xi_1 \) and \( \xi_2 \) it produces the vector of estimates

\[ \begin{pmatrix} \hat{\Xi}_1 \\ \hat{\Xi}_2 \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + T, \]

(24)

where the channel matrix \( A \) is a \( 2 \times 2 \) matrix defined as

\[ A \triangleq D (I_\eta - B_\Sigma)^{-1} A_b, \]

(25)

and the noise vector \( T \) is a zero-mean bivariate Gaussian defined as

\[ T \triangleq D (I_\eta - B_\Sigma)^{-1} (B_1 W_1 + B_2 W_2 + Z), \]

(26)

for \( W_1 \triangleq (W_{1,1}, \ldots, W_{1,\eta})^\top \), \( W_2 \triangleq (W_{2,1}, \ldots, W_{2,\eta})^\top \), and \( Z \triangleq (Z_1, \ldots, Z_\eta)^\top \). (Notice, that since \( B_1 \) and \( B_2 \) are strictly lower-triangular matrices, the matrix \( (I_\eta - B_\Sigma) \) is nonsingular and the inverse exists.) Defining then the \( 2 \times \eta \) matrix

\[ C \triangleq D (I_\eta - B_\Sigma)^{-1}, \]

(27)

the channel matrix in (25) can be expressed as

\[ A = C A_b, \]

(28)

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and the noise vector in (26) can be expressed as
\[ T = C (B_1 W_1 + B_2 W_2 + Z). \] (29)

For fixed \( \eta, B_1, B_2 \) the mapping (27) from \( D \) to \( C \) is one-to-one, and thus we can parameterize our concatenated scheme for noisy feedback by the parameters \( \eta, a_1, a_2, B_1, B_2, C \).

Note that by choosing \( \eta = 1, a_1 = \sqrt{P_1}, a_2 = \sqrt{P_2} \), and \( C \) as the \( 2 \times 1 \) matrix with unit entries, our scheme includes the capacity achieving scheme for the original MAC \( x_1, x_2 \mapsto Y \) without feedback subject to the power constraints (3).

4.2 Noisy and Perfect Partial Feedback

By restricting \( B_1 \) to be the all-zero matrix, the concatenated scheme in the previous subsection 4.1 can also be used in settings with noisy or perfect partial feedback. For noisy partial feedback with feedback-noise variance \( \sigma_2^2 \geq 0 \), the MAC \( \xi_1, \xi_2 \mapsto (\hat{\xi}_1, \hat{\xi}_2) \) induced by the parameters \( \eta, a_1, a_2, B_2, \) and \( D \) is described by the channel law
\[ \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} = A_P \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + T_P, \] (30)

where the channel matrix \( A_P \) is a \( 2 \times 2 \) matrix defined as
\[ A_P \triangleq D (I_\eta - B_2)^{-1} A_b, \] (31)
and the noise vector \( T_P \) is a zero-mean bivariate Gaussian defined as
\[ T_P \triangleq D (I_\eta - B_2)^{-1} (B_2 W_2 + Z). \] (32)

Defining then the \( 2 \times \eta \) matrix
\[ C_P \triangleq D (I_\eta - B_2)^{-1}, \] (33)
the channel matrix in (31) and the noise vector in (32) can be expressed as
\[ \begin{align*}
A_P &= C_P A_b, \\
T_P &= C_P (B_2 W_2 + Z).
\end{align*} \] (34, 35)

For fixed \( \eta \) and \( B_2 \) the mapping (33) from \( D \) to \( C_P \) is one-to-one, and thus we can parameterize our concatenated scheme for noisy partial feedback by the parameters \( \eta, a_1, a_2, B_2, C_P \).

Specializing the power constraints (21) and (22) to \( B_1 = 0 \), we obtain that for noisy partial feedback all choices of parameters \( \eta, a_1, a_2, \) and \( B_2 \) are allowed which satisfy
\[ a_1^\top a_1 \leq \eta P_1 \] (36)
and
\[ \text{tr} \left( (I_\eta - B_2)^{-1} (a_2 a_2^\top + B_2 a_1 a_1^\top B_2^\top + (N + \sigma_2^2) B_2 B_2^\top) (I_\eta - B_2)^{-\top} \right) \leq \eta P_2. \] (37)
4.3 Noisy Feedback with Receiver Side-Information

In this section we extend our concatenated scheme to noisy feedback with receiver side-information.

We use the same outer code and the same inner encoders as in the setting without side-information. The difference is only in the inner decoder. Thus, when fed by the pair of symbols \((\xi_1, \xi_2)\) the inner encoders produce as before sequences of channel inputs described by

\[
X_\nu = a_\nu \xi_\nu + B_\nu V_\nu, \quad \nu \in \{1, 2\},
\]

where \(X_\nu \triangleq (X_{\nu,1}, \ldots, X_{\nu,\eta})^T\), \(V_\nu \triangleq (V_{\nu,1}, \ldots, V_{\nu,\eta-1})^T\), and where \(a_1, a_2\) are \(\eta\)-dimensional vectors and \(B_1, B_2\) are strictly lower-triangular \(\eta \times \eta\) matrices satisfying the power constraints (21) and (22). But, we modify the structure of the inner decoder so that it computes the estimates \((\hat{\Xi}_1, \hat{\Xi}_2)\) not only as a function of the output sequence but also of the feedback-noise sequences. Again, we choose a linear mapping, i.e., for \(Y \triangleq (Y_1, \ldots, Y_\eta)^T\), \(W_1 \triangleq (W_{1,1}, \ldots, W_{1,\eta})^T\), and \(W_2 \triangleq (W_{2,1}, \ldots, W_{2,\eta})^T\), the inner decoder computes

\[
\begin{pmatrix} \hat{\Xi}_1 \\ \hat{\Xi}_2 \end{pmatrix} = D_0 Y + D_1 W_1 + D_2 W_2,
\]

for \(2 \times \eta\) matrices \(D_0, D_1, D_2\) of our choice. Given \(a_1, a_2, B_1, B_2,\) and \(D_0\) an optimal choice for the matrices \(D_1\) and \(D_2\) subtracts off the contributions to \(D_0 Y\) that come about from the feedback-noise sequences, i.e., an optimal choice of \(D_1\) and \(D_2\) satisfies

\[
D_1 = -D_0 (I_\eta - B_\Sigma)^{-1} B_1 \quad \text{and} \quad D_2 = -D_0 (I_\eta - B_\Sigma)^{-1} B_2.
\]

In particular, such a choice leads to the following description of the “new” MAC \(\xi_1, \xi_2 \mapsto (\hat{\Xi}_1, \hat{\Xi}_2)\):

\[
\begin{pmatrix} \hat{\Xi}_1 \\ \hat{\Xi}_2 \end{pmatrix} = A_{SI} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + T_{SI},
\]

where the channel matrix \(A\) is a \(2 \times 2\) matrix defined as

\[
A_{SI} \triangleq D_0 (I_\eta - B_\Sigma)^{-1} A_b,
\]

and the noise vector \(T\) is a zero-mean bivariate Gaussian defined as

\[
T_{SI} \triangleq D_0 (I_\eta - B_\Sigma)^{-1} Z.
\]

In the following we shall always assume that \(D_1\) and \(D_2\) are optimally chosen so that the “new” MAC is given as by (41)–(43). We define the \(2 \times \eta\) matrix

\[
C_{SI} \triangleq D_0 (I_\eta - B_\Sigma)^{-1},
\]

and hence the channel matrix in (42) and the noise vector in (43) can be expressed as

\[
A_{SI} = C_{SI} A_b, \quad \text{and} \quad T_{SI} = C_{SI} Z.
\]

For fixed \(\eta, B_1, B_2\) the mapping (44) from \(D_0\) to \(C_{SI}\) is one-to-one, and thus we can parameterize our concatenated scheme for noisy feedback with receiver side-information by \(\eta, a_1, a_2, B_1, B_2, C_{SI}\).

All parameters \(\eta, a_1, a_2, B_1, B_2\) are allowed which satisfy the power constraints (21) and (22).
5 Extensions

In the following three subsections we present three extensions of our concatenated schemes by rate-splitting them with other schemes. The idea of rate-splitting was introduced in [2] and [9]. In each subsection, we first describe the extension for noisy feedback, followed by describing how to modify it to apply also for noisy or perfect partial feedback. The description of similar extensions for noisy feedback with receiver side-information is omitted.

5.1 Rate-Splitting with No-Feedback Scheme

In this first extension we combine our scheme with a no-feedback scheme employing IID Gaussian codewords. This extension was inspired by the rate-splitting scheme proposed by Ozarow for perfect feedback [9]. Only one transmitter applies the rate-splitting. For the description we assume it is Transmitter 1. Thus, Transmitter 1 splits Message $M_1$ of rate $R_1$ into two independent parts: Message $M_{1,\text{NF}}$ of rate $R_{1,\text{NF}}$ and Message $M_{1,\text{CS}}$ of rate $R_{1,\text{CS}}$, where $R_{1,\text{NF}}$ and $R_{1,\text{CS}}$ sum up to $R_1$. Here, NF stands for “no-feedback” and CS stands for “concatenated scheme”.

We first present a rough overview of the scheme. We start with the encodings. Transmitter 1 uses a fraction of its available power $P_1'$, for some $0 \leq P_1' \leq P_1$, to produce a sequence by encoding Message $M_{1,\text{NF}}$ using Gaussian codewords\(^6\) (without using the feedback). With the rest of the power ($P_1 - P_1'$) it produces a sequence of the same length by encoding Message $M_{1,\text{CS}}$ using our concatenated scheme. It then sends the sum of the two produced sequences over the channel. If the concatenated scheme is of parameter $\eta$ and its outer code is of blocklength $n$, then both sequences are of length $\eta n$. Transmitter 2 produces a sequence of equal length by encoding Message $M_2$ with power $P_2$ using the concatenated scheme and sends this sequence.

We next present a rough overview of the decoding at the receiver. The receiver first decodes the pair $(M_{1,\text{CS}}, M_2)$ by using the inner and the outer decoder of our concatenated scheme and treating the transmission of Message $M_{1,\text{NF}}$ as additional noise. From its guess of $(M_{1,\text{NF}}, M_2)$ the receiver cannot recover the sequences produced by our concatenated scheme because it is incognizant of the feedback noise. Nevertheless, it can form an estimate of both produced sequences (pretending that its guess of $(M_{1,\text{CS}}, M_2)$ is correct) and subtract the sum of the estimates from the received signal. Based on the resulting difference the receiver finally decodes message $M_{1,\text{NF}}$, which concludes the decoding.

In the following we describe the scheme in more detail. Given $M_{1,\text{NF}} = m_{1,\text{NF}}$, Transmitter 1 picks the codeword $u(m_{1,\text{NF}}) \triangleq (u_1, \ldots, u_m)\dagger$ corresponding to $m_{1,\text{NF}}$ from its Gaussian codebook. Given $M_{1,\text{CS}} = m_{1,\text{CS}}$, Transmitter 1 feeds $m_{1,\text{CS}}$ to Outer Encoder 1, which picks the codeword $\xi_1(m_{1,\text{CS}}) \triangleq (\xi_{1,1}, \ldots, \xi_{1,n})\dagger$ corresponding to $m_{1,\text{CS}}$ from its codebook and feeds it to Inner Encoder 1. Similarly, given $M_2 = m_2$, Transmitter 2 feeds $m_2$ to Outer Encoder 2, which picks the codeword $\xi_2(m_2) \triangleq (\xi_{2,1}, \ldots, \xi_{2,n})\dagger$ corresponding to $m_2$ and feeds it to Inner Encoder 2. Denoting the parameters of the inner encoders by $a_1, a_2, B_1, \text{ and } B_2$, respectively, Inner Encoder 1 forms the $\eta$-dimensional vectors

\begin{equation}
 a_1 \xi_{1,i} + B_1 V_{1,i}, \quad i \in \{1, \ldots, n\},
\end{equation}

\(^6\)To satisfy the powers constraints the Gaussian codewords should be of variance slightly less than $P_1'$. However, this is a technicality which we ignore.
and Inner Encoder 2 forms the $\eta$-dimensional vectors
\[ a_2 \xi_{2,i} + B_2 V_{2,i}, \quad i \in \{1, \ldots, n\}, \]
where
\[ V_{\nu,i} \triangleq (V_{\nu,(i-1)\eta+1}, \ldots, V_{\nu,i\eta})^T, \quad i \in \{1, \ldots, n\}, \nu \in \{1, 2\}. \]
The signal transmitted by Transmitter 1 is then described by the sum of the vectors in (47) and the vectors
\[ u_i \triangleq (u_{(i-1)\eta+1}, \ldots, u_{i\eta})^T, \quad i \in \{1, \ldots, n\}, \]
as follows:
\[ X_{1,i} = u_i + a_1 \xi_{1,i} + B_1 V_{1,i}, \quad i \in \{1, \ldots, n\}, \]
where
\[ X_{1,i} \triangleq (X_{1,(i-1)\eta+1}, \ldots, X_{1,i\eta})^T, \quad i \in \{1, \ldots, n\}. \]
The signal transmitted by Transmitter 2 is described by the vectors in (48) as follows:
\[ X_{2,i} = a_2 \xi_{2,i} + B_2 V_{2,i}, \quad i \in \{1, \ldots, n\}, \]
where
\[ X_{2,i} \triangleq (X_{2,(i-1)\eta+1}, \ldots, X_{2,i\eta})^T, \quad i \in \{1, \ldots, n\}. \]
Notice that if $a_1, a_2, B_1, B_2$ satisfy (21) and (22) for powers ($P_1 - P'_1$) and $P_2$, noise variance $(N + P'_1)$ and feedback-noise covariance matrix $K_{W_1 W_2}$ and if the outer code’s codewords are zero-mean and average block-power constrained to 1, then for sufficiently large blocklength $n$ the input sequences (49) and (50) satisfy the power constraint (3) with arbitrary high probability.

We next describe the decoding. The receiver first decodes the pair $(M_{1,CS}, M_2)$ based on the tuple $(Y_1, \ldots, Y_n)$ by treating the codeword $U(M_{1,NF})$ as additional noise and by applying inner and outer decoders of the concatenated scheme. Let $\hat{M}_{ICS}$ and $\hat{M}_2$ denote the receiver’s guesses of the messages $M_{1,CS}$ and $M_2$, and let $(\hat{\xi}_{1,1}^{(Rx)}, \ldots, \hat{\xi}_{1,n}^{(Rx)})$ and $(\hat{\xi}_{2,1}^{(Rx)}, \ldots, \hat{\xi}_{2,n}^{(Rx)})$ denote the corresponding codewords in the outer codes. The receiver then attempts to estimate and subtract the influence of the concatenated scheme (see (47) and (48)) by computing the differences
\[ \hat{Y}_i \triangleq (I_y - B_1 - B_2) Y_i - a_1 \hat{\xi}_{1,i}^{(Rx)} - a_2 \hat{\xi}_{2,i}^{(Rx)}, \quad i \in \{1, \ldots, n\}, \]
where the $\eta$-dimensional vectors $\{Y_i\}_{i=1}^n$ are defined as
\[ Y_i \triangleq (Y_{(i-1)\eta+1}, \ldots, Y_{i\eta})^T, \quad i \in \{1, \ldots, n\}. \]
If the receiver decoded $M_{1,CS}$ and $M_2$ correctly, i.e., if $\hat{M}_{1,CS} = M_{1,CS}$ and $\hat{M}_2 = M_2$, then (51) corresponds to
\[ U_i + B_1 W_{1,i} + B_2 W_{2,i} + Z_i, \quad i \in \{1, \ldots, n\}. \]
Finally, the receiver decodes Message $M_{1,NF}$ based on the differences $\{\hat{Y}_i\}_{i=1}^n$ using an optimal decoder for a Gaussian $\eta$-input antenna/$\eta$-output antenna channel where the noise sequences are white but correlated across antennas. Notice that because of the correlation of the noise sequences across antennas, the scheme might be improved if correlated Gaussian codewords are used to transmit Message $M_{1,NF}$.

**Remark 9.** This scheme applies also to settings with noisy or perfect partial feedback, if in the concatenated scheme $B_1$ is set to the all-zero matrix.
5.2 Rate-Splitting with Carleial’s Cover-Leung Scheme

Our second extension is based on modifying Carleial’s rate-splitting scheme [2]. Carleial’s scheme combines a variation of the Cover-Leung scheme [4] with a no-feedback scheme by means of rate-splitting. Here, we propose to modify his scheme by replacing the no-feedback scheme with our concatenated scheme. Since for \( \eta = 1, a_1 = \sqrt{P_1} \), and \( a_2 = \sqrt{P_2} \) our concatenated scheme results in an optimal no-feedback scheme the proposed extension includes Carleial’s scheme as a special case.

In the following we roughly sketch the idea of our extended scheme. For more details see Section H.

Our extended scheme is a Block-Markov scheme that is roughly characterized by the following five features. The first feature is that each block of \( n' \) channel uses is divided into \( (B + 1) \) blocks, each of length \( \eta n \) for positive integers \( \eta \) and \( n \), i.e., we assume that \( n' = (B + 1)\eta n \). The second feature is that each transmitter splits its message into two sequences of independent submessages. More precisely, Transmitter 1 splits its message \( M_1 \) into a sequence of independent submessages \( \{M_{1,CL,1}, \ldots, M_{1,CL,B}\} \) of rates \( R_{1,CL} \) and into a sequence of independent submessages \( \{M_{1,CS,1}, \ldots, M_{1,CS,B}\} \) of rates \( R_{1,CS} \). The rates \( R_{1,CL} \) and \( R_{1,CS} \) should be nonnegative and sum to \( R_1 (B+1)/B \), but otherwise can be chosen arbitrary depending on the parameters of the setting. Similarly, for Transmitter 2. (Here, the subscript CL stands for “Cover-Leung” and the subscript CS stands for “concatenated scheme”.)

The third feature is that after each block \( b \in \{1, \ldots, B\} \) Transmitter 1 and Transmitter 2 based on the observed feedback decode the other transmitter’s submessage \( M_{2,CL,b} \) and \( M_{1,CL,b} \), respectively. The two transmitters can accomplish the decodings in two different ways. Transmitter 1 either directly decodes Message \( M_{2,CL,b} \) or it first decodes \( M_{2,CS,b} \) before decoding the desired message \( M_{2,CL,b} \). Similarly, for Transmitter 2. Which alternative is better depends on the specific parameters of the setting.

The fourth feature is that the transmitters use Carleial’s variation of the Cover-Leung scheme to encode messages \( \{M_{1,CL,1}, \ldots, M_{1,CL,B}\} \) and \( \{M_{2,CL,1}, \ldots, M_{2,CL,B}\} \), and they use our concatenated scheme to encode messages \( \{M_{1,CS,1}, \ldots, M_{1,CS,B}\} \) and \( \{M_{2,CS,1}, \ldots, M_{2,CS,B}\} \). More specifically, before the transmission in Block \( b \in \{1, \ldots, B\} \) starts, Transmitter 1 chooses the codewords for messages \( M_{1,CL,b} \), \( M_{1,CL,b-1} \), and \( M_{2,CL,b-1} \) from the corresponding Gaussian codebooks and produces an \( \eta n \)-length sequence of power \( P_1' \), for some \( 0 \leq P_1' \leq P_1 \), by taking a linear combination of the chosen codewords. Transmitter 1 also produces an \( \eta n \)-length sequence of power \( P_2' \) by encoding message \( M_{1,CS,b} \) using the outer and inner encoders of our concatenated scheme where \( \eta \) is the parameter of the inner code and \( n \) is the blocklength of the outer code. Transmitter 1 then sends the sum of the two produced sequences in Block \( b \). In Block \( (B+1) \) Transmitter 1 picks the codewords for messages \( M_{1,CL,B} \) and \( M_{2,CL,B} \) from the corresponding Gaussian codebooks and sends a linear combination of power \( P_1' \) of these codewords. Similarly, for Transmitter 2.

The last feature is that after each Block \( b \in \{1, \ldots, B\} \) the receiver first decodes messages \( M_{1,CS,b} \) and \( M_{2,CS,b} \) using inner and outer decoder of our concatenated scheme and treating the sequences produced by encoding messages \( M_{1,CL,b-1}, M_{2,CL,b-1}, M_{1,CL,b} \) and \( M_{2,CL,b} \) as additional noise. From its guess of \( (M_{1,CS,b}, M_{2,CS,b}) \) the receiver cannot recover the sequences produced by our concatenated scheme because it is ignognizant of the feedback noise. Nevertheless, it can form an estimate of both produced sequences (pretending that its guess is correct) and subtract the sum of the estimates from the received signal. Based on the
resulting difference and based on similar differences which resulted in the previous block, the receiver finally decodes messages \( (M_{1,ICS,b−1}, M_{2,CL,b−1}) \). After Block \((B+1)\) the receiver decodes the pair \( (M_{1,CL,B}, M_{2,CL,B}) \). More general decoding orders at the receiver could be considered, but for simplicity, we restrict attention to this order.

**Remark 10.** The described extension applies also to settings with partial feedback, if \( B_1 \) is set to the all-zero matrix and if Carleial’s variation of the Cover-Leung scheme is specialized to partial feedback.

### 5.3 Interleaving & Rate-Splitting with Carleial’s Cover-Leung Scheme

Our third extension is based on rate-splitting an interleaved version of Carleial’s Cover-Leung scheme with an interleaved version of our concatenated scheme. We only describe here the general structure of the scheme. For more details see Appendix I.

Our extended scheme is a Block-Markov scheme and roughly characterized by the following six features.

- The first feature is that each block of \( n' \) channel uses is divided into \((B+1)\) blocks, each of length \( \eta n \) and each such block is further divided into \( \eta \) subblocks of length \( n \). Thus, \( B, \eta, \) and \( n \) are positive integers and \( n' = (B+1)\eta n \).

- The second feature is that each transmitter splits its message into two sequences of independent submessages. More precisely, Transmitter 1 splits its message \( M_1 \) into a sequence of independent submessages \( \{M_{1,ICL,1}, \ldots, M_{1,ICL,\eta B}\} \) and into a sequence of independent submessages \( \{M_{1,ICS,1}, \ldots, M_{1,ICS,B}\} \). Notice that the first sequence of submessages is of length \( \eta B \), and the second of length \( B \). Message \( M_{1,ICL,(b−1)\eta+\ell}, \) for \( b \in \{1, \ldots, B\} \) and \( \ell \in \{1, \ldots, \eta\} \), is of rate \( R_{1,ICL,\ell} \), and Message \( M_{1,ICS,b} \) for \( b \in \{1, \ldots, B\} \), is of rate \( R_{1,ICS} \). The rates \( R_{1,ICL,1}, \ldots, R_{1,ICL,\eta}, \) and \( R_{1,ICS} \) should be nonnegative and should sum to \( R_1 \frac{B+1}{B} \), but otherwise can be chosen arbitrary depending on the parameters of the setting. Similarly, for Transmitter 2. (The subscript ICL stands for “interleaved Cover-Leung” and the subscript ICS stands for “interleaved concatenated scheme”.)

- The third feature is that after each subblock \( b \in \{1, \ldots, B\eta\} \), Transmitter 1 and Transmitter 2 use the observed feedback to decode the other transmitter’s submessage \( M_{2,ICL,b} \) and \( M_{1,ICL,b} \), respectively.

- The fourth feature is that after each subblock \( b \in \{1, \ldots, B\eta\} \) and after having decoded \( M_{2,ICL,b} \) and \( M_{1,ICL,b} \), respectively, Transmitter 1 and Transmitter 2 compute “cleaned” feedback outputs as follows. Transmitter 1 computes its “cleaned” feedback in the following two steps. In the first step it reconstructs the sequence that was produced by Transmitter 2 in this subblock \( b \) to encode messages \( M_{2,ICL,b}, M_{1,ICL,b−\eta}, \) and \( M_{2,ICL,b−\eta} \) (pretending that its guesses of \( M_{2,ICL,b} \) and \( M_{2,ICL,b−\eta} \) are correct). In the second step it subtracts from the observed feedback the sum of this reconstructed sequence and the sequence itself produced in subblock \( b \) to encode \( M_{1,ICL,b}, M_{1,ICL,b−\eta}, \) and \( M_{2,ICL,b−\eta} \). Similarly, for Transmitter 2.

- The fifth feature is that the transmitters use an interleaved version of Carleial’s Cover-Leung scheme to encode Messages \( \{M_{1,ICL,1}, \ldots, M_{1,ICL,\eta B}\} \) and \( \{M_{2,ICL,1}, \ldots, M_{2,ICL,\eta B}\} \), and they use an interleaved version of our concatenated scheme to encode Messages \( \{M_{1,ICS,1}, \ldots, M_{1,ICS,B}\} \) and \( \{M_{2,ICS,1}, \ldots, M_{2,ICS,B}\} \). In the following we describe these encodings for a fixed block \( b \in \{1, \ldots, B\} \). In Block \( b \), Transmitter 1 sends the sum of two
ηn-length sequences which are produced as follows. The first sequence is of power $P_1'$, for some $0 \leq P_1' \leq P_1$, and is produced as described in the following. To produce the $ℓ$-th sub-block of the sequence, for $ℓ \in \{1, \ldots, η\}$, Transmitter 1 chooses the $n$-length codewords for Messages $M_{1, ICS, b}^{(b−1)η+ℓ}$, $M_{1, ICL, (b−2)η+ℓ}$, and $M_{2, ICL, (b−2)η+ℓ}$ from the corresponding Gaussian codebooks and takes a linear combination of these chosen codewords. We notice that here each pair of messages $(M_{1, ICL, ˜b}, M_{2, ICL, ˜b})$, for $˜b \in \{1, \ldots, Bη\}$, is encoded into Subblocks $˜b$ and $b+η$, and not—as in Carleial’s original scheme—into Subblocks $b$ and $b+1$. The second sequence is of power $(P_1−P_1')$ and produced as described in the following. Transmitter 1 first applies its outer encoder to encode Message $M_{1, ICS, b}$. The produced codeword is then fed to a modified version of Transmitter 1’s inner encoder, where the inner encoder is modified as described by the following two items. 1.) Instead of the original feedback the modified inner encoder uses the “cleaned” feedback described in the fourth feature. 2.) Unlike in the sequence produced by the original inner encoder where the $ℓ$-th fed codeword symbol is encoded into $η$ subsequent symbols at positions $(ℓ−1)η+1$ to $ℓη$, the modified inner encoder encodes the $ℓ$-th fed codeword symbol into the $η$ symbols at positions $ℓ, n+ℓ, \ldots, (η−1)n+ℓ$, for $ℓ \in \{1, \ldots, η\}$. The second sequence produced by Transmitter 1 is then given by the sequence produced by its modified inner encoder. Similarly, for Transmitter 2. Notice that the chosen interleaving of the modified inner encoders preserves the causality of the feedback. Moreover, it implies that in the interleaved sequence the symbols in Subblock $˜b$, for $˜b \in \{(b−1)η+1, \ldots, bη\}$, only depend on feedback outputs of previous subblocks $1, \ldots, b−1$ and not on feedback outputs of the current Subblock $b$. This is the reason why the modified inner encoder can use the “cleaned” feedback instead of the original feedback.

The sixth and last feature is that the receiver first decodes Messages $\{M_{1, ICL, ˜b}\}_{b=1}^{\eta B}$ and $\{M_{2, ICL, ˜b}\}_{b=1}^{\eta B}$ and only thereafter decodes Messages $\{M_{1, ICS, b}\}_{b=1}^{\eta B}$ and $\{M_{2, ICS, b}\}_{b=1}^{\eta B}$. More specifically, the receiver first decodes Messages $\{(M_{1, ICL, (b−1)η+1}, M_{2, ICL, (b−1)η+2})\}_{b=1}^{\eta B}$, followed by Messages $\{(M_{1, ICL, (b−1)η+2}, M_{2, ICL, (b−1)η+1})\}_{b=1}^{\eta B}$, etc. The receiver then reconstructs the sequences produced to encode these messages (pretending its guesses are correct) and subtracts them from the received signal. Based on the resulting difference, which we call the “cleaned” output signal, the receiver decodes Messages $\{M_{1, ICS, b}\}_{b=1}^{\eta B}$ and $\{M_{2, ICS, b}\}_{b=1}^{\eta B}$. To this end, it first reverses the interleaving and then applies the inner and outer decoders of our concatenated scheme.

Notice that in the presented scheme, Messages $\{M_{1, ICS, b}\}_{b=1}^{\eta B}$ and $\{M_{2, ICS, b}\}_{b=1}^{\eta B}$ are decoded based on the “cleaned” output signal and they are encoded using the “cleaned” feedbacks. The “cleaned” output signal and the “cleaned” feedbacks correspond to the output signals and the feedbacks in a situation where only the interleaved concatenated scheme is employed but not the interleaved version of Carleial’s Cover-Leung scheme. Therefore, in the presented rate-splitting scheme there is no degradation in performance of the interleaved concatenated scheme due to the rate-splitting with Carleial’s Cover-Leung scheme.

Further, notice that in a given Block $b \in \{1, \ldots, B\}$ the sum of the two sequences produced to encode Messages $M_{1, ICS, b}$ and $M_{2, ICS, b}$ is of different power in each of the $η$ subblocks. Thus, these sequences introduce different noise levels on the receiver’s decoding of Messages $\{(M_{1, ICL, (b−1)η+ℓ}, M_{2, ICL, (b−1)η+ℓ})\}_{ℓ=1}^{η}$, and consequently the rates $\{R_{1, ICL, ℓ}\}_{ℓ=1}^{η}$ and $\{R_{2, ICL, ℓ}\}_{ℓ=1}^{η}$ should be chosen depending on $ℓ$.

Remark 11. Also this third extension can be used in settings with partial feedback, if in the concatenated scheme the parameter $B_1$ is set to the all-zero matrix and if Carleial’s scheme
is specialized to partial feedback.

6 Main Results

6.1 Achievable Regions for Noisy Feedback

We present several achievability results for the setting with noisy feedback based on the schemes in Section 4 and 5. Our first achievability result is obtained by evaluating the rates achieved by our concatenated scheme for general parameters in Section 4.1. Before stating this result in Theorem 14 ahead, we define the rate region which our concatenated scheme achieves for a fixed choice of parameters \( \eta, a_1, a_2, B_1, B_2, C \), see upcoming Definition 12. (An alternative recursive formulation of this achievable region is presented in Section C.1.)

**Definition 12.** Let \( P_1, P_2, N > 0 \) and \( K_{W_1 W_2} \geq 0 \) be given. Let \( \eta \) be a positive integer, let \( a_1, a_2 \) be \( \eta \)-dimensional vectors, let \( B_1, B_2 \) be \( \eta \times \eta \) strictly lower-triangular matrices, and let \( C \) be a \( 2 \times \eta \) matrix. Depending on the matrix \( C \) the rate region \( R(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \) is defined as follows:

- If \( CC^\top \) is nonsingular,\(^7\) then \( R(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \) is defined as the set of all rate-pairs \((R_1, R_2)\) satisfying

\[
\begin{align*}
R_1 &\leq \frac{1}{2\eta} \log \frac{|C(a_1a_1^\top + \eta I_{\eta} + B_b(K_{W_1 W_2} \otimes I_{\eta})B_b^\top)C^\top|}{|C(N I_{\eta} + B_b(K_{W_1 W_2} \otimes I_{\eta})B_b^\top)C^\top|}, \\
R_2 &\leq \frac{1}{2\eta} \log \frac{|C(a_2a_2^\top + \eta I_{\eta} + B_b(K_{W_1 W_2} \otimes I_{\eta})B_b^\top)C^\top|}{|C(N I_{\eta} + B_b(K_{W_1 W_2} \otimes I_{\eta})B_b^\top)C^\top|}, \\
R_1 + R_2 &\leq \frac{1}{2\eta} \log \frac{|C(A_b A_b^\top + \eta I_{\eta} + B_b(K_{W_1 W_2} \otimes I_{\eta})B_b^\top)C^\top|}{|C(N I_{\eta} + B_b(K_{W_1 W_2} \otimes I_{\eta})B_b^\top)C^\top|},
\end{align*}
\]

where \( A_b, B_b, \) and \( B_{\Sigma} \) are defined in (18), (19), and (20), and where \( \otimes \) denotes the Kronecker product.

- If \( CC^\top \) is singular but \( C \neq 0 \), then \( R(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \) is defined as the set of all rate pairs \((R_1, R_2)\) satisfying (52)-(54) when the \( 2 \times \eta \) matrix \( C \) is replaced by the \( \eta \)-dimensional row-vector obtained by choosing one of the non-zero rows of \( C \).\(^8\)

- If \( C = 0 \), then \( R(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \) is defined as the set containing only the origin.

**Definition 13.** Let \( P_1, P_2, N > 0 \) and \( K_{W_1 W_2} \geq 0 \) be given. Define the region \( R(P_1, P_2, N, K_{W_1 W_2}) \) (or for short \( R \)) as

\[
R(P_1, P_2, N, K_{W_1 W_2}) \triangleq \text{cl} \left( \bigcup_{\eta, a_1, a_2, B_1, B_2, C} R(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \right),
\]

where the union is over all tuples \((\eta, a_1, a_2, B_1, B_2, C)\) satisfying (21) and (22).

---

\(^7\)Whenever \( \eta \in \mathbb{N} \) is larger than 1, there is no loss in optimality in restricting attention to matrices \( C \) so that \( CC^\top \) is nonsingular.

\(^8\)When \( CC^\top \) is singular then the two rows of \( C \) are linearly dependent and it does not matter which non-zero row is chosen.
Theorem 14 (Noisy Feedback). The capacity region of the two-user AWGN MAC with noisy feedback $C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2})$ includes the rate region $\mathcal{R}(P_1, P_2, N, K_{W_1W_2})$, i.e.,

$$C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq \mathcal{R}(P_1, P_2, N, K_{W_1W_2}).$$

Proof. Follows by evaluating the capacity region of the additive Gaussian MAC $\xi_1, \xi_2 \leftrightarrow (\hat{\xi}_1, \hat{\xi}_2)$ in (24) and scaling the region by a factor $\eta^{-1}$. The details are omitted.

Proposition 15 (Monotonicity and Convergence of Region $\mathcal{R}$). Let $N > 0$ be given. The achievable region $\mathcal{R}(P_1, P_2, N, K_{W_1W_2})$ satisfies the following three properties:

1. Given $P_1, P_2 > 0$, it is monotonically decreasing in $K_{W_1W_2}$ with respect to the Loewner order, i.e., for positive semidefinite matrices $K_{W_1W_2}$ and $K'_{W_1W_2}$:

$$K_{W_1W_2} \succeq K'_{W_1W_2} \quad \Rightarrow \quad \mathcal{R}(P_1, P_2, N, K_{W_1W_2}) \subseteq \mathcal{R}(P_1, P_2, N, K'_{W_1W_2}).$$

2. Given $K_{W_1W_2} \succeq 0$, it is continuous in $P_1$ and $P_2$, i.e., for all $P_1, P_2 > 0$:

$$\text{cl} \left( \bigcup_{\delta > 0} \mathcal{R}(P_1 - \delta, P_2 - \delta, N, K_{W_1W_2}) \right) = \mathcal{R}(P_1, P_2, N, K_{W_1W_2}).$$

3. Given $P_1, P_2 > 0$, it converges to the perfect-feedback achievable region $\mathcal{R}(P_1, P_2, N, 0)$ as the feedback-noise variances tend to 0 irrespective of the feedback-noise correlations, i.e.,

$$\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0: \text{tr}(K) \leq \sigma^2} \mathcal{R}(P_1, P_2, N, K) \right) = \mathcal{R}(P_1, P_2, N, 0).$$

Proof. See Appendix 8.1.

Specializing Theorem 14 to symmetric channels, i.e., $P_1 = P_2 = P$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, and to $\eta = 2$ and the choice of parameters $a_1, a_2, B_1, B_2$, and $C$ presented in Section E.1 yields the following corollary.

Corollary 16 (Symmetric Noisy Feedback Channels). The capacity region of the symmetric two-user AWGN MAC with noisy feedback, $C_{\text{NoisyFB}}(P, P, N, K_{W_1W_2})$ where

$$K_{W_1W_2} = \begin{pmatrix} \sigma^2 & \sigma^2 \varrho \\ \sigma^2 \varrho & \sigma^2 \end{pmatrix},$$

includes all rate pairs $(R_1, R_2)$ satisfying

$$R_1, R_2 \leq \frac{1}{4} \log \left( 1 + \frac{2P}{N} \right) + \frac{1}{4} \log \left( 1 + \frac{P^2}{(2P + N)(P + N + \sigma^2 + \frac{2P}{N}(\sigma^2 - \varrho \sigma^2))} \right) \quad \text{and} \quad R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{2P}{N} \right) + \frac{1}{4} \log \left( 1 + \frac{2P^2}{(2P + N)(P + N + \sigma^2 + \frac{2P}{N}(\sigma^2 - \varrho \sigma^2))} \right).$$

In particular, it includes the equal-rate points $(R, R)$ whenever

$$R \leq \frac{1}{4} \log \left( 1 + \frac{2P}{N} \right) + \frac{1}{8} \log \left( 1 + \frac{2P^2}{(2P + N)(P + N + \sigma^2 + \frac{2P}{N}(\sigma^2(1 - \varrho)))} \right).$$

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Specializing Theorem 14 to perfect feedback, i.e., $K_{W_1W_2} = 0$, and to the choice of parameters presented in Section F.1 yields the following remark.

**Remark 17 (Perfect Feedback).** Let $P_1, P_2, N > 0$ be given. For the two-user AWGN MAC with perfect feedback our concatenated scheme achieves all rate pairs inside the region $\mathcal{R}_{\Omega_0}(P_1, P_2, N)$, i.e.,

$$\mathcal{R}(P_1, P_2, N, 0) \supseteq \mathcal{R}_{\Omega_0}(P_1, P_2, N).$$

**Proof.** Is based on the specific choice of parameters in Section F.1, i.e., on the regions $\mathcal{R}_{\eta}(P_1, P_2, N, 0)$ in Remark 56. For details, see Section 8.3.

Our last achievability result for noisy feedback is based on the rate-splitting scheme in Section 5.1. Before stating the result in Proposition 20, we define:

**Definition 19.** Let $P'_1, P''_1, P_2, N > 0$, $K_{W_1W_2} \geq 0$ be given. For fixed $\eta \in \mathbb{N}$ and fixed $\eta$-dimensional vectors $\mathbf{a}_1, \mathbf{a}_2$, $\eta \times \eta$ strictly lower-triangular matrices $B_1, B_2$, and $2 \times \eta$ matrix $C$, define the region $\mathcal{R}_{RS,1}(P'_1, N, K_{W_1W_2}; \eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C)$ as the set of all rate pairs $(R_1, R_2)$ that for some nonnegative $R_{1,CS}, R_{1, NF}$ summing to $R_1$ satisfy the following two conditions:

$$\begin{align*}
(R_1, R_2) &\in \mathcal{R} \left( N + P'_1, K_{W_1W_2}; \eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C \right), \\
R_{1, NF} &\leq \frac{1}{2\eta} \log \left( \frac{|P'_1|\eta + N\eta + B_1(K_{W_1W_2} \otimes I_\eta)B_1^T|}{|N\eta + B_2(K_{W_1W_2} \otimes I_\eta)B_2^T|} \right),
\end{align*}$$

where $A_0, B_0, B_1, B_2$, and $B_{\Sigma}$ are defined in (18), (19), and (20).

Similarly, given $P'_1, P''_1, P_2, N > 0$ and $K_{W_1W_2} \geq 0$, define the region $\mathcal{R}_{RS,2}(P''_2, N, K_{W_1W_2}; \eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C)$ analogously to the region $\mathcal{R}_{RS,1}(P'_1, N, K_{W_1W_2}; \eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C)$, but with exchanged indices 1 and 2.

**Definition 19.** Given $P'_1, P''_1, P_2, N > 0$ and $K_{W_1W_2} \geq 0$, define the region $\mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, K_{W_1W_2})$ (or for short $\mathcal{R}_{RS,1}$) as

$$\mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, K_{W_1W_2}) \triangleq \bigcup_{\eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C} \mathcal{R}_{RS,1}(P'_1, N, K_{W_1W_2}; \eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C),$$

where the union is over all tuples $(\eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C)$ satisfying (21) and (22) for powers $P'_1$ and $P_2$, noise variance $(N + P'_1)$, and feedback-noise covariance matrix $K_{W_1W_2}$. Similarly, given $P'_1, P''_1, P_2, N > 0$ and $K_{W_1W_2} \geq 0$, define the region $\mathcal{R}_{RS,2}(P'_1, P''_1, P_2, N, K_{W_1W_2})$ (or for short $\mathcal{R}_{RS,2}$) as

$$\mathcal{R}_{RS,2}(P'_1, P''_1, P_2, N, K_{W_1W_2}) \triangleq \bigcup_{\eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C} \mathcal{R}_{RS,2}(P'_2, N, K_{W_1W_2}; \eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C),$$

where the union is over all tuples $(\eta, \mathbf{a}_1, \mathbf{a}_2, B_1, B_2, C)$ satisfying (21) and (22) for powers $P'_1$ and $P''_1$, noise variance $(N + P'_1)$, and feedback-noise covariance matrix $K_{W_1W_2}$.

**Proposition 20 (Rate-Splitting for Noisy Feedback).** The capacity region $C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2})$ includes $\mathcal{R}_{RS,1}(P'_1, (P_1 - P'_1), P_2, N, K_{W_1W_2})$ for any $P'_1 \in [0, 1]$, and it includes $\mathcal{R}_{RS,2}(P_1, P'_2, (P_2 - P'_2), N, K_{W_1W_2})$ for any $P'_2 \in [0, P_2]$:

$$C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq \bigcup_{P'_1 \in [0, P_1]} \mathcal{R}_{RS,1}(P'_1, (P_1 - P'_1), P_2, N, K_{W_1W_2}),$$

$$C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq \bigcup_{P'_2 \in [0, P_2]} \mathcal{R}_{RS,2}(P_1, P'_2, (P_2 - P'_2), N, K_{W_1W_2}),$$

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and
\[
C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq \bigcup_{P'_2 \in [0, P_2]} \mathcal{R}_{\text{RS},2} \left( P_1, P'_2, (P_2 - P'_2), N, K_{W_1W_2} \right).
\]

Proof. The rate region is achieved by the rate-splitting scheme in Section 5.1. The analysis is based on Theorem 14, on the capacity of a Gaussian multi-input antenna/multi-output antenna channel with noise sequences that are temporally-white but correlated across the antennas, and on a genie-aided argument as in [10] and [19, p. 419]. The details are omitted.

Proposition 21 (Monotonicity and Convergence of Regions \( \mathcal{R}_{\text{RS},1} \) and \( \mathcal{R}_{\text{RS},2} \)). Let \( N > 0 \) be given. The achievable region \( \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, N, K_{W_1W_2} \right) \) satisfies the following three properties:

1. Given \( P'_1, P''_1, P_2 > 0 \), it is monotonically decreasing in \( K_{W_1W_2} \) with respect to the Loewner order, i.e., for positive semidefinite matrices \( K_{W_1W_2} \) and \( K'_{W_1W_2} \):
   \[
   \left( K_{W_1W_2} \preceq K'_{W_1W_2} \right) \implies \left( \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, P_2, N, K_{W_1W_2} \right) \subseteq \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, P_2, N, K'_{W_1W_2} \right) \right).
   \]

2. Given \( K_{W_1W_2} \geq 0 \), it is continuous in \( P'_1, P''_1, P_2 \), i.e., for all \( P'_1, P''_1, P_2 > 0 \):
   \[
   \text{cl} \left( \bigcup_{\delta > 0} \mathcal{R}_{\text{RS},1} \left( P'_1 - \delta, P''_1 - \delta, P_2 - \delta, N, K_{W_1W_2} \right) \right) = \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, P_2, N, K_{W_1W_2} \right).
   \]

3. Given \( P'_1, P''_1, P_2 > 0 \), it converges to the perfect-feedback achievable region \( \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, P_2, N, 0 \right) \) as the feedback-noise variances tend to 0 irrespective of the feedback-noise correlations, i.e.,
   \[
   \text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \succeq 0 : \text{tr}(K) \leq \sigma^2} \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, P_2, N, K \right) \right) = \mathcal{R}_{\text{RS},1} \left( P'_1, P''_1, P_2, N, 0 \right).
   \]

Similarly, for \( \mathcal{R}_{\text{RS},2} \left( P'_1, P''_1, P_2, N, K_{W_1W_2} \right) \).

Proof. Follows from Proposition 15 and because for fixed \( a_1, a_2, B_1, B_2 \), and \( C \) the right-hand side of (57) satisfies the following three properties. It is monotonically decreasing in \( K_{W_1W_2} \) with respect to the Loewner order, it is continuous in \( P'_1 \), and it converges to \( \frac{1}{2} \log \left( 1 + \frac{P'_1}{N} \right) \) as the feedback-noise variances tend to 0 irrespective of the feedback-noise correlations. The details are omitted.

6.2 Achievable Regions for Noisy Partial Feedback

Evaluating the rates achieved by our concatenated scheme with general parameters for noisy partial feedback leads to an analogous result to Theorem 14. Before stating the result in Theorem 24, we define:
**Definition 22.** Let $P_1, P_2, N > 0$ and $\sigma_2^2 \geq 0$ be given. Let $\eta$ be a positive integer, let $a_1, a_2$ be an $\eta$-dimensional vector, let $B_2$ be a strictly lower-triangular $\eta \times \eta$ matrix, and let $C_P$ be a $2 \times \eta$ matrix. Then, depending on the matrix $C_P$ the rate region $\mathcal{R}_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P)$ is defined as follows:

- If $C_P C_P^T$ is nonsingular,\(^9\) then $\mathcal{R}_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P)$ is defined as the set of all rate pairs $(R_1, R_2)$ satisfying

\[
R_1 \leq \frac{1}{2\eta} \log \frac{|C_P (a_1 a_1^T + N I_{\eta} + \sigma_2^2 B_2 B_2^T) C_P|}{|C_P (N I_{\eta} + \sigma_2^2 B_2 B_2^T) C_P|},
\]

(58)

\[
R_2 \leq \frac{1}{2\eta} \log \frac{|C_P (a_2 a_2^T + N I_{\eta} + \sigma_2^2 B_2 B_2^T) C_P|}{|C_P (N I_{\eta} + \sigma_2^2 B_2 B_2^T) C_P|},
\]

(59)

\[
R_1 + R_2 \leq \frac{1}{2\eta} \log \frac{|C_P (A_b A_b^T + N I_{\eta} + \sigma_2^2 B_2 B_2^T) C_P|}{|C_P (N I_{\eta} + \sigma_2^2 B_2 B_2^T) C_P|},
\]

(60)

where $A_b$ is defined in (18).

- If $C_P C_P^T$ is singular but $C_P \neq 0$, then $\mathcal{R}_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P)$ is defined as the set of all rate pairs $(R_1, R_2)$ satisfying (58)–(60) when the $2 \times \eta$ matrix $C_P$ is replaced by the $\eta$-dimensional row-vector obtained by choosing one of the non-zero rows of $C_P$.

- If $C_P = 0$, then $\mathcal{R}_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P)$ is defined as the set containing only the origin.

(An alternative (recursive) formulation of the region $\mathcal{R}_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P)$ is presented in Section C.2.)

**Definition 23.** Let $P_1, P_2, N > 0$ and $\sigma_2^2 \geq 0$ be given. Define

\[
\mathcal{R}_P(P_1, P_2, N, \sigma_2^2) \triangleq \text{cl}\left( \bigcup_{\eta, a_1, a_2, B_2, C_P} \mathcal{R}_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P) \right),
\]

where the union is over all tuples $(\eta, a_1, a_2, B_2, C_P)$ satisfying (36) and (37).

**Theorem 24 (Noisy Partial Feedback).** The capacity region $C_{\text{NoisyPartialFB}}(P_1, P_2, N, \sigma_2^2)$ of the two-user AWGN MAC with noisy partial feedback to Transmitter 2 includes the rate region $\mathcal{R}_P(P_1, P_2, N, \sigma_2^2)$, i.e.,

\[
C_{\text{NoisyPartialFB}}(P_1, P_2, N, \sigma_2^2) \supseteq \mathcal{R}_P(P_1, P_2, N, \sigma_2^2).
\]

**Proof.** Follows from Theorem 14 by choosing $B_1$ as the all-zero matrix. \(\square\)

Specializing Theorem 24 to equal powers channels, i.e., $P_1 = P_2 = P$, and to $\eta = 2$ and the choice of the parameters presented in Section E.1 yields the following corollary.

---

9Whenever $\eta \in \mathbb{N}$ is larger than 1, there is no loss in optimality in restricting attention to matrices $C_P$ so that $C_P C_P^T$ is nonsingular.
Corollary 25 (Equal Powers and Noisy Partial Feedback). The capacity region $C_{\text{NoisyPartialFB}}(P, P, N, \sigma_2^2)$ of the two-user AWGN MAC with noisy partial feedback to Transmitter 2 and equal powers $P_1 = P_2 = P$ includes all rate pairs $(R_1, R_2)$ satisfying

\begin{align*}
R_1 &\leq \frac{1}{4} \log \left( 1 + \frac{2P}{N} \right) + \frac{1}{4} \log \left( 1 - \frac{P}{2P+N} \frac{P \frac{P^2}{N} \sigma_2^2}{(2P+N+\sigma_2^2+\frac{P}{N} \sigma_2^2)(P+N+\sigma_2^2+\frac{P}{N} \sigma_2^2)} \right), \\
R_2 &\leq \frac{1}{4} \log \left( 1 + \frac{2P}{N} \right) + \frac{1}{4} \log \left( 1 + \frac{P}{2P+N} \cdot \frac{P}{P+N+\sigma_2^2+\frac{P}{N} \sigma_2^2} \right), \\
R_1 + R_2 &\leq \frac{1}{2} \log \left( 1 + \frac{2P}{N} \right) + \frac{1}{4} \log \left( 1 + \frac{2P^2}{2P+N} \right) \left( \sqrt{1 + \frac{P(P+N+\sigma_2^2)}{(P+N+\sigma_2^2+\frac{P}{N} \sigma_2^2)^2}} - 1 \right) + \frac{P}{2P+N} \right)^2 \frac{P(2P+N+\sigma_2^2)}{2P+N+\sigma_2^2+\frac{P}{N} \sigma_2^2}(P+N+\sigma_2^2+\frac{P}{N} \sigma_2^2). \tag{61}
\end{align*}

6.3 Achievable Regions for Perfect Partial Feedback

Specializing Theorem 24 to perfect partial feedback, i.e., to $\sigma_2^2 = 0$, results in the following corollary.

Corollary 26 (Perfect Partial Feedback). The capacity region $C_{\text{PerfectPartialFB}}(P_1, P_2, N)$ of the two-user AWGN MAC with perfect partial feedback to Transmitter 2 includes the rate region $R_P(P_1, P_2, N, 0)$, i.e.,

$$C_{\text{PerfectPartialFB}}(P_1, P_2, N) \supseteq R_P(P_1, P_2, N, 0).$$

Specializing Corollary 26 to $\eta = 2$ and the choice of parameters in Section E.1 yields:

Corollary 27. The capacity region $C_{\text{PerfectPartialFB}}(P_1, P_2, N)$ of the two-user AWGN MAC with perfect partial feedback to Transmitter 2 includes all rate pairs $(R_1, R_2)$ satisfying

\begin{align*}
R_1 &\leq \frac{1}{4} \log \left( 1 + \frac{2P_1}{N} \right), \\
R_2 &\leq \frac{1}{4} \log \left( 1 + \frac{P_2 \left( 2 + \frac{P_2}{P_1+N} \right)}{N} \right), \\
R_1 + R_2 &\leq \frac{1}{4} \log \left( 1 + \frac{P_1 + P_2}{N} \right) + \frac{1}{4} \log \left( 1 + \frac{P_1 \frac{P_2+N}{P_1+N} + P_2 + 2 \sqrt{P_1 P_2 \frac{P_1}{P_1+N} P_2 \frac{P_2}{P_2+N}}}{N} \right). \tag{62}
\end{align*}
The last two achievability results are based on the rate-splitting schemes in Sections 5.2 and 5.3. Characterizing the achievable regions of these rate-splitting scheme is in general cumbersome. However, for perfect partial feedback the regions become tractable.

**Proposition 28 (Rate-Splitting for Perfect Partial Feedback I).** The capacity region $C_{\text{PerfectPartialFB}}(P_1, P_2, N)$ of the two-user AWGN MAC with perfect partial feedback to Transmitter 2 includes all rate pairs $(R_1, R_2)$ which for nonnegative $R_{1,\text{CL}}, R_{1,\text{CS}}$ summing to $R_1$ and nonnegative $R_{2,\text{CL}}, R_{2,\text{CS}}$ summing to $R_2$ and for some choice of $\rho_1, \rho_2 \in [0, 1]$ and $P'_1 \in [0, P_1], P'_2 \in [0, P_2]$ satisfy

$$(R_{1,\text{CL}}, R_{2,\text{CL}}) \in \mathcal{R}_{\text{CL}}^{(\rho_1, \rho_2)}(P'_1, P'_2, N),$$

$$(R_{1,\text{CS}}, R_{2,\text{CS}}) \in \mathcal{R}_P\left((P_1 - P'_1), (P_2 - P'_2), N + P'_1 + P'_2 + 2\sqrt{P'_1 P'_2 \rho_1 \rho_2}, 0\right).$$

**Proof.** The rate region is achieved by the rate-splitting scheme in Section 5.2 (see also Appendix H) when we choose the version where Transmitter 2 decodes the submessages encoded with the concatenated scheme before decoding the submessages encoded with Carleial’s Cover-Leung scheme. The analysis of the rate-splitting scheme is based on a genie-aided argument as in [10] and [19]. The details are omitted.

**Proposition 29 (Rate-Splitting for Perfect Partial Feedback II).** The capacity region $C_{\text{PerfectPartialFB}}(P_1, P_2, N)$ of the two-user AWGN MAC with perfect partial feedback to Transmitter 2 includes all rate pairs $(R_1, R_2)$ which for nonnegative $(R_{1,\text{ICL,1}}, R_{1,\text{ICL,2}}, R_{1,\text{ICS}})$ summing to $R_1$ and nonnegative $(R_{2,\text{ICL,1}}, R_{2,\text{ICL,2}}, R_{2,\text{ICS}})$ summing to $R_2$ and for some choice of $\rho_1, \rho_2 \in [0, 1]$ and $P'_1 \in [0, P_1], P'_2 \in [0, P_2]$ satisfy all the following constraints.

$$R_{1,\text{ICS}} \leq \frac{1}{4} \log \left(1 + \frac{2(P_1 - P'_1)}{N}\right),$$

$$R_{2,\text{ICS}} \leq \frac{1}{4} \log \left(1 + \frac{(P_2 - P'_2) \left(2 + \frac{P_2 - P'_2}{N - P'_2 + N}\right)}{N}\right),$$

$$R_{1,\text{ICS}} + R_{2,\text{ICS}} \leq \frac{1}{4} \log \left(1 + \frac{P_1 - P'_1 + P_2 - P'_2}{N}\right) + \frac{1}{4} \log \left(1 + \frac{N_2}{N}\right),$$

$$R_{1,\text{ICL,1}} \leq \frac{1}{4} \log \left(1 + \frac{(1 - \rho_1^2)P'_1}{P'_1 + N}\right),$$

$$R_{1,\text{ICL,1}} \leq \frac{1}{4} \log \left(1 + \frac{(1 - \rho_1^2)P'_1}{N_1 + N}\right) + \frac{1}{4} \log \left(1 + \frac{\sqrt{\rho_1^2 P'_1 + \rho_2^2 P'_2}^2}{N_1 + N + (1 - \rho_1^2)P'_1 + (1 - \rho_2^2)P'_2}\right)$$

$$R_{2,\text{ICL,1}} \leq \frac{1}{4} \log \left(1 + \frac{(1 - \rho_2^2)P'_2}{N_1 + N}\right),$$

$$R_{1,\text{ICL,1}} + R_{2,\text{ICL,1}} \leq \frac{1}{4} \log \left(1 + \frac{P'_1 + P'_2 + 2\sqrt{P'_1 P'_2 \rho_1 \rho_2}}{N_1 + N}\right),$$

$$R_{1,\text{ICL,2}} \leq \frac{1}{4} \log \left(1 + \frac{(1 - \rho_1^2)P'_1}{P'_1 N_1 + N}\right),$$

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\[ R_{1,\text{ICL}} \leq \frac{1}{4} \log \left( 1 + \frac{(1 - \rho_1^2)P_1'}{N_2 + N} \right) + \frac{1}{4} \log \left( 1 + \frac{\left( \sqrt{\rho_1^2 P_1' + \rho_2^2 P_2'} \right)^2}{N_2 + N + (1 - \rho_1^2)P_1' + (1 - \rho_2^2)P_2'} \right), \]

\[ R_{2,\text{ICL}} \leq \frac{1}{4} \log \left( 1 + \frac{(1 - \rho_2^2)P_2'}{N_2 + N} \right), \]

\[ R_{1,\text{ICL}} + R_{2,\text{ICL}} \leq \frac{1}{4} \log \left( 1 + \frac{P_1' + P_2' + 2 \sqrt{P_1'P_2'\rho_1^2\rho_2^2}}{N_2 + N} \right). \]

Here,

\[ N_1 \triangleq P_1 - P_1' + P_2 - P_2', \]

\[ N_2 \triangleq \frac{(P_1 - P_1')(P_2 - P_2') + N}{(P_1 - P_1' + P_2 - P_2' + N)} + (P_2 - P_2') + 2 \sqrt{\frac{(P_1 - P_1')^2}{(P_1 - P_1' + N)(P_1 - P_1' + P_2 - P_2' + N)}}, \]

Proof. The rate region is achieved by the rate-splitting scheme in Section 5.3 (see also Appendix I, in particular Section I.1) and by choosing the parameters of the concatenated scheme as \( \eta = 2 \) and as presented in Section E.1, Remark 53. The proof follows by accordingly combining Corollary 27 and the rate constraints which arise from the decodings in Carleial’s variation of the Cover-Leung scheme. Again, a genie-aided argument is used in the analysis. The details are omitted. \( \square \)

Remark 30. In the case of perfect partial feedback, for all channel parameters \( P_1, P_2, N > 0 \), the achievable regions by Carleial [2] and Willems et al. [18] (Appendices A and B) correspond to the Cover-Leung region \( R_{\text{CL}}(P_1, P_2, N) \). Since the Cover-Leung region is also contained in the two achievable regions in Propositions 28 and 29, we conclude that Propositions 28 and 29 include also Carleial’s and Willems et al.’s regions for perfect partial feedback.

6.4 Achievable Regions for Noisy Feedback with Receiver Side-Information

Definition 31. Let \( P_1, P_2, N > 0 \) and \( K_{\text{W}1\text{W}2} \geq 0 \) be given. Let \( \eta \) be a positive integer, \( \mathbf{a}_1, \mathbf{a}_2 \) be \( \eta \)-dimensional vectors, \( \mathbf{B}_1, \mathbf{B}_2 \) be strictly lower-triangular \( \eta \times \eta \) matrices, and \( C_{\text{SI}} \) be a \( 2 \times \eta \) matrix. Depending on the matrix \( C_{\text{SI}} \) the rate region \( R_{\text{SI}}(\eta, a_1, a_2, B_1, B_2, C_{\text{SI}}) \) is defined as follows:

- If \( C_{\text{SI}}C_{\text{SI}}^T \) is nonsingular,\(^{10} \) then \( R_{\text{SI}}(N, \sigma_2^2; \eta, a_1, a_2, B_1, B_2, C_{\text{SI}}) \) is defined as the set of all

\(^{10}\)Whenever \( \eta \in \mathbb{N} \) is larger than 1, there is no loss in optimality in restricting attention to matrices \( C_{\text{SI}} \) so that \( C_{\text{SI}}C_{\text{SI}}^T \) is nonsingular.

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Theorem 34 (Feedback is Always Useful).

and Theorem 24 imply that one or two noisy feedback links always increase the capacity.

Our first result in this section regards the usefulness of imperfect feedback. Theorem 14 follows from Theorem 14 by setting \( N, \sigma^2_{2}, \eta, a_1, a_2, b_2, C_{SI} \) is defined as the set containing only the origin.

An alternative recursive formulation of the region \( \mathcal{R}_{SI} (N, K_{W1W2}; \eta, a_1, a_2, b_1, b_2, C_{SI}) \) is presented in Section C.3.)

**Definition 32.** Given \( P_1, P_2, N > 0 \) and \( K_{W1W2} \geq 0 \), define

\[
\mathcal{R}_{SI} (P_1, P_2, N, K_{W1W2}) \triangleq \text{cl} \left( \bigcup_{\eta, a_1, a_2, b_1, b_2, C_{SI}} \mathcal{R}_{SI} (N, K_{W1W2}; \eta, a_1, a_2, b_1, b_2, C_{SI}) \right),
\]

where the union is over all tuples \( (\eta, a_1, a_2, b_1, b_2, C_{SI}) \) satisfying (21) and (22).

**Theorem 33 (Noisy Feedback with Receiver Side-Information).** The capacity region \( C_{\text{NoisyFBSI}}(P_1, P_2, N, K_{W1W2}) \) of the two-user AWGN MAC with noisy feedback where the receiver is cognizant of the realization of the feedback-noise sequences includes the rate region \( \mathcal{R}_{SI} (P_1, P_2, N, K_{W1W2}) \) for positive integers \( \eta \), i.e.,

\[
C_{\text{NoisyFBSI}}(P_1, P_2, N, K_{W1W2}) \supseteq \mathcal{R}_{SI} (P_1, P_2, N, K_{W1W2}).
\]

**Proof.** Follows from Theorem 14 by setting \( \sigma^2_1 = \sigma^2_2 = 0 \) in the rate expressions in (54) (but not in the power constraints (21) and (22)). The reason why in (54) we may set \( \sigma^2_1 = \sigma^2_2 = 0 \) is because prior to the decoding the receiver subtracts off the influence of the feedback-noise sequences \( \{W_{1,t}\} \) and \( \{W_{2,t}\} \). The details of the proof are omitted.

### 6.5 Qualitative Properties of the Capacity Regions

Our first result in this section regards the usefulness of imperfect feedback. Theorem 14 and Theorem 24 imply that one or two noisy feedback links always increase the capacity region—no matter how noisy the feedback links are.

**Theorem 34 (Feedback is Always Useful).** Let \( N, P_1, P_2 > 0 \) be given.

1) For every feedback-noise covariance matrix \( K_{W1W2} \)

\[
C_{\text{MAC}}(P_1, P_2, N) \subset C_{\text{NoisyFB}}(P_1, P_2, N, K_{W1W2}).
\]
2) For every $\sigma^2_2 \geq 0$

$$C_{\text{MAC}}(P_1, P_2, N) \subset C_{\text{NoisyPartialFB}}(P_1, P_2, N, \sigma^2_2).$$  \hspace{1cm} (67)

The inclusions in both cases are strict.

Proof. For the symmetric noisy-feedback setting the theorem follows directly from Corollary 16, and for the equal-powers noisy partial-feedback setting from Corollary 25. For a general proof, see Section 8.2. \hfill \Box

We next consider the noisy-feedback setting in the asymptotic regime where the noise variances on both feedback links vanish. Proposition 35 already exhibits that in this asymptotic regime our achievable regions in Theorem 14 converge to the point of maximum sum-rate in $C_{\text{PerfectFB}}$. More precisely, Proposition 35 establishes that this convergence holds for every sequence of feedback-noise covariance matrices with feedback-noise variances tending to 0 irrespective of the feedback-noise correlations.

**Proposition 35 (Convergence to Maximum Sum-Rate of $C_{\text{PerfectFB}}$).** Let $P_1, P_2, N > 0$ be given. Then,

$$\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \succeq 0; \text{tr}(K) \leq \sigma^2} \mathcal{R}_1(P_1, P_2, N, K) \right) \supseteq \mathcal{R}^\rho_{\text{Oz}}(P_1, P_2, N).$$  \hspace{1cm} (68)

Thus by Remark 8, asymptotically our achievable regions in Theorem 14 approach the point of maximum sum-rate in the perfect-feedback capacity region.

Proof. Follows directly by Proposition 15, Part 3), and by Remark 17. \hfill \Box

**Remark 36.** The following stronger result than Proposition 35 holds. Inclusion (68) remains valid if the region $\mathcal{R}_1(P_1, P_2, N, K)$ is replaced by the union $\left( \bigcup_{\eta \in \mathbb{N}} \tilde{\mathcal{R}}_{\eta}(P_1, P_2, N, K) \right)$, where the regions $\tilde{\mathcal{R}}_{\eta}(P_1, P_2, N, K)$ are defined in Definition 54 in Section F, and represent the regions achieved by our concatenated scheme for the specific choice of parameters presented in Section F.1.

Based on the rate-splitting extension in Section 5.1, Proposition 35 can be generalized to all boundary points of $C_{\text{PerfectFB}}$.

**Proposition 37 (Convergence to Boundary of $C_{\text{PerfectFB}}$).** Let $P_1, P_2, N > 0$ be given. For every $\rho \in [0, \rho^*(P_1, P_2, N)]$ there exists a $P'_1(\rho) \in [0, P_1]$ so that

$$\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \succeq 0; \text{tr}(K) \leq \sigma^2} \mathcal{R}_{\text{RS},1}(P'_1(\rho), (P_1 - P'_1(\rho)), P_2, N, K) \right) \supseteq \mathcal{R}^\rho_{1, \text{Oz}}(P_1, P_2, N).$$  \hspace{1cm} (69)

Similarly, for every $\rho \in [0, \rho^*(P_1, P_2, N)]$ there exists a $P'_2(\rho) \in [0, P_2]$ so that

$$\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \succeq 0; \text{tr}(K) \leq \sigma^2} \mathcal{R}_{\text{RS},2}(P_1, P'_2(\rho), (P_2 - P'_2(\rho)), N, K) \right) \supseteq \mathcal{R}^\rho_{2, \text{Oz}}(P_1, P_2, N).$$  \hspace{1cm} (70)

Thus, by Remark 5 and Definition 6 our achievable regions in Proposition 20 asymptotically approach all boundary points of the perfect-feedback capacity region.
Proof. See Section 8.4.

Propositions 35 and 37 combined with Remark 7 yield the following continuity result.

**Theorem 38 (Continuity of Noisy-Feedback Capacity Region).** For all $P_1, P_2, N > 0$:

$$
\operatorname{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0, \text{tr}(K) \leq \sigma^2} C_{\text{NoisyFB}}(P_1, P_2, N, K) \right) = C_{\text{PerfectFB}}(P_1, P_2, N).
$$

Proof. The theorem follows from Propositions 35 and 37 combined with Remark 7. For a detailed proof see Section 8.5.

Our last result in this section regards the two-user AWGN MAC with perfect partial feedback. Van der Meulen in [13] asked the question whether the Cover-Leung region equals the capacity region of the MAC with perfect partial feedback. The following theorem answers the question in the negative.

**Theorem 39.** Consider a two-user AWGN MAC with perfect partial feedback. There exist powers $P_1$ and $P_2$ and noise variance $N$ such that the inclusion

$$
\mathcal{R}_{\text{CL}}(P_1, P_2, N) \subset C_{\text{PerfectPartialFB}}(P_1, P_2, N)
$$

is strict.

Proof. The inclusion is proved in Section 8.6 by showing that for powers $P_1 = 1, P_2 = 5$ and noise variance $N = 5$ the region in Corollary 27 includes rate points that lie strictly outside the Cover-Leung region.

7 Choice of Parameters for Concatenated Schemes

We present guidelines on how to choose the parameters of our concatenated schemes. For the purpose of describing our guidelines, throughout this section, we replace the symbols $\xi_1$ and $\xi_2$ fed to the inner encoders by the independent standard Gaussians $\Xi_1$ and $\Xi_2$, respectively.

7.1 Noisy Feedback

Let $P_1, P_2, N > 0$ and $K_{W,1W,2} \succeq 0$ be given. We discuss how to choose the parameters $\eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}$ of our concatenated scheme for noisy feedback. Of course, the choice of parameters should depend on the desired rate pair of transmission in $\mathcal{R}(P_1, P_2, N, K_{W,1W,2})$. Ideally, for every rate pair $(R_1, R_2)$ in the interior of $\mathcal{R}(P_1, P_2, N, K_{W,1W,2})$ we would like to determine parameters $\eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}$ so that $(R_1, R_2)$ lies in $\mathcal{R}(N, K_{W,1W,2}; \eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C})$. However, this seems analytically intractable, and therefore, we will only provide general guidelines on how to choose the parameters $\eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2, \mathbf{C}$.

To analyze the benefits of our guidelines, we introduce the following definition. A choice of parameters $\eta^*, \mathbf{a}_1^*, \mathbf{a}_2^*, \mathbf{B}_1^*, \mathbf{B}_2^*, \mathbf{C}^*$ is said to dominate another choice of parameters $\eta', \mathbf{a}_1', \mathbf{a}_2', \mathbf{B}_1', \mathbf{B}_2', \mathbf{C}'$, if both choices satisfy the power constraints (21) and (22) and if

$$
\mathcal{R}(N, K_{W,1W,2}; \eta', \mathbf{a}_1', \mathbf{a}_2', \mathbf{B}_1', \mathbf{B}_2', \mathbf{C}') \subseteq \mathcal{R}(N, K_{W,1W,2}; \eta^*, \mathbf{a}_1^*, \mathbf{a}_2^*, \mathbf{B}_1^*, \mathbf{B}_2^*, \mathbf{C}^*). \quad (71)
$$
If the inclusion in (71) is strict, we say that the choice of parameters \( \eta^*, a_1^*, a_2^*, B_1^*, B_2^*, C^* \) strictly dominates the choice \( \eta', a_1', a_2', B_1', B_2', C' \). We further say that a condition on the parameters \( \eta, a_1, a_2, B_1, B_2, C \) is an optimality condition, if every choice of parameters violating the condition is dominated by a choice of parameters which satisfies the condition. Thus, there is no loss in optimality, i.e., in achievable rate region, if one restricts attention to parameters \( \eta, a_1, a_2, B_1, B_2, C \) satisfying the optimality conditions. Similarly, we say that a condition is a strict optimality condition, if every choice of parameters violating the condition is strictly dominated by a choice of parameters satisfying the condition.

Our first guideline provides the following first optimality condition. Given \( \eta, a_1, a_2, B_1, \) and \( B_2 \), the parameter \( C \) should be chosen as

\[
C = A_b^T (A_b A_b^T + N I_\eta + B_b (K_{W_1 W_2} \otimes I_\eta) B_b^*)^{-1}.
\]

By (23), (28), and (29), choosing \( C \) as in (72) implies that

\[
\begin{pmatrix}
\Xi_1 \\
\Xi_2
\end{pmatrix} = E \begin{pmatrix}
\Xi_1 \\
\Xi_2
\end{pmatrix} \bigg| Y_1, \ldots, Y_\eta
\]

and therefore, we shall call the choice of \( C \) in (72) the LMMSE-estimation matrix. The proof of optimality of choosing \( C \) as in (72) is based on the fact that when (73) holds, then even additionally revealing \( Y \) (or any linear combinations thereof) to the outer decoder does not increase the set of achievable rates. The details of the proof are omitted.

Our second guideline provides a second optimality condition, which we will see is a strict optimality condition. The parameters \( \eta, a_1, a_2, B_1, B_2 \) should be chosen so that they satisfy both power constraints (21) and (22) with equality. In particular, every choice of parameters that satisfies either (21) or (22) with strictly inequality is strictly dominated by some choice of parameters satisfying both (21) and (22) with equality. This readily follows from the recursive formulation of \( R (N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \) in Section C.1, because the right-hand sides of (182)–(184) (which determine \( R (N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \)) can always be increased by changing the last entry of \( a_1 \), i.e., \( a_1, \eta \), or the last entry of \( a_2 \), i.e., \( a_2, \eta \).

Our third guideline describes a strict optimality condition for the special case of perfect feedback. This special case is in view of Ozarow’s capacity result [9] only of limited interest, but it provides insight on how to choose the parameters for other settings, e.g., the perfect partial-feedback setting and the noisy feedback-setting with receiver side-information. For perfect feedback the parameters \( a_1, a_2, B_1, B_2 \) should be chosen such that Inner Encoder \( \nu \), for \( \nu \in \{1, 2\} \), produces as its \( \ell \)-th channel input a scaled version of the LMMSE-estimation error of \( \Xi_\nu \) when observing \( Y_1, \ldots, Y_{\ell-1} \), i.e.,

\[
X_{1, \ell} = \pi_{1, \ell} (\Xi_1 - E[\Xi_1 | Y_1, \ldots, Y_{\ell-1}]), \quad \ell \in \{1, \ldots, \eta\},
\]

and

\[
X_{2, \ell} = \pi_{2, \ell} (\Xi_2 - E[\Xi_2 | Y_1, \ldots, Y_{\ell-1}]), \quad \ell \in \{1, \ldots, \eta\},
\]

for some real numbers \( \pi_{1,1}, \ldots, \pi_{1,\eta} \) and \( \pi_{2,1}, \ldots, \pi_{2,\eta} \). In particular, every choice of parameters \( \eta, a_1, a_2, B_1, B_2, C \) not of the form (74) and (75) is dominated by some choice of parameters \( \eta, a_1', a_2', B_1', B_2', C' \) of the form (74) and (75). For an optimality proof of the presented strict optimality condition for perfect feedback, see Section D.
This third optimality condition does not generalize to noisy feedback, i.e., choosing $\mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2$ as in (74) and (75) when the channel outputs $Y_1, \ldots, Y_{\ell-1}$ are replaced by the feedback outputs $V_{1,1}, \ldots, V_{1,\ell-1}$ and $V_{2,1}, \ldots, V_{2,\ell-1}$, respectively, is not an optimality condition for noisy feedback. Intuitively, the lack of optimality of such parameters is due to the fact that for these parameters the inner encoders introduce too much feedback noise into the forward communication.

Our fourth guideline considers again general noisy feedback. Numerical results indicate that the larger the feedback-noise variances are, the smaller the parameter $\eta$ should be chosen if the goal is to maximize the sum-rate of our concatenated scheme. Indeed, it is easily proved that in the extreme case of no feedback the sum-rate is maximized by choosing $\eta = 1$. In contrast, in the extreme case of perfect feedback we prove in Section 8.3 that with the choice of parameters suggested in Section F.1 the maximum sum-rate of our concatenated scheme converges to the perfect-feedback sum-rate capacity as the parameter $\eta$ tends to infinity. To maximize the single-rates, the choice $\eta = 1$ is optimal, i.e., no feedback should be used.

In the remaining, we discuss two specific choices of the parameters $\mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2, C$ given $\eta \in \mathbb{N}$. These choices allow to derive most of the results in Section 6. The choices are presented in Sections E.1 and F.1. For both choices, the parameter $C$ is the LMMSE-estimation matrix and the parameters $\mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2$ are so that when specialized to perfect feedback they satisfy (74) and (75). In the first choice, i.e., the choice in Section E.1, each inner encoder produces all its channel inputs of the same expected power. The achievable region corresponding to this choice is presented in Corollary 52 in Section E. It includes as special cases the results in Corollaries 16, 25, and 27, which are used to prove Theorem 34. In the second choice, i.e., the choice in Section F.1, each inner encoder produces its $\eta$ channel inputs of different expected powers according to the power-allocation strategy suggested in [8]. We again present the corresponding achievable region, see Corollary 55 in Section F. This region includes as special case the achievable region for perfect feedback in Remark 56, which is used in the proof of Propositions 35 and 37 and Theorem 38.

### 7.2 Noisy and Perfect Partial Feedback

Let $P_1, P_2, N > 0$, $\sigma_2^2 \geq 0$ be given. We present guidelines on how to choose the parameters $\eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_2, C_P$. Let the notions of (strict) dominating parameters and (strict) optimality conditions be defined in analogy to the previous subsection 7.1.

Our first guideline describes an optimality condition analogous to the first optimality condition discussed in the previous subsection 7.1. Given $\eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_2$ the parameter $C_P$ should be chosen as

$$C_P = A_1^\dagger (A_1 A_1^\dagger + N I_\eta + \sigma_2^2 B_2 B_2^\dagger)^{-1}, \quad (76)$$

since every choice of parameters not satisfying (76) is dominated by a choice of parameters satisfying (76). We call the matrix $C_P$ in (76) the LMMSE-estimation matrix, since by (30), (34), and (35), the choice in (76) implies that

$$\begin{bmatrix} \hat{\Xi}_1 \\ \hat{\Xi}_2 \end{bmatrix} = \mathbb{E} \left[ \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix} \bigg| Y_1, \ldots, Y_\eta \right].$$

Our second guideline describes a strict optimality condition similar to the second optimality condition in the previous subsection 7.1. The parameters $\eta, \mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_2, C_P$ should be
chosen so that both power constraints (36) and (37) are satisfied with equality, since every choice of parameters not satisfying (36) and (37) with equality is strictly dominated by a choice satisfying them with equality.

Our third guideline describes a strict optimality condition for the special case of perfect partial feedback and is similar to the third optimality condition in the previous subsection 7.1. For perfect partial feedback the parameters $a_1, a_2, B_2$ should be chosen so that (75) holds for some real numbers $\pi_{2,1}, \ldots, \pi_{2,\eta}$, since every choice not satisfying (75) is strictly dominated by a choice which satisfies (75).

The optimality proofs of the three presented optimality conditions parallel the proofs in the previous subsection 7.1, see also Section C.2 and Remark 49 in Section D. The details are omitted.

In Section E.1, Remark 53, we present a specific (suboptimal) choice of the parameters $a_1, a_2, B_2, C_P$ for given $\eta \in \mathbb{N}$. For this specific choice, the parameter $C_P$ is the LMMSE-estimation matrix, $\eta, a_1, a_2, B_2$ satisfy the power constraints (36) and (37) with equality, and when specialized to perfect partial feedback $a_1, a_2, B_2$ satisfy (75). We present the corresponding achievable region for $\eta = 2$ and equal powers, i.e., $P_1 = P_2 = P$, in Corollary 25 and for $\eta = 2$ and perfect partial feedback in Corollary 27.

7.3 Noisy Feedback with Receiver Side-Information

Let $P_1, P_2, N > 0, K_{W_1W_2} \succeq 0$ be given. We present guidelines on how to choose the parameters $\eta, a_1, a_2, B_1, B_2, C_{SI}$. Let the notions of (strict) dominating parameters and (strict) optimality conditions be defined in analogy to Subsection 7.1.

Our first guideline describes an optimality condition similar to the first optimality condition discussed in Subsection 7.1. Given $\eta, a_1, a_2, B_1, B_2$ the parameter $C_{SI}$ should be chosen as

$$C_{SI} = A_b^T (A_b A_b^T + N I_\eta)^{-1},$$

since every choice of parameters not satisfying (77) is dominated by a choice of parameters satisfying (77). We call the matrix $C_{SI}$ in (77) the LMMSE-estimation matrix with side-information, since by (41), (45), and (46) the choice in (77)—combined with the optimal choices of $D_1$ and $D_2$ defined by (39), (40), and (44)—implies that

$$\left(\bar{z}_1, \bar{z}_2\right) = E\left[\left(\Xi_1, \ldots, \Xi_\eta, W_{1,1}, \ldots, W_{1,\eta}, W_{2,1}, \ldots, W_{2,\eta}\right) \bigg| \right].$$

Our second guideline describes a strict optimality condition similar to the second optimality condition in Subsection 7.1. The parameters $\eta, a_1, a_2, B_1, B_2, C_{SI}$ should be chosen so that Constraints (21) and (22) are satisfied with equality. In particular, every choice of parameters not satisfying (21) and (22) with equality is strictly dominated by a choice which satisfies them with equality.

Our third guideline describes a strict optimality condition for the following two special cases:

a) $\eta \in \mathbb{N}$ is arbitrary and $\varrho = 1$, i.e., the feedback noises are perfectly correlated,

b) $\eta = 2$ and $\varrho \in [-1, 1)$ arbitrary.
This third optimality condition is similar to the third optimality condition in Subsection 7.1: For special cases a) and b) the parameters \(a_1, a_2, B_1, B_2\) should be chosen such that the inner encoders produce

\[
X_{1,\ell} = \pi_{1,\ell}(\Xi_1 - \mathbb{E}[\Xi_1|V_{1,1}, \ldots, V_{1,\ell-1}]), \quad \ell \in \{1, \ldots, \eta\},
\]

and

\[
X_{2,\ell} = \pi_{2,\ell}(\Xi_2 - \mathbb{E}[\Xi_2|V_{2,1}, \ldots, V_{2,\ell-1}]), \quad \ell \in \{1, \ldots, \eta\},
\]

for some real numbers \(\pi_{1,1}, \ldots, \pi_{1,\eta}\) and \(\pi_{2,1}, \ldots, \pi_{2,\eta}\). In particular, every choice of parameters \(\eta, a_1, a_2, B_1, B_2, C_{\text{SI}}\) not of the form (78) and (79) is strictly dominated by some choice of parameters of the form (78) and (79).

The optimality proofs of the three presented optimality conditions parallel the proofs in Subsection 7.1, see also Section C.3 and Remark 50 in Section D.

In Section G, we present a specific choice of the parameters \(a_1, a_2, B_1, B_2, C_{\text{SI}}\) so that \(C_{\text{SI}}\) is the LMMSE-estimation matrix with side-information, the power constraints (21) and (22) are satisfied with equality, and \(\eta, a_1, a_2, B_1, B_2\) satisfy (78) and (79) for all \(\eta \in \mathbb{N}\) and \(\varrho \in [-1, 1]\). We present the corresponding achievable region in Corollary 58.

8 Proofs

8.1 Proof of Proposition 15

Fix \(P_1, P_2, N > 0\). We first prove Part 1). To this end, we show that for every fixed \(\eta \in \mathbb{N}\) and fixed \(\eta\)-dimensional vectors \(a_1, a_2, \eta \times \eta\)-dimensional matrices \(B_1, B_2\), and \(2 \times \eta\)-dimensional matrix \(C\), the following two statements hold:

i) For all positive semidefinite matrices \(K_{W_1W_2}\) and \(K'_{W_1W_2}\):

\[
\left( K_{W_1W_2} \succeq K'_{W_1W_2} \right) \implies \left( \mathcal{R}(N, K_{W_1W_2}; \eta, a_1, a_2, B_1, B_2, C) \subseteq \mathcal{R}(N, K'_{W_1W_2}; \eta, a_1, a_2, B_1, B_2, C) \right).
\]

ii) If the choice of parameters \(\eta, a_1, a_2, B_1, B_2, C\) satisfies the power constraints (21) and (22) for a covariance matrix \(K_{W_1W_2}\), then it also satisfies these power constraints for all covariance matrices \(K'_{W_1W_2}\) for which \(K_{W_1W_2} \succeq K'_{W_1W_2}\).

By Definition 13, Statements i) and ii) imply that

\[
\left( K_{W_1W_2} \succeq K'_{W_1W_2} \right) \implies \left( \mathcal{R}(P_1, P_2, N, K_{W_1W_2}) \subseteq \mathcal{R}(P_1, P_2, N, K'_{W_1W_2}) \right),
\]

and thus conclude the proof of Part 1).

We start by proving Statement i). Fix a tuple \((\eta, a_1, a_2, B_1, B_2, C)\). We only prove Statement i) for the case where \(CC^\top\) is nonsingular. For the case where \(CC^\top\) is singular the proof is analogous and therefore omitted. To establish Statement i) when \(CC^\top\) is nonsingular, it suffices to show that all three right-hand sides of (52)–(54) are monotonically decreasing in \(K_{W_1W_2}\) with respect to the Loewner order. We only prove the monotonicity of the right-hand
side of (52); the monotonicities of the right-hand sides of (53) and (54) can be shown analogously. Thus, in the following we fix two positive semidefinite $2 \times 2$ matrices $K_{W_1W_2}$ and $K'_{W_1W_2}$ satisfying $K_{W_1W_2} \succeq K'_{W_1W_2}$ and we show that:

$$
1 \over 2\eta \log \left( \frac{|C(a_ia_i^T + N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T)|}{|C(N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T)|} \right) \geq \frac{1}{2\eta} \log \left( \frac{|C(a_ia_i^T + N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)|}{|C(N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)|} \right).
$$

(80)

Before proving (80) we recall the following well-known properties of positive semidefinite matrices. For all positive semidefinite $n \times n$ matrices $K, K_1, K_2$ satisfying $K_1 \succeq K_2$ and for all $m \times n$ matrices $M$ the following properties hold:

$$MK_1M^T \succeq MK_2M^T,$$

(81)

$$K + K_1 \succeq K + K_2,$$

(82)

$$KK_1 \succeq KK_2,$$

(83)

$$K_1^{-1} \succeq K_2^{-1},$$

(84)

and

$$|K_1| \geq |K_2|,$$

(85)

$$\text{tr}(K_1) \geq \text{tr}(K_2).$$

(86)

Based on these properties and the definition $A_1 \triangleq a_ia_i^T$ the following sequence of implications can be proved:

$$K_{W_1W_2} \succeq K'_{W_1W_2} \implies (K_{W_1W_2} \otimes I_{\eta}) \succeq (K'_{W_1W_2} \otimes I_{\eta})$$

(87)

$$\implies (B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T) \succeq (B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)$$

(88)

$$\implies (N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T) \succeq (N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)$$

(89)

$$\implies (C(N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T)^T) \succeq (C(N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)^T)$$

(90)

$$\implies (C(N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T)^{-1}) \succeq (C(N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)^{-1})$$

(91)

$$\implies \left( I_2 + CA_1C^T(C(N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T)^{-1}) \right) \leq \left( I_2 + CA_1C^T(C(N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)^{-1}) \right)$$

(92)

$$\implies 1 \over 2\eta \log \left| I_2 + CA_1C^T(C(N I_{\eta} + B_b(K_{W_1W_2} \otimes I_{\eta})B_{b}^T)^{-1}) \right| \leq 1 \over 2\eta \log \left| I_2 + CA_1C^T(C(N I_{\eta} + B_b(K'_{W_1W_2} \otimes I_{\eta})B_{b}^T)^{-1}) \right|,$$

(93)

where (87) follows by the linearity of the Kronecker product $\otimes$ and because for every positive semidefinite matrix $K$ also the Kronecker product $K \otimes I_{\eta}$ is positive semidefinite\footnote{That $K \succeq 0$ implies $(K \otimes I) \succeq 0$ can be seen as follows. For every $2\eta$-dimensional vector $x \triangleq (x_1, \ldots, x_{2\eta})^T$, where we define $x_i \triangleq (x_{2i-1}, x_{2i})^T$ for $i \in [1, \eta]$, and every $2 \times 2$ positive semidefinite matrix $K$ the term $x^T(K \otimes I_{\eta})x$ can be written as $\sum_{i=1}^{\eta} x_i^T K x_i$, which is nonnegative since $K$ is positive semidefinite.}; where (87)
follows by (81); where (89) follows by (82) and because \( N I_\eta \succeq 0 \); where (90) follows by (81); where (91) follows by (84); where (92) follows by (83) and (84) and because \( A_1 \succeq 0 \), and thus, by (81), also \( CA_1 C^T \succeq 0 \); where (93) follows by (85) and by the monotonicity of the log-function.

The matrix inequality (80) follows then from (93), since for every \( 2 \times 2 \) positive semidefinite matrix \( K \) and for \( A_1 \) as defined above, when \( C C^T \) is nonsingular:

\[
\frac{1}{2\eta} \log \left( \frac{\left| C(a_1 a_1^T + N I_\eta + B_0 (K \otimes I_\eta) B_0^T) C^T \right|}{\left| C(N I_\eta + B_0 (K \otimes I_\eta) B_0^T) C^T \right|} \right) = \frac{1}{2\eta} \log \left( \left| l_2 + C A_1 C^T (C(N I_\eta + B_0 (K_{W_1 W_2} \otimes I_\eta) B_0^T) C^T)^{-1} \right| \right),
\]

because for all nonsingular square matrices \( M_1 \) and \( M_2 \) of the same dimension \( |M_1| = |M_1 M_2^{-1}| \). This concludes the proof of Statement i).

We next prove Statement ii). It suffices to show that for fixed parameters \( \eta, a_1, a_2, B_1, B_2, C \), the left-hand side of the power constraints (21) and (22) are monotonically increasing in \( K_{W_1 W_2} \) with respect to the Loewner order. Similarly to the proof of Statement i), this can be shown by a sequence of implications based on (82), on (83), on (86), on the fact that \( K_{W_1 W_2} \succeq K'_{W_1 W_2} \) implies \( (K_{W_1 W_2} \otimes I_\eta) \succeq (K'_{W_1 W_2} \otimes I_\eta) \), and on the fact that the trace of a sum equals the sum of the traces. The details are omitted.

We prove Part 2). The inclusion of the left-hand side in the right-hand side is trivial, because for every positive \( \delta \) all choices of parameters \( \eta, a_1, a_2, B_1, B_2, C \) satisfying the power constraints (21) and (22) for powers \( (P_1 - \delta) \) and \( (P_2 - \delta) \) satisfy the power constraints also for powers \( P_1 \) and \( P_2 \).

In the following we prove the inclusion of the right-hand side in the left-hand side. Choose a rate pair \( (R_1^\circ, R_2^\circ) \) in the interior of \( \mathcal{R}(P_1, P_2, N, K_{W_1 W_2}) \), i.e.

\[
(R_1^\circ, R_2^\circ) \in \mathcal{R}(P_1, P_2, N, K_{W_1 W_2}). \tag{94}
\]

We show that for all sufficiently small \( \delta > 0 \)

\[
(R_1^\circ, R_2^\circ) \in \mathcal{R}(P_1 - \delta, P_2 - \delta, N, K_{W_1 W_2}). \tag{95}
\]

By (94), there exists a choice of parameters \( \eta, a_1, a_2, B_1, B_2, C \) that satisfies (21) and (22) for powers \( P_1 \) and \( P_2 \) and so that

\[
(R_1^\circ, R_2^\circ) \in \mathcal{R}(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C). \tag{96}
\]

We can assume that \( a_1 \) and \( a_2 \) are both not equal to the zero-vector, because when either \( a_1 \) or \( a_2 \) equals the zero-vector the region \( \mathcal{R}(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \) is strictly included in the region \( \mathcal{R}(N, K_{W_1 W_2}; \eta', a_1', a_2', B_1', B_2', C') \) for some choice of parameters \( \eta', a_1', a_2', B_1', B_2', C' \) where \( a_1' \) and \( a_2' \) are both not equal to the zero-vector.

For every \( \delta > 0 \) for which the left-hand sides of (97) and (98) are both positive, choose two numbers \( \kappa_1(\delta), \kappa_2(\delta) \in (0, 1) \) so that

\[
\text{tr} \left( A_1^T (l_2 \eta - B_B)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (l_2 \eta - B_B)^{-1} A_1 \right) - \delta \eta 
\geq \text{tr} \left( \begin{pmatrix} \kappa_1(\delta) a_1^T & 0 \\ 0 & \kappa_2(\delta) a_2^T \end{pmatrix} (l_2 \eta - B_B)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (l_2 \eta - B_B)^{-1} \begin{pmatrix} \kappa_1(\delta) a_1^T & 0 \\ 0 & \kappa_2(\delta) a_2^T \end{pmatrix} \right). \tag{97}
\]
and
\[ \text{tr} \left( A_d^T (I_2 - B_B)^{-T} \begin{pmatrix} 0 & 0 \\ 0 & I_\eta \end{pmatrix} (I_2 - B_B)^{-1} A_d \right) - \delta \eta \geq \text{tr} \left( \begin{pmatrix} \kappa_1(a_1^T) & 0 \\ 0 & \kappa_2(a_2^T) \end{pmatrix} (I_2 - B_B)^{-T} \begin{pmatrix} 0 & 0 \\ 0 & I_\eta \end{pmatrix} (I_2 - B_B)^{-1} \begin{pmatrix} \kappa_1(a_1^T) & 0 \\ 0 & \kappa_2(a_2^T) \end{pmatrix} \right), \] (98)
and so that both sequences \( \kappa_1(\delta) \) and \( \kappa_2(\delta) \) tend to 1 as \( \delta \downarrow 0 \). Such sequences \( \kappa_1(\delta) \) and \( \kappa_2(\delta) \) always exist, because we assumed that \( a_1 \) and \( a_2 \) are non-zero and because, when defining the \( \eta \times \eta \) matrices \( Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4} \) as
\[ (I_2 - B_B)^T \begin{pmatrix} I_\eta & 0 \\ 0 & 0 \end{pmatrix} (I_2 - B_B)^{-1} = \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{1,3} & Q_{1,4} \end{pmatrix} \]
and the \( \eta \times \eta \) matrices \( Q_{2,1}, Q_{2,2}, Q_{2,3}, Q_{2,4} \) as
\[ (I_2 - B_B)^T \begin{pmatrix} 0 & 0 \\ 0 & I_\eta \end{pmatrix} (I_2 - B_B)^{-1} = \begin{pmatrix} Q_{2,1} & Q_{2,2} \\ Q_{2,3} & Q_{2,4} \end{pmatrix}, \]
then the matrices \( Q_{1,1}, Q_{1,4}, Q_{2,1}, Q_{2,4} \) are positive definite and for all real numbers \( \kappa_1 \) and \( \kappa_2 \)
\[ \text{tr} \left( \begin{pmatrix} \kappa_1(a_1^T) & 0 \\ 0 & \kappa_2(a_2^T) \end{pmatrix} (I_2 - B_B)^T \begin{pmatrix} 0 & 0 \\ 0 & I_\eta \end{pmatrix} (I_2 - B_B)^{-1} \begin{pmatrix} \kappa_1(a_1^T) & 0 \\ 0 & \kappa_2(a_2^T) \end{pmatrix} \right) = \kappa_1^2 a_1^T Q_{1,1} a_1 + \kappa_2^2 a_2^T Q_{1,4} a_2, \]
and
\[ \text{tr} \left( \begin{pmatrix} \kappa_1(a_1^T) & 0 \\ 0 & \kappa_2(a_2^T) \end{pmatrix} (I_2 - B_B)^T \begin{pmatrix} 0 & 0 \\ 0 & I_\eta \end{pmatrix} (I_2 - B_B)^{-1} \begin{pmatrix} \kappa_1(a_1^T) & 0 \\ 0 & \kappa_2(a_2^T) \end{pmatrix} \right) = \kappa_1^2 a_1^T Q_{2,1} a_1 + \kappa_2^2 a_2^T Q_{2,4} a_2. \]

By (97) and (98) and since the parameters \( \eta, a_1, a_2, B_1, B_2, C \) satisfy Constraints (21) and (22) for powers \( P_1 \) and \( P_2 \), for all sufficiently small \( \delta > 0 \) the parameters \( \eta, \kappa_1(\delta) a_1, \kappa_2(\delta) a_2, B_1, B_2, C \) satisfy (21) and (22) for powers \( (P_1 - \delta) \) and \( (P_2 - \delta) \). Moreover, since the right-hand sides of Constraints (52)–(54)—which determine the achievable region \( \mathcal{R}(N, K_{W_1 W_2}; \eta, a_1, a_2, B_1, B_2, C) \)—are continuous in the entries of \( a_1 \) and \( a_2 \), and since \( \kappa_1(\delta) \) and \( \kappa_2(\delta) \) tend to 1 as \( \delta \downarrow 0 \), by (96) for all sufficiently small \( \delta > 0 \)
\[ (P_1', P_2') \in \mathcal{R}(N, K_{W_1 W_2}; \eta, \kappa_1(\delta) a_1, \kappa_2(\delta) a_2, B_1, B_2, C). \]
This proves (95) and thus concludes the proof of Part 2).

We finally prove Part 3), i.e., Equality (55). The inclusion of the left-hand side in the right-hand side is trivial, because replacing the intersection on the left-hand side by the specific choice \( K = 0 \) can only increase the region, and because the region \( \mathcal{R}(P_1, P_2, N, 0) \) is closed. The interesting inclusion is that the left-hand side contains the right-hand side. To prove this inclusion, we first notice that
\[
\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0; \text{tr}(K) \leq \sigma^2} \mathcal{R}(P_1, P_2, N, K) \right) \supseteq \text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0; \text{tr}(K) \leq \sigma^2} \mathcal{R}(P_1, P_2, N, \text{tr}(K) I_2) \right) = \text{cl} \left( \bigcup_{\sigma^2 > 0} \mathcal{R}(P_1, P_2, N, \sigma^2 I_2) \right),
\]
38
where the inclusion and the equality both follow by the monotonicity proved in Part 1). Thus, it remains to show that
\[
\text{cl} \left( \bigcup_{\sigma^2 > 0} \mathcal{R} \left( P_1, P_2, N, \sigma^2 I_2 \right) \right) \supseteq \mathcal{R} \left( P_1, P_2, N, 0 \right). \tag{99}
\]

To prove (99), choose a rate pair \((R_1^0, R_2^0)\) in the interior of \(\mathcal{R} \left( P_1, P_2, N, 0 \right)\). By Part 2), for all sufficiently small \(\delta > 0\) the pair \((R_1^0, R_2^0)\) lies also in the interior of \(\mathcal{R} \left( P_1 - \delta, P_2 - \delta, N, 0 \right)\), i.e.,
\[
(R_1^0, R_2^0) \in \mathcal{R} \left( P_1 - \delta, P_2 - \delta, N, 0 \right). \tag{100}
\]
Fix a \(\delta > 0\) so that (100) holds. Further, fix parameters \(\eta, a_1, a_2, B_1, B_2, C\) satisfying the power constraints (21) and (22) for \(K_{W_1W_2} = 0\) and for powers \((P_1 - \delta)\) and \((P_2 - \delta)\), and so that the pair \((R_1^0, R_2^0)\) also lies in the interior of \(\mathcal{R} \left( N, 0; \eta, a_1, a_2, B_1, B_2, C \right)\), i.e.,
\[
(R_1^0, R_2^0) \in \mathcal{R} \left( N, 0; \eta, a_1, a_2, B_1, B_2, C \right).
\]

We assume in the following that \(CC^T\) is nonsingular, and thus the region \(\mathcal{R} \left( N, 0; \eta, a_1, a_2, B_1, B_2, C \right)\) is defined by Constraints (52)–(54). The proof for the case where \(CC^T\) is singular is analogous and therefore omitted. Since for \(K_{W_1W_2} = \sigma^2 I_2\) the right-hand sides of (52)–(54) are continuous in \(\sigma^2\), for all sufficiently small \(\sigma^2 > 0\)
\[
(R_1^0, R_2^0) \in \mathcal{R} \left( N, \sigma^2 I_2; \eta, a_1, a_2, B_1, B_2, C \right).
\]
Moreover, since for \(K_{W_1W_2} = \sigma^2 I_2\) the left-hand sides of (21) and (22) are continuous in \(\sigma^2\), for all sufficiently small \(\sigma^2 > 0\) the parameters \(\eta, a_1, a_2, B_1, B_2, C\) satisfy the power constraints (21) and (22) for \(K_{W_1W_2} = \sigma^2 I_2\) and for powers \(P_1\) and \(P_2\). Therefore, for all sufficiently small \(\sigma^2 > 0\)
\[
(R_1^0, R_2^0) \in \mathcal{R} \left( P_1, P_2, N, \sigma^2 I_2 \right),
\]
which concludes the proof of (99).

### 8.2 Proof of Theorem 34
If \(K_{W_1W_2} = \begin{pmatrix} \sigma_1^2 & \varrho \sigma_1 \sigma_2 \\ \varrho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}\) then
\[
C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq C_{\text{NoisyPartialFB}}(P_1, P_2, N, \sigma_2^2), \tag{101}
\]
because in the noisy feedback setting Transmitter 1 can ignore its feedback and thus reduce the setting to a noise partial feedback setting with feedback-noise variance \(\sigma_2^2\). Since Part 1) follows from Part 2) and (101), it suffices to prove Part 2).

To prove Part 2) we distinguish between the case of equal powers and of unequal powers. In the case of equal powers, \(P_1 = P_2 = P\) we first notice that in the achievable region in Corollary 25 the right-hand side of (63) is smaller than the sum of the right-hand side of (61) and of (62), for all \(P, N > 0\) and \(\sigma_2^2 \geq 0\). Thus, the region is a pentagon and not a rectangle and there exist achievable pairs \((R_1, R_2)\) whose sum is equal to the right-hand side of (63). The sum-rate constraint (63) exhibits that the achievable region in Corollary 25 includes rate pairs which are of sum-rate strictly larger than \(\frac{1}{2} \log \left( 1 + \frac{2P}{N} \right)\), and which thus
lie strictly outside the no-feedback capacity region. This concludes the proof in the case of equal powers.

To prove Part 2) in the case of unequal powers, $P_1 \neq P_2$, we use the following rate-splitting/time-sharing strategy. We assume $P_1 > P_2$; the case $P_1 < P_2$ can analogously be treated. Transmitter 1 splits its message $M_1$ into two independent submessages: submessage $M_{1,1}$ of rate $R_{1,1}$ and submessage $M_{1,2}$ of rate $R_{1,2}$. During a fraction of time $\frac{P_1 - P_2}{P_1 + P_2}$, Transmitter 1 sends Message $M_{1,2}$ using an optimal no-feedback scheme of power $(P_1 + P_2)$ while Transmitter 2 is quiet. During the remaining fraction of time $\frac{2P_2}{P_1 + P_2}$ Transmitters 1 and 2 use equal powers $\frac{P_1 + P_2}{2}$ to send messages $M_{1,1}$ and $M_2$ with the concatenated scheme in Section 4. Choosing the parameters of the concatenated scheme as proposed in Remark 53 in Section E.1, by Corollary 25 (where we replace $P$ by $\frac{P_1 + P_2}{2}$) and by the capacity of a Gaussian single-user channel, with the described rate-splitting/time-sharing scheme the rate pair $(R_1 = R_{1,1} + R_{1,2}, R_2)$ is achievable where

$$R_{1,1} = \frac{P_2}{2(P_1 + P_2)} \log \left( 1 + \frac{P_1 + P_2}{N} \right),$$

$$R_{1,2} = \frac{P_1}{2(P_1 + P_2)} \log \left( 1 + \frac{P_1 + P_2}{N} \right) + \frac{P_2}{2(P_1 + P_2)} \log \left( 1 + \frac{P_1 + P_2}{N} \left( 1 + \frac{\frac{P_1 + P_2}{2} + N + \sigma_2^2}{\left( \frac{P_1 + P_2}{2} + N + \sigma_2^2 + \frac{P_1 + P_2}{2N} \sigma_2^2 \right)^2} - 1 \right) \frac{\frac{P_1 + P_2}{2}}{(P_1 + P_2 + N)} \frac{(P_1 + P_2 + N + \sigma_2^2)}{(P_1 + P_2 + N + \sigma_2^2 + \frac{P_1 + P_2}{2N} \sigma_2^2)},$$

$$R_2 = \frac{P_1}{2(P_1 + P_2)} \log \left( 1 + \frac{P_1 + P_2}{N} \right) + \frac{P_2}{2(P_1 + P_2)} \log \left( 1 + \frac{\frac{P_1 + P_2}{2}}{(P_1 + P_2 + N)} \left( 1 + \frac{\frac{P_1 + P_2}{2} + N + \sigma_2^2}{\left( \frac{P_1 + P_2}{2} + N + \sigma_2^2 + \frac{P_1 + P_2}{2N} \sigma_2^2 \right)^2} - 1 \right) \frac{\frac{P_1 + P_2}{2}}{(P_1 + P_2 + N)^2} \frac{(P_1 + P_2 + N + \sigma_2^2)}{(P_1 + P_2 + N + \sigma_2^2 + \frac{P_1 + P_2}{2N} \sigma_2^2)},$$

The proof of (67) follows then by noting that for every $P_1, P_2, N > 0$ and every $\sigma_2^2 \geq 0$ the rate pair $(R_1, R_2)$ has a sum-rate which is strictly larger than $\frac{1}{2} \log \left( 1 + \frac{\frac{P_1 + P_2}{N}}{N} \right)$, and therefore lies strictly outside the no-feedback capacity region $C_{\text{MAC}}(P_1, P_2, N)$.

### 8.3 Proof of Remark 17

Fix $P_1, P_2, N > 0$. Specializing our concatenated scheme to the specific choice of parameters in Remark 56, Section F, obviously cannot outperform our concatenated scheme for general
parameters. Thus,

$$\text{cl} \left( \bigcup_{\eta \in \mathbb{N}} \tilde{R}_\eta(P_1, P_2, N, 0) \right) \subseteq R(P_1, P_2, N, 0).$$

(102)

We shall show in the following that

$$\mathcal{R}^{\sigma'}_{\text{Cl}}(P_1, P_2, N) \subseteq \text{cl} \left( \bigcup_{\eta \in \mathbb{N}} \tilde{R}_\eta(P_1, P_2, N, 0) \right),$$

(103)

which combined with (102) establishes the remark.

Recall that for fixed $\eta \in \mathbb{N}$ the region $\tilde{R}_\eta(P_1, P_2, N, 0)$ is defined as the set of all rate pairs $(R_1, R_2)$ satisfying

$$R_1 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_1}{N} \right) + \sum_{\ell=2}^{\eta} \frac{1}{2\eta} \log \left( 1 + \frac{P_1(1 - \rho_{\ell-1}^2)}{N} \right),$$

$$R_2 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_2}{N} \right) + \sum_{\ell=2}^{\eta} \frac{1}{2\eta} \log \left( 1 + \frac{P_2(1 - \rho_{\ell-1}^2)}{N} \right),$$

$$R_1 + R_2 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_1 + rP_2}{N} \right) + \sum_{\ell=2}^{\eta} \frac{1}{2\eta} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1P_2(1 - \rho_{\ell-1}^2)}}{N} \right),$$

where

$$\rho_1 = -\rho^*(P_1, P_2, N)$$

(104)

and

$$\rho_\ell = \frac{\rho_{\ell-1}N - (1)^{\ell-1}\sqrt{P_1P_2(1 - \rho_{\ell-1}^2)}}{\sqrt{P_1(1 - \rho_{\ell-1}^2) + N\sqrt{P_2(1 - \rho_{\ell-1}^2) + N}}}, \quad \ell \in \{2, \ldots, \eta - 1\},$$

(105)

and where $r$ is the unique solution in $[0, 1]$ to (229), i.e., to

$$\sqrt{\frac{r^2P_1P_2}{(rP_1 + N)(rP_2 + N)}} = \rho^*(P_1, P_2, N).$$

We shall shortly prove that the solution to the recursion (104)–(105) is

$$\rho_\ell = (-1)^\ell \rho^*(P_1, P_2, N), \quad \ell \in \mathbb{N}.$$

(106)

This implies that for all $\ell \in \mathbb{N}$ larger than 1:

$$\rho_{\ell-1}^2 = \rho^{*2},$$

$$(-1)^{\ell-1}\rho_{\ell-1} = \rho^*,$$

and hence for fixed $\eta \in \mathbb{N}$ the region $\tilde{R}_\eta(P_1, P_2, N, 0)$ includes all rate pairs $(R_1, R_2)$ satisfying:

$$R_1 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_1}{N} \right) + \frac{\eta - 1}{2\eta} \log \left( 1 + \frac{P_1(1 - \rho^{*2})}{N} \right),$$

(107)

$$R_2 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_2}{N} \right) + \frac{\eta - 1}{2\eta} \log \left( 1 + \frac{P_2(1 - \rho^{*2})}{N} \right),$$

(108)

$$R_1 + R_2 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_1 + rP_2}{N} \right) + \frac{\eta - 1}{2\eta} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1P_2\rho^*}}{N} \right).$$

(109)
Notice that when $\eta$ tends to infinity, the right-hand sides of (107)–(109) tend to the right-hand sides of the three Constraints (8)–(10) evaluated for $\rho = \rho^*$. Since Constraints (8)–(10) evaluated for $\rho = \rho^*$ determine the region $\mathcal{R}_{O_3}(P_1, P_2, N)$, Inclusion (103) follows immediately by (107)–(109) and by letting $\eta$ tend to infinity.

In the remaining, we prove (106) in two steps. In the first step we show that $\rho^*(P_1, P_2, N)$ is a fix point of the function $h(\cdot)$ defined as

$$
h : [0, 1] \to \mathbb{R},
$$

$$
h(\rho) = \frac{\sqrt{P_1 P_2 (1 - \rho^2)} - \rho N}{\sqrt{P_1 (1 - \rho^2)} + N \sqrt{P_2 (1 - \rho^2)} + N}.
$$

Notice that $h(\cdot)$ has at least one fix point in $[0, 1]$ because $h(0) > 0$ whereas $h(1) < 0$ and because $h(\cdot)$ is continuous. Further notice that every fix point of $h(\cdot)$ must also be a solution to

$$
1 - h(\rho)^2 = 1 - \rho^2,
$$

i.e., a solution to

$$
\frac{N(N + P_1 + P_2 + 2 \sqrt{P_1 P_2 \rho})}{(N + P_1 (1 - \rho^2))(N + P_2 (1 - \rho^2))}(1 - \rho^2) = (1 - \rho^2).
$$

The solutions in $[0, 1]$ to (110) are given by $\rho = 1$ and by the solutions to

$$
N(N + P_1 + P_2 + 2 \sqrt{P_1 P_2 \rho}) = (N + P_1 (1 - \rho^2))(N + P_2 (1 - \rho^2)).
$$

Since $\rho = 1$ is not a fix point of $h(\cdot)$ and since $\rho^*(P_1, P_2, N)$ is the unique solution in $[0, 1]$ to (111) (see Definition 1), $\rho^*(P_1, P_2, N)$ must be a fix point of $h(\cdot)$. This concludes the first step.

In the second step we use the derived fix-point property of $h(\cdot)$ to prove (106). The proof is lead by induction. For $\ell = 1$ Condition (106) holds by definition. Assuming that (106) holds for some fixed $\ell \geq 1$, we have

$$
\rho_{\ell+1} = \frac{-(1)\ell \sqrt{P_1 P_2 (1 - |\rho_\ell|^2)} + \rho_\ell N}{\sqrt{P_1 (1 - |\rho_\ell|^2)} + N \sqrt{P_2 (1 - |\rho_\ell|^2)} + N}
$$

(112)

$$
= (1)^{\ell+1} h(|\rho_\ell|)
$$

(113)

$$
= (1)^{\ell+1} \rho^*(P_1, P_2, N),
$$

(114)

where (112) follows by the definition of the sequence $\{\rho_\ell\}$ for $\ell > 1$; (113) follows because by the induction assumption $\text{sign}(\rho_\ell) = (1)\ell$; (114) follows by the definition of the function $h(\cdot)$; and finally (115) follows because by the induction assumption $|\rho_\ell| = \rho^*$ and because $\rho^*$, as shown in the first step, is a fix point of $h(\cdot)$. Thus, (106) holds also for $(\ell + 1)$, which concludes the induction step and the proof of the remark.
8.4 Proof of Proposition 37

We only prove Inclusion (69); Inclusion (70) can be proved analogously.

Fix \( \rho \in [0, \rho^*] \), and define \( \alpha(\rho) \) as the unique solution in \([0,1]\) to

\[
\frac{P_1 + P_2 + 2\sqrt{P_1P_2}\rho + N}{P_1(1 - \rho^2) + N} = 1 + \frac{P_2\left(1 - \frac{\rho^2}{1-\alpha}\right)}{\alpha P_1 + N}. \tag{116}
\]

That (116) has exactly one solution in \([0,1]\) follows by the Intermediate Value Theorem and the following observations: The right-hand side of (116) is continuous and strictly decreasing in \(\alpha\); for \(\alpha = 0\) the right-hand side of (116) is larger or equal to the left-hand side because \(0 \leq \rho \leq \rho^*\) and by Remark 3; and for \(\alpha\) tending to 1 the right-hand side tends to \(-\infty\) and thus is smaller than the left-hand side.

Further, define

\[
P'_1 \triangleq \alpha(\rho)P_1, \quad P''_1 \triangleq (1 - \alpha(\rho))P_1, \quad N' \triangleq P'_1 + N, \quad \rho' \triangleq \frac{\rho}{\sqrt{1 - \alpha(\rho)}}
\]

and notice that by these definitions:

\[
N'(N' + P''_1 + P_2 + 2\sqrt{P''_1P_2}\rho') = (N' + P''_1(1 - \rho^2))(N' + P_2(1 - \rho^2)), \tag{117}
\]

and hence

\[\rho' = \rho^*(P''_1, P_2, N').\]

To establish Inclusion (69) we shall show that for the parameters as defined above

\[
\text{cl}\left(\bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0: \text{tr}(K) \leq \sigma^2} \mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, K)\right) \supseteq \mathcal{R}_{1, Oz}^\rho(P_1, P_2, N). \tag{118}
\]

To this end, notice that by Proposition 21, Part 2.:

\[
\text{cl}\left(\bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0: \text{tr}(K) \leq \sigma^2} \mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, K)\right) = \mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, 0),
\]

and thus we have to prove:

\[
\mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, 0) \supseteq \mathcal{R}_{1, Oz}^\rho(P_1, P_2, N). \tag{119}
\]

Before proving Inclusion (119), we define \((R^\rho_{1, Oz}, R^\rho_{2, Oz})\) as the dominant corner point of the rectangle \(\mathcal{R}_{1, Oz}^\rho(P_1, P_2, N)\). The following two remarks on \((R^\rho_{1, Oz}, R^\rho_{2, Oz})\) are from [9], and based on (117).
Remark 40. The rate point \((R^p_{1,Oz}, R^p_{2,Oz})\) can be expressed as
\[
R^p_{1,Oz} = R^p_{1,1,Oz} + R^p_{1,2,Oz},
\]
\[
R^p_{2,Oz} = \frac{1}{2} \log \left( 1 + \frac{P_2(1 - \rho^2)}{N'} \right),
\]
where
\[
R^p_{1,1,Oz} \triangleq \frac{1}{2} \log \left( 1 + \frac{P'_1}{N} \right),
\]
\[
R^p_{1,2,Oz} \triangleq \frac{1}{2} \log \left( 1 + \frac{P''_1(1 - \rho^2)}{N'} \right).
\]

Remark 41. The rate point \((R^p_{1,2,Oz}, R^p_{2,Oz})\) corresponds to the dominant corner point of the rectangle \(\mathcal{R}^p_{Oz}(P'_1, P_2, N')\), where \(\rho' = \rho''(P''_1, P_2, N')\).

We are now ready to prove Inclusion (119). For \(K_{W_1W_2} = 0\) the right-hand side of (57) equals \(\frac{1}{2} \log \left( 1 + \frac{P'_1}{N} \right)\), irrespective of the parameters \(a_1, a_2, B_1, B_2, C\). Therefore, the region \(\mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, 0)\) is given by the set of all rate pairs \((R_1, R_2)\) which for some nonnegative \(R_{1,CS}, R_{1,NF}\) summing to \(R_1\) satisfy
\[
(R_{1,CS}, R_2) \in \mathcal{R}(P''_1, P_2, N', 0),
\]
\[
R_{1,NF} \leq \frac{1}{2} \log \left( 1 + \frac{P'_1}{N} \right). \tag{120}
\]
Since by Remark 41 and Remark 17:
\[
(R^p_{1,2,Oz}, R^p_{2,Oz}) \in \mathcal{R}(P''_1, P_2, N', 0)
\]
and by Remark 40:
\[
R^p_{1,1,Oz} \leq \frac{1}{2} \log \left( 1 + \frac{P'_1}{N} \right),
\]
the triple \((R^p_{1,1,Oz}, R^p_{1,2,Oz}, R^p_{2,Oz})\) satisfies (120) and (121), and hence
\[
(R^p_{1,Oz}, R^p_{2,Oz}) \in \mathcal{R}_{RS,1}(P'_1, P''_1, P_2, N, 0). \tag{122}
\]
Inclusion (119) finally follows because \((R^p_{1,Oz}, R^p_{2,Oz})\) is the dominant corner point of the rectangle \(\mathcal{R}^p_{1,Oz}(P_1, P_2, N)\), and therefore (122) implies that the entire region \(\mathcal{R}^p_{1,Oz}(P_1, P_2, N)\) is contained in the right-hand side of (122).

8.5 Proof of Theorem 38

Fix \(P_1, P_2, N > 0\). The proof of the \(\subseteq\)-direction follows trivially because replacing the intersection on the left-hand side by the specific choice \(K = 0\) can only increase the region, because \(C_{\text{NoisyFB}}(P_1, P_2, N, 0) = C_{\text{PerfectFB}}(P_1, P_2, N)\), and because by definition the region \(C_{\text{PerfectFB}}(P_1, P_2, N)\) is closed.
The $\supseteq$-direction, i.e.,

$$\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0 : \text{tr}(K) \leq \sigma^2} C_{\text{NoisyFB}} (P_1, P_2, N, K) \right) \supseteq C_{\text{PerfectFB}} (P_1, P_2, N),$$

follows from the following sequence of inclusions:

$$\text{cl} \left( \bigcup_{\sigma^2 > 0} \bigcap_{K \geq 0 : \text{tr}(K) \leq \sigma^2} \left( \bigcup_{P_1' \in [0, P_1]} \mathcal{R}_{\text{RS,1}} (P_1', (P_1 - P_1'), P_2, N, K) \right) \cup \left( \bigcup_{P_2' \in [0, P_2]} \mathcal{R}_{\text{RS,2}} (P_1, P_2', (P_2 - P_2'), N, K) \right) \right) \supseteq \left( \bigcup_{P_1' \in [0, P_1]} \mathcal{R}_{\text{RS,1}} (P_1', (P_1 - P_1'), P_2, N, K) \right) \supseteq \left( \bigcup_{P_2' \in [0, P_2]} \mathcal{R}_{\text{RS,2}} (P_1, P_2', (P_2 - P_2'), N, K) \right) \supseteq \text{cl} \left( \bigcup_{\rho \in [0, \rho^*(P_1, P_2, N)]} \mathcal{R}_{1,0z}^\rho (P_1, P_2, N) \right) \cup \text{cl} \left( \bigcup_{\rho \in [0, \rho^*(P_1, P_2, N)]} \mathcal{R}_{2,0z}^\rho (P_1, P_2, N) \right)$$

$$= C_{\text{PerfectFB}} (P_1, P_2, N),$$

where (123) follows from Proposition 20; (124) follows by basic rules on sets; (125) follows from Proposition 37; and (126) follows by Remark 7.

### 8.6 Proof of Theorem 39

We consider an AWGN MAC with powers $P_1 = 1, P_2 = 5$, noise variance $N = 5$, and with perfect partial feedback. We prove the theorem by showing that for this channel the rate point $(\tilde{R}_1, \tilde{R}_2)$,

$$\tilde{R}_1 = \frac{1}{4} \log \left( \frac{7}{5} \right),$$

$$\tilde{R}_2 = \frac{1}{4} \log \left( 3 + \frac{3}{7} + \frac{2}{7} \sqrt{\frac{11}{6}} \right),$$

—which by Corollary 27 is achievable—lies outside the Cover-Leung region $\mathcal{R}_{\text{CL}} (P_1, P_2, N)$. This implies that the capacity region $C_{\text{PerfectPartialFB}} (P_1, P_2, N)$ is a strict superset of the Cover-Leung region $\mathcal{R}_{\text{CL}} (P_1, P_2, N)$ for $P_1 = 1$ and $P_2 = N = 5$.

Before starting with the proof, we have a closer look at the region $\mathcal{R}_{\text{CL}} (P_1, P_2, N)$ and show the following lemma.
Lemma 42. For \( P_1, P_2, N > 0 \) and for every \( \rho_1 \in [0, 1] \) which satisfies
\[
\frac{P_2}{N} \geq \frac{\rho_1^2}{1 - \rho_1^2},
\] (127)
the rate point \((R_1(\rho_1), R_2(\rho_1))\) given by
\[
R_1(\rho_1) = \frac{1}{2} \log \left( 1 + \frac{P_1 (1 - \rho_1^2)}{N} \right),
\] (128)
\[
R_2(\rho_1) = \max_{\rho_2 \in [0, 1]} \left\{ \min \left\{ \frac{1}{2} \log \left( 1 + \frac{P_2 (1 - \rho_2^2)}{N} \right), \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2\sqrt{P_1 P_2 \rho_1 \rho_2} + N}{P_1 (1 - \rho_1^2) + N} \right) \right\} \right\},
\] (129)
lies on the boundary of \( R_{CL}(P_1, P_2, N) \) in the sense that for every \( \epsilon > 0 \)
\[(R_1(\rho_1), R_2(\rho_1) + \epsilon) \notin R_{CL}(P_1, P_2, N).\]

Proof. As a first step we examine Expression (129) and characterize \( R_2(\rho_1) \) more explicitly. To this end, we consider a fixed \( \rho_1 \in [0, 1] \) which satisfies (127). Then, we notice that in the minimization in (129) the first term is strictly decreasing in \( \rho_2 \in [0, 1] \) whereas the second term is strictly increasing in \( \rho_2 \). Also, for \( \rho_2 = 1 \) the first term in the maximization in (129) is smaller than the second term, whereas by Condition (127) for \( \rho_2 = 0 \) the second term is smaller. Thus, for fixed \( \rho_1 \in [0, 1] \) satisfying (127) the maximum in (129) is achieved when both terms are equal, i.e., for \( \tilde{\rho}_2 \) given by the unique solution in \([0, 1]\) to
\[
\frac{1}{2} \log \left( 1 + \frac{P_2 (1 - \tilde{\rho}_2^2)}{N} \right) = \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2\sqrt{P_1 P_2 \rho_1 \tilde{\rho}_2} + N}{P_1 (1 - \rho_1^2) + N} \right).
\]
This implies that the rate pair \((R_1(\rho_1), R_2(\rho_1))\) satisfies all three rate constraints defining the rectangle \( R_{CL}^{(\rho_1, \rho_2)}(P_1, P_2, N) \) with equality, i.e.,
\[
R_1(\rho_1) = \frac{1}{2} \log \left( 1 + \frac{P_1 (1 - \rho_1^2)}{N} \right),
\] (130)
\[
R_2(\rho_1) = \frac{1}{2} \log \left( 1 + \frac{P_2 (1 - (\tilde{\rho}_2)^2)}{N} \right),
\] (131)
\[
R_1(\rho_1) + R_2(\rho_1) = \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2 \rho_1 \tilde{\rho}_2}}{N} \right).
\] (132)

Hence, \((R_1(\rho_1), R_2(\rho_1))\) is the dominant corner point of the rectangle \( R_{CL}^{(\rho_1, \rho_2)}(P_1, P_2, N) \), and for all \( \epsilon > 0 \) the rate point \((R_1(\rho_1), R_2(\rho_1) + \epsilon)\) lies outside the rate region \( R_{CL}^{(\rho_1, \rho_2)}(P_1, P_2, N) \). In the remaining we show that the rate point \((R_1(\rho_1), R_2(\rho_1))\) also lies outside the regions \( R_{CL}^{(\rho_1', \rho_2')} (P_1, P_2, N) \) for all \( \rho_1', \rho_2' \in [0, 1] \) not equal to the pair \((\rho_1, \tilde{\rho}_2)\), and therefore also \((R_1(\rho_1), R_2(\rho_1) + \epsilon)\) lies outside these regions for every \( \epsilon > 0 \). This will then conclude the proof of the proposition. We distinguish the following three cases: 1) \( \rho_1' > \rho_1 \) and \( \rho_2' \) arbitrary; 2) \( \rho_1' = \rho_1 \) and \( \rho_2' > \tilde{\rho}_2 \); and 3) \( \rho_1' \leq \rho_1 \) and \( \rho_2' < \tilde{\rho}_2 \). In case 1) the rate point \((R_1(\rho_1), R_2(\rho_1))\) lies outside the region \( R_{CL}^{(\rho_1', \rho_2')} (P_1, P_2, N) \) because \( R_1(\rho_1) \) violates the single-rate constraint, see (13) and (130). Similarly, in case 2) the rate point lies outside the region
\[ R_{\CL}^{(\rho'_1, \rho'_2)}(P_1, P_2, N) \] because in this case \( R_2(\rho_1) \) violates the single-rate constraint, see (14) and (131). Finally, in case 3) the rate point lies outside the region \( R_{\CL}^{(\rho'_1, \rho'_2)}(P_1, P_2, N) \) because the product \( \rho'_1 \cdot \rho'_2 \) is strictly smaller than the product \( \rho_1 \cdot \bar{\rho}_2 \), and thus the sum \( R_1(\rho_1) + R_2(\rho_1) \) violates the sum-rate constraint, see (15) and (132).

Now we are ready to prove that the achievable rate point \((\bar{R}_1, \bar{R}_2)\) lies strictly outside the Cover-Leung region \( R_{\CL}(P_1, P_2, N) \). To this end, we choose \( \rho_1 = \sqrt{6 - \sqrt{35}} \) and notice that it satisfies Condition (127) for \( P_2 = N = 5 \). Hence, Lemma 42 applies and the rate point \((R^B_1, R^B_2)\),

\[
R^B_1 \triangleq \frac{1}{2} \log \left( \sqrt{\frac{7}{5}} \right), \\
R^B_2 \triangleq \max_{\rho_2 \in [0, 1]} \left\{ \min \left\{ \frac{1}{2} \log \left( 1 + (1 - \rho_2^3) \right), \frac{1}{2} \log \left( \frac{11 + 2 \sqrt{5(6 - \sqrt{35})\rho_2}}{\sqrt{35}} \right) \right\} \right\},
\]

lies on the boundary of the Cover-Leung region \( R_{\CL}(P_1, P_2, N) \), and in particular for every \( \epsilon > 0 \) the rate point \((R^B_1, R^B_2 + \epsilon)\) lies strictly outside the Cover-Leung region \( R_{\CL}(P_1, P_2, N) \). Since

\[
R^B_1 = \bar{R}_1,
\]
in order to show that the rate point \((\bar{R}_1, \bar{R}_2)\) lies strictly outside \( R_{\CL}(P_1, P_2, N) \) it suffices to show that

\[
R^B_2 < \bar{R}_2. \tag{135}
\]

To prove (135) we could compute \( \bar{\rho}_2 \)—the value of \( \rho_2 \) which maximizes (134)—and \( R^B_2 \) and then check Condition (135). However, it is easier—and sufficient—to show that for all \( \rho_2 \in [0, 1] \) either

\[
\frac{1}{2} \log \left( 2 - \rho_2^3 \right) < \bar{R}_2 \tag{136}
\]
or

\[
\frac{1}{2} \log \left( \frac{11 + 2 \sqrt{5(6 - \sqrt{35})\rho_2}}{\sqrt{35}} \right) < \bar{R}_2. \tag{137}
\]

To this end, note first that the left-hand side of (136) is decreasing in \( \rho_2 \in [0, 1] \), and therefore for all \( \sqrt{\frac{1}{7}} \leq \rho_2 \leq 1 \) it follows that

\[
\frac{1}{2} \log \left( 2 - \rho_2^3 \right) \leq \frac{1}{4} \log \left( 3 + \frac{3}{7} + \frac{1}{49} \right) < \bar{R}_2.
\]

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On the other hand, the left-hand side of (137) is increasing in \(\rho_2\), and thus for all \(0 \leq \rho_2 \leq \sqrt{\frac{1}{7}}\)

\[
\frac{1}{2} \log \left( \frac{11 + 2\sqrt{5(6 - \sqrt{35})\rho_2}}{\sqrt{35}} \right)
\leq \frac{1}{4} \log \left( 3 + \frac{3}{7} + \frac{1}{35} + \frac{44}{7} \frac{(6 - \sqrt{35})}{35} + \frac{4}{7} \frac{(30 - 5\sqrt{35})}{35} \right)
\leq \frac{1}{4} \log \left( 3 + \frac{3}{7} + \frac{1}{7} \left( \frac{1}{10} + \frac{22}{\sqrt{35}} \sqrt{(6 - \sqrt{35}) + \frac{12}{7} - 2\sqrt{\frac{5}{7}}} \right) \right)
< \bar{R}_2,
\]

where the inequality follows because

\[
\frac{1}{10} + \frac{22}{\sqrt{35}} \sqrt{(6 - \sqrt{35}) + \frac{12}{7} - 2\sqrt{\frac{5}{7}}} < \sqrt{\frac{11}{6}}.
\]

This concludes the proof of the theorem.

9 Summary

We have studied four different kinds of two-user AWGN MACs with imperfect feedback:

- **noisy feedback**, where the feedback links to both transmitters are corrupted by additive white Gaussian noise;

- **noisy partial feedback**, where one transmitter has noisy feedback and the other no feedback;

- **perfect partial feedback**, where one transmitter has noise-free feedback and the other no feedback; and

- **noisy feedback with receiver side-information**, where both transmitters have noisy feedback and the feedback-noise sequences are perfectly known to the receiver.

For each of these settings we have presented a coding scheme with general parameters, and we have stated the corresponding achievable regions (Theorem 14, Theorem 24, Corollary 26, and Theorem 33). For the case of noisy feedback we have further improved the scheme by rate-splitting it with a simple no-feedback scheme, see Proposition 20 for the corresponding achievable rate region.

The two achievable regions for noisy feedback (Theorem 14 and Proposition 20) exhibit the following three properties: 1. They are monotonically decreasing in the feedback-noise covariance matrix with respect to the Loewner order (Propositions 15 and 21). 2. They are continuous in the transmit-powers (Propositions 15 and 21). 3. They converge to Ozarow’s perfect-feedback regions when the feedback noise-variances tend to 0, irrespective of the feedback-noise correlations (Propositions 35 and 37).

Our achievable regions for noisy feedback, for noisy partial feedback, and for perfect partial feedback allowed us to infer:
1) Feedback—no matter how noisy—is strictly better than no feedback. I.e., irrespective of the feedback-noise variances, the capacity region with one or two noisy feedback links is strictly larger than the no-feedback capacity region (Theorem 34).

2) The noisy-feedback capacity region converges to the perfect-feedback capacity region as the feedback-noise variances on both links tend to 0—irrespective of the feedback-noise correlations (Theorem 38).

3) The Cover-Leung region in general does not equal capacity for perfect partial feedback channels (Theorem 39). This answers in the negative a question posed by van der Meulen in [13].

We have further presented guidelines for choosing the parameters of our concatenated scheme (Section 7) and we have suggested (suboptimal) specific choices for the parameters (Sections E.1, F.1, and G.1). The achievable rate regions corresponding to our specific choices are stated in Corollary 16, Corollary 25, Corollary 27, Corollary 52 (Section E), Corollary 55 and Remark 56 (Section F), and Corollary 58 (Section G). These achievable regions were helpful in deriving Conclusions 1)–3) above.

Moreover, we have extended our concatenated schemes for noisy feedback, noisy partial feedback, and perfect partial feedback by rate-splitting them with Carleial’s version of the Cover-Leung scheme. Propositions 28 and 29 state the corresponding achievable regions for perfect partial feedback.

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### A Carleial’s region

Carleial [2] proved that the region $R_{\text{ach}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$, which is defined in Definition 43 ahead, is achievable for the two-user AWGN MAC with noisy feedback of powers $P_1$ and $P_2$, noise variance $N$, and feedback-noise covariance matrix $K_{W_1W_2} = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$ for all $\rho \in [-1, 1]$.

**Definition 43.** Let $P_1, P_2, N > 0$ and $\sigma_1^2, \sigma_2^2 \geq 0$ be given. Define the rate region $R_{\text{Car}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ as the set of all rate pairs $(R_1, R_2)$ which for some nonnegative numbers $R_{1,0}, R_{1,1}$ summing to $R_1$, for some nonnegative numbers $R_{2,0}, R_{2,2}$ summing to $R_2$, and for some choice of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda \in [0, 1]$ satisfy the following 13 conditions:

\[
R_{1,0} \leq \frac{1}{2} \log \left( 1 + \frac{\alpha_1 \beta_1 P_1}{\alpha_1 \beta_1 P_1 + N + \sigma_2^2} \right), \tag{138}
\]

\[
R_{2,0} \leq \frac{1}{2} \log \left( 1 + \frac{\alpha_2 \beta_2 P_2}{\alpha_2 \beta_2 P_2 + N + \sigma_1^2} \right), \tag{139}
\]

\[
R_{1,1} \leq \frac{1}{2} \log \left( 1 + \frac{\alpha_1 \beta_1 P_1}{N} \right). \tag{140}
\]
Lemma 44. The rate region $R_{\text{Car}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ collapses to the no-feedback capacity region $C_{\text{MAC}}(P_1, P_2, N)$ when the feedback-noise variances $\sigma_1^2, \sigma_2^2$ exceed a certain threshold depending on the parameters $P_1, P_2,$ and $N$. In particular,

$$R_{\text{Car}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2) = C_{\text{MAC}}(P_1, P_2, N),$$

where for $x \in [0, 1]$ we define $\bar{x} \triangleq (1 - x)$.
for $\sigma_1^2 \geq P_1 \left( \frac{3}{2} + \frac{P_1}{N} \right)$ and $\sigma_2^2 \geq P_2 \left( \frac{3}{2} + \frac{P_1}{N} \right)$.

**Proof.** For all values of $\sigma_1^2, \sigma_2^2$ the region $R_{\text{Car}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ trivially includes the no-feedback capacity region $R_{\text{MAC}}(P_1, P_2, N)$ because the region defined by $a_1 = a_2 = b_1 = b_2 = 1$ and (138)–(150) coincides with $R_{\text{MAC}}(P_1, P_2, N)$. Thus, it remains to prove that $R_{\text{Car}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ is included in $R_{\text{MAC}}(P_1, P_2, N)$ for all $\sigma_1^2, \sigma_2^2$ exceeding some threshold depending on $P_1, P_2, N$.

To this end, we choose $\sigma_1^2, \sigma_2^2 > 0$ and we fix a rate pair $(R_1, R_2)$ in $R_{\text{Car}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$. We then fix parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda \in [0, 1]$, two nonnegative numbers $P_{1,0}$ and $P_{1,1}$ summing to $R_1$, and two nonnegative numbers $R_{2,0}$ and $R_{2,2}$ summing to $R_2$ so that (138)–(150) are satisfied. We show in the following that if $\sigma_1^2, \sigma_2^2 > 0$ are sufficiently large, then $(R_1, R_2)$ lies in $R_{\text{MAC}}(P_1, P_2, N)$. This will then conclude the proof of the lemma.

By (138) and (140) the rate $R_1$ satisfies

$$R_1 \leq \frac{1}{2} \log \left( 1 + \frac{\alpha_1 \beta_1 P_1}{\alpha_1 \beta_1 P_1 + N + \sigma_2^2} \right) + \frac{1}{2} \log \left( 1 + \frac{\alpha_1 \beta_1 P_1}{N} \right),$$

and by (139) and (141) the rate $R_2$ satisfies

$$R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\alpha_2 \beta_2 P_2}{\alpha_2 \beta_2 P_2 + N + \sigma_1^2} \right) + \frac{1}{2} \log \left( 1 + \frac{\alpha_2 \beta_2 P_2}{N} \right),$$

Furthermore, by (138), (139), and (145) the sum of the rates $R_1 + R_2$ satisfies

$$R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{\alpha_1 \beta_1 P_1}{\alpha_1 \beta_1 P_1 + N + \sigma_2^2} \right) + \frac{1}{2} \log \left( 1 + \frac{\alpha_2 \beta_2 P_2}{\alpha_2 \beta_2 P_2 + N + \sigma_1^2} \right)$$

$$+ \frac{1}{2} \log \left( 1 + \frac{\alpha_1 \beta_1 P_1 + \alpha_2 \beta_2 P_2}{N} \right)$$

$$+ \frac{\alpha_1 \beta_1 P_1}{N} \left( N + \alpha_1 \beta_1 P_1 + \alpha_2 \beta_2 P_2 \right) + \left( \alpha_1 \beta_1 P_1 + \frac{1}{2} N \right) \frac{\alpha_2 \beta_2 P_2}{\alpha_2 \beta_2 P_2 + N + \sigma_1^2}$$

$$+ \frac{\alpha_2 \beta_2 P_2}{N} \left( N + \alpha_1 \beta_1 P_1 + \alpha_2 \beta_2 P_2 \right) + \left( \alpha_2 \beta_2 P_2 + \frac{1}{2} N \right) \frac{\alpha_1 \beta_1 P_1}{\alpha_1 \beta_1 P_1 + N + \sigma_2^2}.$$ (153)

Notice that for $\sigma_1^2, \sigma_2^2$ larger than some threshold depending on $P_1, P_2, N$—and in particular for $\sigma_1^2 > P_1 \left( \frac{3}{2} + \frac{P_1}{N} \right)$ and $\sigma_2^2 > P_2 \left( \frac{3}{2} + \frac{P_1}{N} \right)$—irrespective of the chosen parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda$:

$$\frac{N + \alpha_1 \beta_1 P_1 + \alpha_2 \beta_2 P_2 + \left( \alpha_1 \beta_1 P_1 + \frac{1}{2} N \right) \frac{\alpha_2 \beta_2 P_2}{\alpha_2 \beta_2 P_2 + N + \sigma_1^2}}{\alpha_1 \beta_1 P_1 + N + \sigma_2^2} < 1,$$ (154)

$$\frac{N + \alpha_1 \beta_1 P_1 + \alpha_2 \beta_2 P_2 + \left( \alpha_2 \beta_2 P_2 + \frac{1}{2} N \right) \frac{\alpha_1 \beta_1 P_1}{\alpha_1 \beta_1 P_1 + N + \sigma_2^2}}{\alpha_2 \beta_2 P_2 + N + \sigma_1^2} < 1.$$ (155)
Thus, when $\sigma_1^2, \sigma_2^2$ exceed a certain threshold depending on $P_1, P_2,$ and $N$, the right-hand side of (153) is upper bounded by $\frac{1}{2} \log \left( 1 + \frac{P_1 + P_2}{N} \right)$. We conclude that when $\sigma_1^2, \sigma_2^2$ are sufficiently large, then by (151)–(155) the rate pair $(R_1, R_2)$ satisfies (4)–(6) and hence lies in the no-feedback capacity region $C_{\text{MAC}}(P_1, P_2, N)$. This concludes the proof. 

\[ \square \]

\section{Willems et al.’s region}

Willems et al. \cite{Willems} proved an achievability result for the discrete memoryless MAC with imperfect feedback. The result can easily be extended to the two-user AWGN MAC with noisy feedback. This extension establishes that the region $R_{\text{Wil}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$, which is defined in Definition 45 ahead, is achievable for the AWGN MAC with noisy feedback of powers $P_1$ and $P_2$, noise variance $N$, and feedback-noise covariance matrix $K_{\text{Wil}} = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$ for all $\rho \in [-1, 1]$.

\begin{definition}
Let $P_1, P_2, N > 0$ and $\sigma_1^2, \sigma_2^2 > 0$ be given. Define the rate region $R_{\text{Wil}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ as the set of all rate pairs $(R_1, R_2)$ which for some nonnegative numbers $R_{1,1}$ and $R_{1,2}$ summing to $R_1$, for some nonnegative numbers $R_{2,1}$ and $R_{2,2}$ summing to $R_2$, and for some parameters $\delta_1, \delta_2, \rho_1, \rho_2 \in [0, 1]$ satisfy the following five constraints:

\begin{align*}
R_{1,1} &\leq \frac{1}{2} \log \left( 1 + \frac{\delta_1 P_1}{N} \right), \\
R_{1,0} &\leq \frac{1}{2} \log \left( 1 + \frac{\delta_1 P_1 (1 - \rho_1^2)}{\delta_1 P_1 + N + \sigma_2^2} \right), \\
R_{2,0} &\leq \frac{1}{2} \log \left( 1 + \frac{\delta_2 P_2 (1 - \rho_2^2)}{\delta_2 P_2 + N + \sigma_1^2} \right), \\
R_{2,2} &\leq \frac{1}{2} \log \left( 1 + \frac{\delta_2 P_2}{N} \right), \\
R_{1,1} + R_{2,2} &\leq \frac{1}{2} \log \left( 1 + \frac{\delta_1 P_1 + \delta_2 P_2}{N} \right), \\
R_1 + R_2 &\leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2 \sqrt{\delta_1 \delta_2 P_1 P_2 \rho_1 \rho_2}}{N} \right).
\end{align*}

Notice that again the rate region is independent of the feedback-noise correlation $\rho$.

\begin{lemma}
The rate region $R_{\text{Wil}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ collapses to the no-feedback capacity region $C_{\text{MAC}}(P_1, P_2, N)$ when the feedback-noise variances $\sigma_1^2, \sigma_2^2$ exceed a certain threshold depending on the parameters $P_1, P_2,$ and $N$. In particular,

\[ R_{\text{Wil}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2) = C_{\text{MAC}}(P_1, P_2, N), \]

for $\sigma_1^2 \geq P_1 \left( \frac{\delta_1^2}{2} + \frac{\rho_1^2}{N} \right)$ and $\sigma_2^2 \geq P_2 \left( \frac{\delta_2^2}{2} + \frac{\rho_2^2}{N} \right)$.

\begin{proof}
Follows along similar lines as the proof of Lemma 44 and is omitted. \hfill \square
\end{proof}
\end{lemma}

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C Recursive Formulation of Achievable Regions

We derive a recursive formulation for the achievable region by our concatenated scheme \( R(N, K_{W_1}, W_2; \eta, a_1, a_2, B_1, B_2, C) \) when \( C \) is the LMMSE-estimation matrix and \( \eta, a_1, a_2, B_1, B_2 \) are arbitrary. Recall that there is no loss in optimality in restricting attention to this choice of \( C \), see Section 7. Similarly, we derive a recursive formulation for the achievable region \( R_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P) \) when \( C_P \) is the LMMSE-estimation matrix and \( \eta, a_1, a_2, B_2 \) are arbitrary, and a recursive formulation for the achievable region \( R_{SI}(N, K_{W_1}, \eta, a_1, a_2, B_1, B_2, C_{SI}) \) when \( C_{SI} \) is the LMMSE-estimation matrix with side-information and \( \eta, a_1, a_2, B_1, B_2 \) are arbitrary. These recursive formulations simplify the analysis of the achievable regions corresponding to our specific choices of parameters suggested in Sections E, F, and G. Moreover, they are useful for establishing certain properties of the regions \( R(N, K_{W_1}, W_2; \eta, a_1, a_2, B_1, B_2, C) \), \( R_P(N, \sigma_2^2; \eta, a_1, a_2, B_2, C_P) \), and \( R_{SI}(N, K_{W_1}, W_2; \eta, a_1, a_2, B_1, B_2, C_{SI}) \), see again Section 7.

C.1 Noisy Feedback

Let \( P_1, P_2, N > 0 \), and \( K_{W_1}W_2 \geq 0 \) be given. Fix parameters \( \eta, a_1, a_2, B_1, B_2, C \) satisfying the power constraints (21) and (22) and so that \( C \) equals the LMMSE-estimation matrix. We derive a recursive formulation for the region achieved by our concatenated scheme for parameters \( \eta, a_1, a_2, B_1, B_2, C \).

Notice first that our concatenated scheme achieves all nonnegative rate pairs \((R_1, R_2)\) satisfying

\[
R_1 \leq \frac{1}{\eta} I(\Xi_1; \hat{\Xi}_1, \hat{\Xi}_2|\Xi_2), \tag{156}
\]

\[
R_2 \leq \frac{1}{\eta} I(\Xi_2; \hat{\Xi}_1, \hat{\Xi}_2|\Xi_1), \tag{157}
\]

\[
R_1 + R_2 \leq \frac{1}{\eta} I(\Xi_1, \Xi_2; \hat{\Xi}_1, \hat{\Xi}_2), \tag{158}
\]

where \( \Xi_1, \Xi_2 \) are independent standard Gaussians and where the conditional law of \((\hat{\Xi}_1, \hat{\Xi}_2)\) given \( \Xi_1 = \xi_1 \) and \( \Xi_2 = \xi_2 \) is determined by the channel law \( \xi_1, \xi_2 \mapsto (\hat{\xi}_1, \hat{\xi}_2) \) in (35).

Before presenting a recursive formulation of Constraints (156)–(158), we introduce some assumptions and definitions, and we derive an alternative formulation of Constraints (156)–(158).

To simplify notation in this section we assume that Inner Encoder 1 and Inner Encoder 2 are fed by independent zero-mean unit-variance Gaussian random variables and therefore we denote them by \( \Xi_1 \) and \( \Xi_2 \) instead of \( \xi_1 \) and \( \xi_2 \). Since \( C \) is the LMMSE-estimation matrix, the pair \((\hat{\Xi}_1, \hat{\Xi}_2)\), which is defined in (23) can be expressed as

\[
\begin{pmatrix}
\hat{\Xi}_1 \\
\hat{\Xi}_2
\end{pmatrix} = \mathbb{E} \left[ \begin{pmatrix}
\Xi_1 \\
\Xi_2
\end{pmatrix} \bigg| Y_1, \ldots, Y_\eta \right]. \tag{159}
\]

Define the errors \( E_1 \triangleq (\Xi_1 - \hat{\Xi}_1) \) and \( E_2 \triangleq (\Xi_2 - \hat{\Xi}_2) \), their variances \( \alpha_1 \triangleq \text{Var}(E_1) \), \( \alpha_2 \triangleq \text{Var}(E_2) \), and their correlation coefficient \( \rho \triangleq \frac{\text{Cov}(E_1, E_2)}{\sqrt{\alpha_1 \alpha_2}} \). By (159) and by the Gaussianity
of the involved random variables, the errors \( E_1 \) and \( E_2 \) are independent of the pair \((\Xi_1, \hat{\Xi}_2)\), and the following nonrecursive alternative formulation of Constraints (156)–(158) is obtained:

\[
R_1 \leq \frac{1}{2\eta} \log \left( \frac{1}{\alpha_1 (1 - \rho^2)} \right), \tag{160}
\]

\[
R_2 \leq \frac{1}{2\eta} \log \left( \frac{1}{\alpha_2 (1 - \rho^2)} \right), \tag{161}
\]

\[
R_1 + R_2 \leq \frac{1}{2\eta} \log \left( \frac{1}{\alpha_1 \alpha_2 (1 - \rho^2)} \right). \tag{162}
\]

In the following, we derive a recursive formulation of Constraints (160)–(162). To this end, we define the LMMSE-estimation errors

\[
E_{1,\ell} \triangleq \Xi_1 - E[\Xi_1 | Y_1, \ldots, Y_{\ell-1}], \quad \ell \in \{1, \ldots, \eta\},
\]

\[
E_{2,\ell} \triangleq \Xi_2 - E[\Xi_2 | Y_1, \ldots, Y_{\ell-1}], \quad \ell \in \{1, \ldots, \eta\},
\]

their variances \( \alpha_{1,\ell} \triangleq \text{Var}(E_{1,\ell}) \), \( \alpha_{2,\ell} \triangleq \text{Var}(E_{2,\ell}) \), and their correlation coefficient \( \rho_{\ell} \triangleq \frac{\text{Cov}(E_{1,\ell}, E_{2,\ell})}{\sqrt{\alpha_{1,\ell} \alpha_{2,\ell}}} \), for \( \ell \in \{1, \ldots, \eta\} \). Notice that \( \alpha_{1,\eta} = \alpha_1 \), \( \alpha_{2,\eta} = \alpha_2 \), and \( \rho_\eta = \rho \), and thus,

\[
\frac{1}{\alpha_1 (1 - \rho^2)} = \prod_{\ell=1}^{\eta} \frac{\alpha_{1,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{1,\ell} (1 - \rho_{\ell}^2)},
\]

\[
\frac{1}{\alpha_2 (1 - \rho^2)} = \prod_{\ell=1}^{\eta} \frac{\alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{2,\ell} (1 - \rho_{\ell}^2)},
\]

\[
\frac{1}{\alpha_1 \alpha_2 (1 - \rho^2)} = \prod_{\ell=1}^{\eta} \frac{\alpha_{1,\ell-1} \alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{1,\ell} \alpha_{2,\ell} (1 - \rho_{\ell}^2)},
\]

where we define \( \alpha_{1,0} \triangleq 1 \), \( \alpha_{2,0} \triangleq 1 \), and \( \rho_0 \triangleq 0 \). Therefore, Constraints (160)–(162) are equivalent to:

\[
R_1 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( \frac{\alpha_{1,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{1,\ell} (1 - \rho_{\ell}^2)} \right), \tag{165}
\]

\[
R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( \frac{\alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{2,\ell} (1 - \rho_{\ell}^2)} \right), \tag{166}
\]

\[
R_1 + R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( \frac{\alpha_{1,\ell-1} \alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{1,\ell} \alpha_{2,\ell} (1 - \rho_{\ell}^2)} \right). \tag{167}
\]

We shall derive a more explicit formulation of Constraints (165)–(167) in terms of the entries of the parameters \( a_1, a_2, B_1, \) and \( B_2 \). Let \( a_{\nu,\ell} \) denote the \( \ell \)-th entry of the vector \( a_\nu \) and \( b_{\nu,\ell,j} \) denote the row-\( \ell \) column-\( j \) entry of the matrix \( B_\nu \), for \( j, \ell \in \{1, \ldots, \eta\} \) and \( \nu \in \{1, 2\} \). We notice that, when the inner encoders are fed by \( \Xi_1 \) and \( \Xi_2 \), the \( \eta \) channel outputs produced by the original MAC \( x_1, x_2 \mapsto Y \) are given by

\[
Y_\ell = a_{1,\ell} \Xi_1 + a_{2,\ell} \Xi_2 + \sum_{j=1}^{\ell-1} b_{1,\ell,j} V_{1,j} + \sum_{j=1}^{\ell-1} b_{2,\ell,j} V_{2,j} + Z_\ell, \quad \ell \in \{1, \ldots, \eta\}. \tag{168}
\]
We define the innovations
\[
I_\ell \triangleq Y_\ell - \mathbb{E}[Y_\ell|Y_1, \ldots, Y_{\ell-1}], \quad \ell \in \{1, \ldots, \eta\},
\]
(169)
which by the Gaussianity of all involved random variables and by the following two definitions
\[
W_{\Sigma, \ell-1} \triangleq \sum_{j=1}^{\ell-1} b_{1, \ell, j} W_{1, j} + \sum_{j=1}^{\ell-1} b_{2, \ell, j} W_{2, j}, \quad \ell \in \{1, \ldots, \eta\},
\]
(170)
\[
W_{\perp, \ell-1} \triangleq W_{\Sigma, \ell-1} - \mathbb{E}[W_{\Sigma, \ell-1}|E_{1, \ell-1}, E_{2, \ell-1}, Y_1, \ldots, Y_{\ell-1}], \quad \ell \in \{1, \ldots, \eta\},
\]
for \( \ell \in \{1, \ldots, \eta\} \) can be expressed as
\[
I_\ell = (a_{1, \ell} + \kappa_{1, \ell-1}) E_{1, \ell-1} + (a_{2, \ell} + \kappa_{2, \ell-1}) E_{2, \ell-1} + W_{\perp, \ell-1} + Z_\ell,
\]
(171)
where \( \kappa_{1,0} \triangleq 0, \kappa_{2,0} \triangleq 0 \), and for \( \ell \in \{1, \ldots, \eta - 1\} \):
\[
k_{1, \ell} \triangleq \frac{\alpha_{1, \ell} \text{Cov}[E_{1, \ell-1}, W_{\Sigma, \ell}]}{(1 - \rho_\ell^2)\alpha_{1, \ell} \alpha_{2, \ell}} - \frac{\rho_\ell \sqrt{\alpha_{1, \ell} \alpha_{2, \ell}} \text{Cov}[E_{2, \ell}, W_{\Sigma, \ell}]}{(1 - \rho_\ell^2)\alpha_{1, \ell} \alpha_{2, \ell}},
\]
(172)
\[
k_{2, \ell} \triangleq \frac{\alpha_{1, \ell} \text{Cov}[E_{2, \ell-1}, W_{\Sigma, \ell}]}{(1 - \rho_\ell^2)\alpha_{1, \ell} \alpha_{2, \ell}} - \frac{\rho_\ell \sqrt{\alpha_{1, \ell} \alpha_{2, \ell}} \text{Cov}[E_{1, \ell}, W_{\Sigma, \ell}]}{(1 - \rho_\ell^2)\alpha_{1, \ell} \alpha_{2, \ell}}.
\]
(173)
Since the triple \((E_{1, \ell-1}, E_{2, \ell-1}, I_\ell)\) is independent of the tuple \((Y_1, \ldots, Y_{\ell-1}, I_1, \ldots, I_{\ell-1})\), the LMMSE-estimation errors \(E_{1, \ell}\) and \(E_{2, \ell}\) can be computed recursively as
\[
E_{1, \ell} = E_{1, \ell-1} - \frac{\text{Cov}[E_{1, \ell-1}, I_\ell]}{\text{Var}(I_\ell)} I_\ell, \quad \ell \in \{1, \ldots, \eta\},
\]
(174)
\[
E_{2, \ell} = E_{2, \ell-1} - \frac{\text{Cov}[E_{2, \ell-1}, I_\ell]}{\text{Var}(I_\ell)} I_\ell, \quad \ell \in \{1, \ldots, \eta\},
\]
(175)
which implies that
\[
\alpha_{1, \ell} = \alpha_{1, \ell-1} - \frac{\text{Cov}[E_{1, \ell-1}, I_\ell]^2}{\text{Var}(I_\ell)},
\]
(176)
\[
\alpha_{2, \ell} = \alpha_{2, \ell-1} - \frac{\text{Cov}[E_{2, \ell-1}, I_\ell]^2}{\text{Var}(I_\ell)},
\]
(177)
\[
\rho_\ell = \sqrt{\frac{\alpha_{1, \ell-1} \alpha_{2, \ell-1}}{\alpha_{1, \ell} \alpha_{2, \ell}}} \rho_{\ell-1} - \frac{\text{Cov}[E_{1, \ell-1}, I_\ell] \text{Cov}[E_{2, \ell-1}, I_\ell]}{\sqrt{\alpha_{1, \ell} \alpha_{2, \ell}} \text{Var}(I_\ell)}.
\]
(178)
By (171),
\[
\text{Cov}[E_{1, \ell-1}, I_\ell] = (a_{1, \ell} + \kappa_{1, \ell-1}) \alpha_{1, \ell-1} + (a_{2, \ell} + \kappa_{2, \ell-1}) \rho_{\ell-1} \sqrt{\alpha_{1, \ell-1} \alpha_{2, \ell-1}},
\]
\[
\text{Cov}[E_{2, \ell-1}, I_\ell] = (a_{1, \ell} + \kappa_{1, \ell-1}) \rho_{\ell-1} \sqrt{\alpha_{1, \ell-1} \alpha_{2, \ell-1}} + (a_{2, \ell} + \kappa_{2, \ell-1}) \alpha_{2, \ell-1},
\]
\[
\text{Var}(I_\ell) = (a_{1, \ell} + \kappa_{1, \ell-1})^2 \alpha_{1, \ell-1} + (a_{2, \ell} + \kappa_{2, \ell-1})^2 \alpha_{2, \ell-1} + 2(a_{1, \ell} + \kappa_{1, \ell-1})(a_{2, \ell} + \kappa_{2, \ell-1}) \rho_{\ell-1} \sqrt{\alpha_{1, \ell-1} \alpha_{2, \ell-1}} + \text{Var}(W_{\perp, \ell-1}) + N,
\]
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and thus by (176)–(178),

\[
\alpha_{1,\ell} - \alpha_{1,\ell-1} \left( \frac{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{1,\ell-1} + (a_{2,\ell} + \kappa_{2,\ell-1})^2 \alpha_{2,\ell-1}}{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N} \right) \\
+ \frac{2(a_{1,\ell} + \kappa_{1,\ell-1})(a_{2,\ell} + \kappa_{2,\ell-1}) \rho_{\ell-1} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}}}{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N} \\
+ \frac{(a_{2,\ell} + \kappa_{2,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N}{\text{Var}(W_{\perp,\ell-1}) + N} \right)^{-1},
\]

(179)

\[
\alpha_{2,\ell} = \alpha_{2,\ell-1} \left( \frac{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{1,\ell-1} + (a_{2,\ell} + \kappa_{2,\ell-1})^2 \alpha_{2,\ell-1}}{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N} \right) \\
+ \frac{2(a_{1,\ell} + \kappa_{1,\ell-1})(a_{2,\ell} + \kappa_{2,\ell-1}) \rho_{\ell-1} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}}}{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N} \\
+ \frac{(a_{2,\ell} + \kappa_{2,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N}{\text{Var}(W_{\perp,\ell-1}) + N} \right)^{-1},
\]

(180)

\[
\rho_{\ell} = \left( - (a_{1,\ell} + \kappa_{1,\ell-1})(a_{2,\ell} + \kappa_{2,\ell-1}) \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2)} + \rho_{\ell-1}(\text{Var}(W_{\perp,\ell-1}) + N) \right) \\
\cdot \frac{1}{\sqrt{(a_{1,\ell} + \kappa_{1,\ell-1})^2 \alpha_{1,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N}} \\
\cdot \frac{1}{\sqrt{(a_{2,\ell} + \kappa_{2,\ell-1})^2 \alpha_{2,\ell-1}(1 - \rho_{\ell-1}^2) + \text{Var}(W_{\perp,\ell-1}) + N}}
\]

(181)

After some algebraic manipulations the following recursions are obtained for \( \ell \in \{1, \ldots, \eta\} \):

\[
\frac{\alpha_{1,\ell-1} \alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{1,\ell} \alpha_{2,\ell} (1 - \rho_{\ell}^2)} = 1 + \frac{\alpha_{1,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})^2 + \alpha_{2,\ell-1}(a_{2,\ell} + \kappa_{2,\ell-1})^2}{\text{Var}(W_{\perp,\ell-1}) + N} \\
+ \frac{2 \rho_{\ell-1} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})(a_{2,\ell} + \kappa_{2,\ell-1})}}{\text{Var}(W_{\perp,\ell-1}) + N},
\]

and

\[
\frac{\alpha_{1,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{1,\ell} (1 - \rho_{\ell}^2)} = 1 + \frac{\alpha_{1,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})^2 (1 - \rho_{\ell-1}^2)}{\text{Var}(W_{\perp,\ell-1}) + N},
\]

and

\[
\frac{\alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2)}{\alpha_{2,\ell} (1 - \rho_{\ell}^2)} = 1 + \frac{\alpha_{2,\ell-1}(a_{2,\ell} + \kappa_{2,\ell-1})^2 (1 - \rho_{\ell-1}^2)}{\text{Var}(W_{\perp,\ell-1}) + N}.
\]

Plugging the last three identities into (165)–(167) finally results in the desired recursive formulation of Constraints (156)–(158):

\[
R_1 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{\alpha_{1,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})^2 (1 - \rho_{\ell-1}^2)}{\text{Var}(W_{\perp,\ell-1}) + N} \right); \quad (182)
\]

\[
R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{\alpha_{2,\ell-1}(a_{2,\ell} + \kappa_{2,\ell-1})^2 (1 - \rho_{\ell-1}^2)}{\text{Var}(W_{\perp,\ell-1}) + N} \right); \quad (183)
\]
\[ R_1 + R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{\alpha_{1,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})^2 + \alpha_{2,\ell-1}(a_{2,\ell} + \kappa_{2,\ell-1})^2}{\text{Var}(W_{\perp,\ell-1}) + N} \right. \\
\left. + 2\rho_{\ell-1} \sqrt{\alpha_{1,\ell-1}\alpha_{2,\ell-1}(a_{1,\ell-1} + \kappa_{1,\ell-1})(a_{2,\ell} + \kappa_{2,\ell-1})} \right) \text{ Var}(W_{\perp,\ell-1}) + N \right), \tag{184} \]

where \( \{\alpha_{1,\ell}\}_{\ell=0}^{\eta}, \{\alpha_{2,\ell}\}_{\ell=0}^{\eta}, \{\rho_{\ell}\}_{\ell=0}^{\eta}, \{\kappa_{1,\ell}\}_{\ell=0}^{\eta}, \{\kappa_{2,\ell}\}_{\ell=0}^{\eta}, \) and \( \{W_{\perp,\ell}\}_{\ell=0}^{\eta} \) are defined by (168)–(181).

### C.2 Noisy or Perfect Partial Feedback

The desired recursive formulation of \( \mathcal{R}_P \) \( (N, \sigma_d^2; \eta, a_1, a_2, B_1, B_2, C_P) \) can be derived along the lines shown in the previous subsection C.1. We omit the details and only present the result.

Fix a choice of parameters \( \eta, a_1, a_2, B_1, B_2, C_P \) satisfying the power constraints (36) and (37) and so that \( C_P \) is the LMMSE-estimation matrix. Denote the \( \ell \)-th entry of the vector \( a_v \) by \( a_{v,\ell} \) and denote the row-\( \ell \) column-\( j \) entry of the matrix \( B_2 \) by \( b_{2,\ell,j} \) for \( j, \ell \in \{1, \ldots, \eta\} \) and \( \nu \in \{1, 2\} \). Our concatenated scheme for noisy partial feedback and parameters \( \eta, a_1, a_2, B_2, C_P \) achieves all rate pairs \( (R_1, R_2) \) satisfying

\[
R_1 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{\alpha_{1,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})^2 \left( 1 - \rho_{\ell-1}^2 \right)}{\text{Var}(W_{\perp,\ell-1}) + N} \right),
\]

\[
R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{\alpha_{2,\ell-1}(a_{2,\ell} + \kappa_{2,\ell-1})^2 \left( 1 - \rho_{\ell-1}^2 \right)}{\text{Var}(W_{\perp,\ell-1}) + N} \right),
\]

\[
R_1 + R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{\alpha_{1,\ell-1}(a_{1,\ell} + \kappa_{1,\ell-1})^2 + \alpha_{2,\ell-1}(a_{2,\ell} + \kappa_{2,\ell-1})^2 \left( 1 - \rho_{\ell-1}^2 \right)}{\text{Var}(W_{\perp,\ell-1}) + N} \right)
\]
\[+ 2\rho_{\ell-1} \sqrt{\alpha_{1,\ell-1}\alpha_{2,\ell-1}(a_{1,\ell-1} + \kappa_{1,\ell-1})(a_{2,\ell} + \kappa_{2,\ell-1})} \right) \text{ Var}(W_{\perp,\ell-1}) + N \right),
\]

where \( \{\alpha_{1,\ell}\}_{\ell=0}^{\eta}, \{\alpha_{2,\ell}\}_{\ell=0}^{\eta}, \{\rho_{\ell}\}_{\ell=0}^{\eta}, \{\kappa_{1,\ell}\}_{\ell=0}^{\eta}, \{\kappa_{2,\ell}\}_{\ell=0}^{\eta}, \{E_1,\ell\}_{\ell=1}^{\eta}, \{E_2,\ell\}_{\ell=1}^{\eta}, \) and \( \{W_{\perp,\ell}\}_{\ell=0}^{\eta} \) are defined as in the previous subsection C.1 except that (168) should be replaced by

\[
Y_{\ell} = a_1,\ell \Xi_1 + a_2,\ell \Xi_2 + \sum_{j=1}^{\ell-1} b_{2,\ell,j} V_{2,j} + Z_{\ell}, \quad \ell \in \{1, \ldots, \eta\},
\]

and (170) should be replaced by

\[
W_{\perp,\ell-1} \triangleq \sum_{j=1}^{\ell-1} b_{2,\ell,j} W_{2,j}, \quad \ell \in \{1, \ldots, \eta\}.
\]

### C.3 Noisy Feedback with Receiver Side-Information

We derive a recursive formulation of the rate region achieved by our concatenated scheme \( \mathcal{R}_{SI} \) \( (N, K_{W,\ell} W_0; \eta, a_1, a_2, B_1, B_2, C_{SI}) \) for all choices of parameters \( \eta, a_1, a_2, B_1, B_2, C_{SI} \) where \( C_{SI} \) is the LMMSE-estimation matrix with side-information.
Fix a choice of parameters \(\eta, a_1, a_2, B_1, B_2, C\) satisfying the power constraints (21) and (22) and so that \(C\) is the LMMSE-estimation matrix with side-information. Denote the \(\ell\)-th entry of the vector \(a_\nu\) by \(a_{\nu,\ell}\) and denote the row-\(\ell\) column-\(j\) entry of the matrix \(b_\nu\) by \(b_{\nu,\ell,j}\), for \(j, \ell \in \{1, \ldots, \eta\}\) and \(\nu \in \{1, 2\}\). The desired recursive formulation of \(R_{SI}(N, K_{W1,W2}; \eta, a_1, a_2, B_1, B_2, C_{SI})\) can be derived along the lines described in Subsection C.1 but with the following two modifications. Instead of being defined as in (163) and (164), the LMMSE-estimation errors \(E_{1,\ell}\) and \(E_{2,\ell}\) for \(\ell \in \{1, \ldots, \eta\}\) are defined as

\[
E_{1,\ell} \triangleq \Xi_1 - E[\Xi_1 | Y_1, \ldots, Y_\ell, W_{1,1}, \ldots, W_{1,\ell}, W_{2,1}, \ldots, W_{2,\ell}], \quad (185)
\]

\[
E_{2,\ell} \triangleq \Xi_2 - E[\Xi_2 | Y_1, \ldots, Y_\ell, W_{1,1}, \ldots, W_{1,\ell}, W_{2,1}, \ldots, W_{2,\ell}], \quad (186)
\]

and instead of being defined as in (169), the innovation \(I_\ell\), for \(\ell \in \{1, \ldots, \eta\}\), is defined as

\[
I_\ell \triangleq Y_\ell - E[Y_\ell | Y_1, \ldots, Y_{\ell-1}, W_{1,1}, \ldots, W_{1,\ell}, W_{2,1}, W_{2,\ell}]. \quad (187)
\]

Notice that by (185), (186), and (187):

\[
I_\ell = a_{1,\ell}E_{1,\ell-1} + a_{2,\ell}E_{2,\ell-1} + Z_\ell, \quad \ell \in \{1, \ldots, \eta\}.
\]

We omit the details of the derivation and only state the resulting recursive formulation of the region \(R_{SI}(N, K_{W1,W2}; \eta, a_1, a_2, B_1, B_2, C_{SI})\). Our concatenated scheme for noisy feedback with receiver side-information and parameters \(\eta, a_1, a_2, B_1, B_2, C_{SI}\) achieves all rate pairs \((R_1, R_2)\) satisfying

\[
R_1 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{a_{1,\ell}^2 \alpha_{1,\ell-1} \left( 1 - \rho_{\ell-1}^2 \right)}{N} \right), \quad (188)
\]

\[
R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{a_{2,\ell}^2 \alpha_{2,\ell-1} \left( 1 - \rho_{\ell-1}^2 \right)}{N} \right), \quad (189)
\]

\[
R_1 + R_2 \leq \frac{1}{\eta} \sum_{\ell=1}^{\eta} \frac{1}{2} \log \left( 1 + \frac{a_{1,\ell}^2 \alpha_{1,\ell-1} + a_{2,\ell}^2 \alpha_{2,\ell-1} + 2a_{1,\ell}a_{2,\ell} \rho_{\ell-1} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}}}{N} \right), \quad (190)
\]

where \(\alpha_{1,0} = 1, \alpha_{2,0} = 1, \rho_0 = 0\), and for \(\ell \in \{1, \ldots, \eta - 1\}\):

\[
\alpha_{1,\ell} = \alpha_{1,\ell-1} \left( a_{1,\ell}^2 \alpha_{1,\ell-1} + a_{2,\ell}^2 \alpha_{2,\ell-1} + 2a_{1,\ell} a_{2,\ell} \rho_{\ell-1} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}} + N \right)^{-1},
\]

\[
\alpha_{2,\ell} = \alpha_{2,\ell-1} \left( a_{1,\ell}^2 \alpha_{1,\ell-1} + a_{2,\ell}^2 \alpha_{2,\ell-1} + 2a_{1,\ell} a_{2,\ell} \rho_{\ell-1} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}} + N \right)^{-1},
\]

\[
\rho_{\ell} = \left( -a_{1,\ell} a_{2,\ell} \sqrt{\alpha_{1,\ell-1} \alpha_{2,\ell-1}} (1 - \rho_{\ell-1}^2) + \rho_{\ell-1} N \right)
\]

\[
\frac{1}{\sqrt{a_{1,\ell}^2 \alpha_{1,\ell-1} (1 - \rho_{\ell-1}^2) + N / a_{2,\ell}^2 \alpha_{2,\ell-1} (1 - \rho_{\ell-1}^2) + N}}.
\]

**D Optimal of LMMSE-Estimation Error Parameters for Perfect Feedback**

We show that for perfect feedback every choice of parameters \(\eta, a_1, a_2, B_1, B_2, C\) violating (74) and (75) is strictly dominated by some other choice of parameters satisfying (74) and (75).
Proposition 47. Let $P_1, P_2, N > 0$ be given and assume that $K_{W_1W_2} = 0$, i.e., assume perfect feedback. For every choice of parameters $\eta, a_1, a_2, B_1, B_2, C$ satisfying the power constraints (21)–(22) but not Conditions (74)–(75), there exists a choice of parameters $\eta, a'_1, a'_2, B'_1, B'_2, C'$ satisfying both (21)–(22) and (74)–(75), and so that

$$R(N, 0; \eta, a_1, a_2, B_1, B_2, C) \subset R(N, 0; \eta, a'_1, a'_2, B'_1, B'_2, C')$$

with the inclusion being strict.

The proof is postponed after the following lemma.

Lemma 48. Let $P_1, P_2, N > 0$ be given and assume that $K_{W_1W_2} = 0$, i.e., assume perfect feedback. For every choice of parameters $\eta, a'_1, a'_2, B'_1, B'_2, C$ satisfying (21)–(22) but violating (74)–(75) there exist parameters $\eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}$ satisfying the following three conditions:

1. the parameters $\eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}$ satisfy both (74) and (75);

2. $R(N, 0; \eta, a'_1, a'_2, B'_1, B'_2, C') = R\left(N, 0; \eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}\right)$;

3. the parameters $\eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}$ satisfy both (21) and (22), and at least one of them with strict inequality.

Proof. Fix parameters $\eta, a'_1, a'_2, B'_1, B'_2, C'$ satisfying (21)–(22) but violating (74)–(75). Choose

- $\hat{a}_1 = a'_1$ and $\hat{a}_2 = a'_2$,
- $\hat{B}_1$ and $\hat{B}_2$ so that $\hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2$ satisfy (74) and (75),
- $\hat{C} = C'$.

By construction our choice $\hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}$ trivially satisfies Condition 1 in the lemma. Moreover, since for $K_{W_1W_2} = 0$ the region $R(N, 0; \eta, a_1, a_2, B_1, B_2, C)$ depends only on $a_1, a_2$, and $C$ but not on $B_1$ and $B_2$, see Definition 12, the regions $R(N, 0; \eta, a'_1, a'_2, B'_1, B'_2, C')$ and $R\left(N, 0; \eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}\right)$ coincide. Thus also Condition 2 is satisfied.

We are left with proving that our choice of parameters $\hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}$ satisfies Condition 3. Before doing so, we introduce some helpful assumptions and notation. Assume in the following that Inner Encoder 1 and Inner Encoder 2 are fed by the independent standard Gaussians $\Xi_1$ and $\Xi_2$, respectively. Then, let $Y'_1, \ldots, Y'_\eta$ denote the $\eta$ channel outputs of the original MAC $x_1, x_2 \mapsto Y$ when the inner encoders use parameters $\eta, a'_1, a'_2, B'_1, B'_2, C'$, and similarly, let $\hat{Y}_1, \ldots, \hat{Y}_\eta$ denote the $\eta$ channel outputs of the original MAC $x_1, x_2 \mapsto \hat{Y}$ when the inner encoders use parameters $\eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C}$. Also, let $a'_{1,\ell}, a'_{2,\ell}, \hat{a}_{1,\ell}, \hat{a}_{2,\ell}$ denote the $\ell$-th entry of the vectors $a'_1, a'_2, \hat{a}_1, \hat{a}_2$, and let $b'_{1,\ell,j}, b'_{2,\ell,j}, \hat{b}_{1,\ell,j}, \hat{b}_{2,\ell,j}$ denote the row-$\ell$ column-$j$ entry of the matrices $B'_1, B'_2, \hat{B}_1$, and $\hat{B}_2$, respectively, for $j, \ell \in \{1, \ldots, \eta\}$ and $\nu \in \{1, 2\}$. 

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In order to prove that our choice \( \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C} \) satisfies Condition 3, we shall show shortly that for all \( \ell \in \{1, \ldots, \eta\} \):

\[
\text{Var} \left( a_{1,\ell}^2 \Xi_1 - \sum_{j=1}^{\ell-1} b_{1,\ell,j}^2 Y_j' \right) \geq \text{Var} \left( \hat{a}_{1,\ell}^2 \Xi_1 - \sum_{j=1}^{\ell-1} \hat{b}_{1,\ell,j} \hat{Y}_j \right),
\]

(191)

\[
\text{Var} \left( a_{2,\ell}^2 \Xi_2 - \sum_{j=1}^{\ell-1} b_{2,\ell,j}^2 Y_j' \right) \geq \text{Var} \left( \hat{a}_{2,\ell}^2 \Xi_2 - \sum_{j=1}^{\ell-1} \hat{b}_{2,\ell,j} \hat{Y}_j \right),
\]

(192)

where equality holds in (191) if, and only if,

\[
\left( a_{1,\ell}^2 \Xi_1 - \sum_{j=1}^{\ell-1} b_{1,\ell,j}^2 Y_j' \right) = \left( \hat{a}_{1,\ell}^2 \Xi_1 - \sum_{j=1}^{\ell-1} \hat{b}_{1,\ell,j} \hat{Y}_j \right)
\]

(193)

and equality holds in (192) if, and only if,

\[
\left( a_{2,\ell}^2 \Xi_2 - \sum_{j=1}^{\ell-1} b_{2,\ell,j}^2 Y_j' \right) = \left( \hat{a}_{2,\ell}^2 \Xi_2 - \sum_{j=1}^{\ell-1} \hat{b}_{2,\ell,j} \hat{Y}_j \right).
\]

(194)

Since the pairs \((\mathcal{B}_1', \mathcal{B}_2')\) and \((\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2)\) differ, and thus not for all \( \ell \in \{1, \ldots, \eta\} \) both equalities (193) and (194) can hold, this will then conclude the proof of Condition 3. Indeed, since for the original parameters \( \eta, \mathcal{A}_1', \mathcal{A}_2', \mathcal{B}_1', \mathcal{B}_2' \) the produced channel inputs are average block-power constrained to \( P_1 \) and \( P_2 \), by (191) and (192) also for parameters \( \eta, \hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2 \) the produced channel inputs are average block-power constrained to \( P_1 \) and \( P_2 \). Moreover, since for some \( \ell \in \{1, \ldots, \eta\} \) either (191) or (192) holds with strict inequality, for parameters \( \eta, \hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2 \) (at least) one of the two channel input sequences satisfies the power constraint with strict inequality. Consequently, the parameters \( \eta, \hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2 \) satisfy both (21) and (22) and they satisfy (at least) one of them with strict inequality.

In the remaining we prove that Inequality (191) is satisfied, and that it is satisfied with equality if, and only if, (193) holds. The proof of Inequality (192) and that it holds with equality if, and only if, (194) holds, follows along similar lines and is omitted. Fix \( \ell \in \{1, \ldots, \eta\} \). Notice that the set of all possible linear combinations \( \left( \hat{a}_{1,\ell} \Xi_1 + \sum_{j=1}^{\ell-1} \hat{b}_{1,\ell,j} \hat{Y}_j \right) \) over all choices of \( \{b_{1,\ell,j}\}_{j=1}^{\ell-1} \) depends only on \( \hat{a}_{1,\ell}, \ldots, \hat{a}_{1,\ell} \) and \( \hat{a}_{2,\ell}, \ldots, \hat{a}_{2,\ell} \), but not on \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \). In particular, since \( \hat{a}_1 = \mathcal{A}_1' \) and \( \hat{a}_2 = \mathcal{A}_2' \), there exists a choice \( \{\hat{b}_{1,\ell,j}\}_{j=1}^{\ell-1} \) so that

\[
\left( \hat{a}_{1,\ell} \Xi_1 + \sum_{j=1}^{\ell-1} \hat{b}_{1,\ell,j} \hat{Y}_j \right) = \left( \mathcal{A}_1' \Xi_1 + \sum_{j=1}^{\ell-1} \mathcal{A}_1' \Xi_1 + \sum_{j=1}^{\ell-1} \hat{b}_{1,\ell,j} \hat{Y}_j \right).
\]

By the variance-minimizing property of the LMMSE-estimation error, it then follows that the variance on the right-hand side of (191)—which corresponds to the variance of a scaled version of the LMMSE-estimation error—is smaller or equal to the variance on its left-hand side, and that equality holds if, and only if, (193) is satisfied. This concludes the proof.

Proof of Proposition 47. Fix parameters \( \eta, \mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C} \) which satisfy (21) and (22) but violate (74) and (75). By Lemma 48 there exists a choice of parameters \( \eta, \hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2, \hat{\mathcal{C}} \) satisfying (21), (22), (74), and (75), so that either (21) or (22) is satisfied with strict inequality.
and so that
\[
\mathcal{R} \left( N, K_{w_2}; \eta, a_1, a_2, B_1, B_2, C \right) = \mathcal{R} \left( N, K_{w_2}; \eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C} \right). \tag{195}
\]

By the second optimality property described in Section 7.1, there exist parameters \( \eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C} \) that satisfy both (21) and (22) with equality and that strictly dominate the parameters \( \eta, \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2, \bar{C} \), and thus by (195) also the original parameters \( \eta, a_1, a_2, B_1, B_2, C \), i.e.,
\[
\mathcal{R} \left( N, K_{w_2}; \eta, a_1, a_2, B_1, B_2, C \right) \subset \mathcal{R} \left( N, K_{w_2}; \eta, \hat{a}_1, \hat{a}_2, \hat{B}_1, \hat{B}_2, \hat{C} \right) \tag{196}
\]
with the inclusion being strict.

Applying Lemma 48 to parameters \( \eta, \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2, \bar{C} \), we conclude that there exists a choice of parameters \( \eta, a_1^*, a_2^*, B_1^*, B_2^*, C^* \) satisfying both (21)–(22) and (74)–(75) and so that
\[
\mathcal{R} \left( N, K_{w_2}; \eta, \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2, \bar{C} \right) = \mathcal{R} \left( N, K_{w_2}; \eta, a_1^*, a_2^*, B_1^*, B_2^*, C^* \right).
\]

By (196) this implies:
\[
\mathcal{R} \left( N, K_{w_2}; \eta, a_1, a_2, B_1, B_2, C \right) \subset \mathcal{R} \left( N, K_{w_2}; \eta, a_1^*, a_2^*, B_1^*, B_2^*, C^* \right)
\]
with the inclusion being strict. \( \square \)

**Remark 49 (Perfect Partial Feedback).** An analogous result to Proposition 47 where the new choice \( \eta, \hat{a}_1, \hat{a}_2, \hat{B}_2, \hat{C} \) only satisfies (75) but not necessarily (74) holds for perfect partial feedback. The proof follows along the same lines as for perfect feedback.

**Remark 50 (Noisy Feedback with Receiver Side-Information).** An analogous result to Proposition 47 when (74)–(75) are replaced by (78)–(79) holds for the following special cases of noisy feedback with receiver side-information:

a) \( \varphi = 1 \) and \( \eta \in \mathbb{N} \) arbitrary,

b) \( \varphi \in [-1, 1] \) arbitrary and \( \eta = 2 \).

The proof follows along the same lines as for perfect feedback.

### E Choice of Parameters I

In Section E.1, we present a specific choice of the parameters \( a_1, a_2, B_1, B_2, C \) for given \( \eta \in \mathbb{N} \). We treat the noisy-feedback setting and the noisy or perfect partial-feedback setting. We denote our choice for noisy feedback by \( \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2, \bar{C} \) and our choice for partial feedback by \( \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C} \).

As we shall see, our choice is so that \( \bar{C} \) is the LMMSE-estimation matrix. Thus, the achievable region of our concatenated scheme with parameters \( \eta, \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2, \bar{C} \) is obtained by plugging the parameters into the right-hand sides of (182)–(184), i.e., into the recursive formulation of the constraints which determine \( \mathcal{R} \left( N, K_{w_2}; \eta, \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2, \bar{C} \right) \). The resulting achievable region is presented in Corollary 52 ahead.
Definition 51. For a positive integer $\eta$, define $\mathcal{R}_\eta(P_1, P_2, N, K_{W_1, W_2})$ as the set of all rate-pairs $(R_1, R_2)$ satisfying

\begin{align}
R_1 &\leq \frac{1}{2\eta} \sum_{\ell=1}^{\eta} \log \left( 1 + \frac{P_1^{\alpha_1, \ell-1}(1 + \bar{\kappa}_{1, \ell-1})^2 (1 - \rho_{\ell-1}^2)}{\lambda_{\ell-1} + N} \right), \\
R_2 &\leq \frac{1}{2\eta} \sum_{\ell=1}^{\eta} \log \left( 1 + \frac{P_2^{\alpha_2, \ell-1}(1 + \bar{\kappa}_{2, \ell-1})^2 (1 - \rho_{\ell-1}^2)}{\lambda_{\ell-1} + N} \right),
\end{align}

and

\begin{align}
R_1 + R_2 &\leq \frac{1}{2\eta} \sum_{\ell=1}^{\eta} \log \left( 1 + \frac{P_1^{\alpha_1, \ell-1}(1 + \bar{\kappa}_{1, \ell-1})^2 + P_2^{\alpha_2, \ell-1}(1 + \bar{\kappa}_{2, \ell-1})^2}{\lambda_{\ell-1} + N} \right) \\
&\quad + \frac{2\sqrt{P_1 P_2^{\alpha_2, \ell-1}}}{P_1^{\alpha_1, \ell-1} P_2^{\alpha_2, \ell-1}} \left( 1 - \rho_{\ell-1}^2 (1 + \bar{\kappa}_{1, \ell-1}) (1 + \bar{\kappa}_{2, \ell-1}) \right) \sqrt{\lambda_{\ell-1} + N},
\end{align}

where $\alpha_{1,0} = 1$, $\alpha_{2,0} = 1$, $\rho_0 = 0$, and for $\ell \in \{1, \ldots, \eta - 1\}$:

\begin{align}
\alpha_{1,\ell} &= \alpha_{1,\ell-1} \left( P_1^{\alpha_1, \ell-1}(1 + \bar{\kappa}_{1, \ell-1})^2 + P_2^{\alpha_2, \ell-1}(1 + \bar{\kappa}_{2, \ell-1})^2 \right) \left( 1 + \bar{\kappa}_{2, \ell-1} \right)^2 \left( 1 - \rho_{\ell-1}^2 \right) + \lambda_{\ell-1} + N \\
\alpha_{2,\ell} &= \alpha_{2,\ell-1} \left( P_1^{\alpha_1, \ell-1}(1 + \bar{\kappa}_{1, \ell-1})^2 + P_2^{\alpha_2, \ell-1}(1 + \bar{\kappa}_{2, \ell-1})^2 \right) \left( 1 + \bar{\kappa}_{1, \ell-1} \right)^2 \left( 1 - \rho_{\ell-1}^2 \right) + \lambda_{\ell-1} + N
\end{align}

and

\begin{align}
\rho_{\ell} &= \left( \rho_{\ell-1} (\bar{\lambda}_{\ell-1} + N) - \left( \sqrt{P_1^{\alpha_1, \ell-1}(1 + \bar{\kappa}_{1, \ell-1})} \sqrt{P_2^{\alpha_2, \ell-1}(1 + \bar{\kappa}_{2, \ell-1})} \right) (1 - \rho_{\ell-1}^2) \right) \\
&\quad \cdot \frac{1}{\sqrt{P_1^{\alpha_1, \ell-1}(1 + \bar{\kappa}_{1, \ell-1})^2 (1 - \rho_{\ell-1}^2) + \bar{\lambda}_{\ell-1} + N}} \\
&\quad \cdot \frac{1}{\sqrt{P_2^{\alpha_2, \ell-1}(1 + \bar{\kappa}_{2, \ell-1})^2 (1 - \rho_{\ell-1}^2) + \bar{\lambda}_{\ell-1} + N}},
\end{align}

where $\bar{\beta}_{1,0} \triangleq 1$, $\bar{\beta}_{2,0} \triangleq 1$, $\{\bar{\beta}_{1,\ell}\}_{\ell=1}^{\eta-1}$ are defined in (212), $\{\bar{\beta}_{2,\ell}\}_{\ell=1}^{\eta-1}$ are defined in (213),
Encoder 1 and Inner Encoder 2 map the fed symbols to the sequences of channel inputs as the LMMSE-estimation matrix. For the purpose of describing our choice we replace the feedback setting is treated only shortly in Remark 53 at the end of this section.

Let \( \bar{\kappa}_0 \triangleq 0, \bar{\kappa}_2.0 \triangleq 0, \bar{\lambda}_0 \triangleq 0, \) and for \( \ell \in \{1, \ldots, \eta - 1\} \):

\[
\bar{\kappa}_{1,\ell} \triangleq \sqrt{\frac{\beta_{1,\ell}}{P_1\alpha_{1,\ell}}} \left( \alpha_{2,\ell}\text{Cov}(E_{1,\ell}, W_{\Sigma,\ell}) - \rho_{\ell}\sqrt{\alpha_{1,\ell}\alpha_{2,\ell}\text{Cov}(E_{2,\ell}, W_{\Sigma,\ell})} \right),
\]

(203)

\[
\bar{\kappa}_{2,\ell} \triangleq \sqrt{\frac{\beta_{2,\ell}}{P_2\alpha_{2,\ell}}} \left( \alpha_{1,\ell}\text{Cov}(E_{2,\ell}, W_{\Sigma,\ell}) - \rho_{\ell}\sqrt{\alpha_{1,\ell}\alpha_{2,\ell}\text{Cov}(E_{1,\ell}, W_{\Sigma,\ell})} \right),
\]

(204)

\[
\bar{\lambda}_{\ell} \triangleq \text{Var}(W_{\Sigma,\ell} - E[W_{\Sigma,\ell}E_{1,\ell}E_{2,\ell}Y_1, \ldots, Y_{\eta}]),
\]

(205)

and where recall that \( \Xi_1 \) and \( \Xi_2 \) are independent standard Gaussians and

\[
E_{1,\ell} = \Xi_1 - E[\Xi_1 | Y_1, \ldots, Y_{\ell}],
\]

\[
E_{2,\ell} = \Xi_2 - E[\Xi_2 | Y_2, \ldots, Y_{\ell}].
\]

Moreover,

\[
W_{\Sigma,\ell} = \sqrt{\frac{P_1}{\beta_{1,\ell}}} \bar{b}_{1,\ell} V_{1,\ell} + (-1)^\ell \sqrt{\frac{P_2}{\beta_{2,\ell}}} \bar{b}_{2,\ell} V_{2,\ell},
\]

(206)

where \( \{\bar{b}_{1,\ell}\}_{\ell=1}^{\eta-1} \) and \( \{\bar{b}_{2,\ell}\}_{\ell=1}^{\eta-1} \) are defined in (215), \( \{\bar{M}_{\ell}\}_{\ell=1}^{\eta-1} \) is defined in (214). The channel inputs \( \{X_{1,\ell}\}_{\ell=1}^\eta \) and \( \{X_{2,\ell}\}_{\ell=1}^\eta \) are defined by (207)-(220) in Subsection E.1 ahead.

**Corollary 52 (Noisy Feedback).** Let \( P_1, P_2, N > 0 \) and \( K_{W_1W_2} \geq 0 \) be given. The capacity region of the two-user AWGN MAC with noisy feedback includes all rate regions \( \mathcal{R}_\eta(P_1, P_2, N, K_{W_1W_2}) \) for positive integers \( \eta \), i.e.,

\[
\mathcal{C}_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq \text{cl} \left( \bigcup_{\eta \in \mathbb{N}} \mathcal{R}_\eta(P_1, P_2, N, K_{W_1W_2}) \right).
\]

### E.1 Description of Parameters

Let \( P_1, P_2, N > 0, K_{W_1W_2} \geq 0, \) and \( \eta \in \mathbb{N} \) be given. We present our specific choice of the parameters \( \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2 \), and \( \bar{C} \). We first consider the noisy-feedback setting; the partial-feedback setting is treated only shortly in Remark 53 at the end of this section.

Instead of directly describing our choice \( \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2 \), and \( \bar{C} \), we will describe how Inner Encoder 1 and Inner Encoder 2 map the fed symbols to the sequences of channel inputs \( X_{1,1}, \ldots, X_{1,\eta} \) and \( X_{2,1}, \ldots, X_{2,\eta} \). This, then determines \( \bar{a}_1, \bar{a}_2, \bar{B}_1, \bar{B}_2 \). The matrix \( \bar{C} \) is chosen as the LMMSE-estimation matrix. For the purpose of describing our choice we replace the pair of input symbols \( \xi_1 \) and \( \xi_2 \) by the independent standard Gaussians \( \Xi_1 \) and \( \Xi_2 \). We choose the inner encoders to produce

\[
X_{1,1} = \sqrt{P_1} \Xi_1,
\]

(207)

\[
X_{1,\ell} = \sqrt{\frac{P_1}{\beta_{1,\ell-1}}} (\Xi_1 - \bar{b}_{1,\ell-1} V_{1,\ell-1}), \quad \ell \in \{2, \ldots, \eta\},
\]

(208)

and

\[
X_{2,1} = \sqrt{P_2} \Xi_2,
\]

(209)

\[
X_{2,\ell} = (-1)^{\ell-1} \sqrt{\frac{P_2}{\beta_{2,\ell-1}}} (\Xi_2 - \bar{b}_{2,\ell-1} M_{\ell-1} V_{2,\ell-1}), \quad \ell \in \{2, \ldots, \eta\},
\]

(210)
where
\[ V_{\nu}^\ell \triangleq (V_{\nu,1}, \ldots, V_{\nu,\ell})^T, \quad \ell \in \{1, \ldots, \eta - 1\}, \nu \in \{1, 2\}, \] (211)
\[ \overset{\text{b}}{\beta}_1^\ell \triangleq \text{Var}(\Xi_1 - \bar{B}_1^\ell V_1^\ell), \quad \ell \in \{1, \ldots, \eta - 1\}, \] (212)
\[ \overset{\text{b}}{\beta}_2^\ell \triangleq \text{Var}(\Xi_2 - \bar{B}_2^\ell M_2 V_2^\ell), \quad \ell \in \{1, \ldots, \eta - 1\}, \] (213)
\[ M_\ell \triangleq \text{diag}(1, -1, 1, \ldots, (-1)^{\ell-1}), \quad \ell \in \{1, \ldots, \eta - 1\}, \] and where \( \bar{B}_1^\ell, \bar{B}_2^\ell \in \mathbb{R}^\ell \), for \( \ell \in \{1, \ldots, \eta - 1\} \) are given by
\[ \bar{B}_{\nu,\ell} \triangleq \left( \frac{P_{\nu}}{N} + K_{V_\nu}^\ell \right)^{-1} K_{V_\nu,\ell}, \quad \ell \in \{1, \ldots, \eta - 1\}, \nu \in \{1, 2\}. \] (215)

The matrix \( \Sigma_\ell \) for \( \ell \in \{1, \ldots, \eta - 1\} \) is defined as
\[ \Sigma_\ell \triangleq \text{Var}(W_\ell - E[W_\ell|E_{\Sigma,\ell}, Y_1, \ldots, Y_\ell]), \] (216)
where for \( \ell \in \{1, \ldots, \eta - 1\}:
\[ Y^\ell \triangleq (Y_1, \ldots, Y_\ell)^T, \] (217)
\[ W_{\nu}^\ell \triangleq (W_{\nu,1}, \ldots, W_{\nu,\ell})^T, \quad \nu \in \{1, 2\}, \] (218)
\[ E_{\Sigma,\ell} \triangleq \sqrt{\frac{P_1}{\beta_1,\ell}} E_{1,\ell} + (-1)^\ell \sqrt{\frac{P_2}{\beta_2,\ell}} E_{2,\ell}, \] (219)
\[ W_\ell \triangleq W_1^\ell + (-1)^\ell M_\ell W_2^\ell. \] (220)

Notice that Inner Encoder 2 modulates its inputs with an alternating sequence of +1 or -1 (which is inspired by the Fourier-MEC scheme in [8]), and it multiplies the noisy feedback vectors by the matrix \( M_{\ell-1} \) before further processing it (which accounts for the modulation of past inputs). The presented choice of the inner encoders ensures that the input sequences to the original MAC \( x_1, x_2 \mapsto Y \) satisfy the average block-power constraints (3). In particular, with the presented choice all input symbols \( X_{1,1}, \ldots, X_{1,\eta} \) have the same expected power \( P_1 \), and all input symbols \( X_{2,1}, \ldots, X_{2,\eta} \) have the same expected power \( P_2 \).

The described encodings correspond to the following parameters of the concatenated scheme:
\[ \bar{a}_1 \triangleq \left( \sqrt{\frac{P_1}{\beta_{1,1}}} \right)^T \quad \sqrt{\frac{P_1}{\beta_{1,2}}} \cdots \sqrt{\frac{P_1}{\beta_{1,\eta-1}}} \right)^T, \] (221)
\[ \bar{a}_2 \triangleq \left( \sqrt{\frac{P_2}{\beta_{2,1}}} \right)^T \quad -\sqrt{\frac{P_2}{\beta_{2,2}}} \cdots (-1)^{\eta-1} \sqrt{\frac{P_2}{\beta_{2,\eta-1}}} \right)^T, \] (222)
and
\[ \bar{B}_1 \triangleq \left( 0 \quad -\sqrt{\frac{P_2}{\beta_{2,1}}} \bar{b}^{(0)}_{1,1} \cdots -\sqrt{\frac{P_2}{\beta_{2,\eta-1}}} \bar{b}^{(0)}_{1,\eta-1} \right)^T, \] (223)
\[ \bar{B}_2 \triangleq \left( 0 \quad \sqrt{\frac{P_2}{\beta_{2,1}}} \bar{b}^{(0)}_{2,1} \cdots (-1)^\eta \sqrt{\frac{P_2}{\beta_{2,\eta-1}}} \bar{b}^{(0)}_{2,\eta-1} \right)^T, \] (224)
where the vectors \( \{\bar{B}_{1,\ell}\}_{\ell=1}^\eta \) and \( \{\bar{B}_{2,\ell}\}_{\ell=1}^{\eta-1} \) are defined as the \( \eta \)-dimensional vector obtained by stacking the vector \( \bar{b}_{\nu,\ell} \) on top of an \((\eta - \ell)\)-dimensional zero-vector, i.e.,
\[ \bar{b}_{\nu,\ell} \triangleq \begin{pmatrix} \bar{b}_{\nu,\ell} \\ 0 \end{pmatrix}, \quad \ell \in \{1, \ldots, \eta - 1\}, \quad \nu \in \{1, 2\}. \] (225)
The parameter $\bar{C}$ is chosen as the LMMSE-estimation matrix, i.e.,
\[
\bar{C} = \bar{A}_b^T (\bar{A}_b \bar{A}_b^T + \nu I_N + \bar{B}_b (K_{W_1W_2} \otimes I_N) \bar{B}_b^T)^{-1},
\]
where $\bar{A}_b \triangleq (\bar{a}_1 \ \bar{a}_2)$ and $\bar{B}_b \triangleq (\bar{B}_1 \ \bar{B}_2)$.

**Remark 53.** A similar choice of the parameters can also be made in the case of partial feedback. Then, we choose parameters corresponding to the inner encoders and the inner decoder as in (207)–(226) except for replacing (208) by
\[
X_{1,\ell} = \sqrt{P_1} \Xi_1, \quad \ell \in \{2, \ldots, \eta\},
\]
and replacing (220) by
\[
W_\ell \triangleq (-1)^\ell M_\ell W_\ell^T.
\]
We denote the parameters of the concatenated scheme corresponding to this choice by $\bar{a}_{1,p}, \bar{a}_{2,p}, \bar{B}_{2,p}$, and $\bar{C}_p$.

## F Choice of Parameters II

In Section F.1 we present a second choice of the parameters $a_1, a_2, B_1, B_2$, and $C$ given $\eta \in \mathbb{N}$. We only treat the noisy-feedback setting. The choice we propose is based on extending the choice of parameters in Section E.1 with a form of power allocation as suggested in [8]. We denote the choice by $\tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}$.

As we shall see, for our choice $\tilde{C}$ is the LMMSE-estimation matrix. Thus, the achievable region of our concatenated scheme with parameters $\eta, \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}$ is obtained by plugging the parameters $\tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2$ into the right-hand sides of (182)–(184), i.e., into the recursive formulation of the constraints which determine $\mathcal{R}(N, K_{W_1W_2}; \eta, \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C})$.

The resulting achievable region is presented in Corollary 55 ahead.

**Definition 54.** Let $P_1, P_2, N > 0$, $K_{W_1W_2} \succeq 0$, and $\eta \in \mathbb{N}$ be given. Define the rate region $\mathcal{R}_\eta(P_1, P_2, N, K_{W_1W_2})$ in the same way as the rate region $\mathcal{R}_\eta(P_1, P_2, N, K_{W_1W_2})$ but with $X_{1,1}$ and $X_{2,1}$ rather than being defined by (207) and (209) now being defined by
\[
X_{\nu,1} = \sqrt{rP_\nu} \Xi, \quad \nu \in \{1, 2\};
\]
with $\alpha_{1,1}$ and $\alpha_{2,1}$ rather than being defined by (200) and (201) (with $\ell = 1$) now being defined by
\[
\begin{align*}
\alpha_{1,1} &= rP_1 \frac{r P_2 + N}{r P_1 + r P_2 + N}, \\
\alpha_{2,1} &= rP_2 \frac{r P_1 + N}{r P_1 + r P_2 + N},
\end{align*}
\]
where $r$ is the unique solution in $[0, 1]$ to
\[
\sqrt{\frac{r^2 P_1 P_2}{(r P_1 + N)(r P_2 + N)}} = \rho^*(P_1, P_2, N);
\]
and with $\rho_1$ rather than being defined by (202) (with $\ell = 1$) now being defined by $\rho_1 = -\rho^*(P_1, P_2, N)$. 

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Corollary 55. Let $P_1, P_2, N > 0$ and $K_{W_1W_2} \geq 0$ be given. For the two-user AWGN MAC with noisy feedback our concatenated scheme with the parameters described in Section F.1 achieves all rate pairs in the regions $\mathcal{R}_\eta(P_1, P_2, N, K_{W_1W_2})$ for positive integers $\eta$, i.e.,

$$C_{\text{NoisyFB}}(P_1, P_2, N, K_{W_1W_2}) \supseteq \text{cl} \left( \bigcup_{\eta \in \mathbb{N}} \mathcal{R}_\eta(P_1, P_2, N, K_{W_1W_2}) \right).$$

Remark 56. Let $P_1, P_2, N > 0$ and $\eta \in \mathbb{N}$ be given. Specializing the region in Definition 54 to perfect feedback, i.e., to $K_{W_1W_2} = 0$, results in the region $\hat{\mathcal{R}}_\eta(P_1, P_2, N, 0)$, which is defined as the set of all rate pairs $(R_1, R_2)$ satisfying

$$R_1 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_1}{N} \right) + \sum_{\ell=2}^{\eta} \frac{1}{2\eta} \log \left( 1 + \frac{P_1(1 - \rho_{\ell-1}^2)}{N} \right),$$

$$R_2 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_2}{N} \right) + \sum_{\ell=2}^{\eta} \frac{1}{2\eta} \log \left( 1 + \frac{P_2(1 - \rho_{\ell-1}^2)}{N} \right),$$

$$R_1 + R_2 \leq \frac{1}{2\eta} \log \left( 1 + \frac{rP_1 + rP_2}{N} \right) + \sum_{\ell=2}^{\eta} \frac{1}{2\eta} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1P_2(-1)^{\ell-1}\rho_{\ell-1}}}{N} \right),$$

where the sequence $\{\rho_\ell\}_{\ell=1}^{\eta-1}$ is recursively defined by $\rho_1 = -\rho^*(P_1, P_2, N)$ and

$$\rho_\ell = \frac{\rho_{\ell-1}N - (-1)^{\ell-1}\sqrt{P_1P_2(1 - \rho_{\ell-1}^2)}}{\sqrt{P_1(1 - \rho_{\ell-1}^2) + N}} \frac{\sqrt{P_2(1 - \rho_{\ell-1}^2) + N}}, \quad \ell \in \{1, \ldots, \eta - 1\},$$

and where $r$ is the unique solution in $[0, 1]$ to (229).

Proof. Notice that for $K_{W_1W_2} = 0$ and for $\ell \in \{1, \ldots, \eta - 1\}$ and $\nu \in \{1, 2\}$ Definitions (203), (204), (205), (212), (213), and (215) result in

$$\bar{b}_{\nu,\ell} = K_{Y^\nu}\bar{K}_{Y^\nu,\Xi_{\nu}},$$

$$\bar{\beta}_{\nu,\ell} = \text{Var}(\Xi_{\nu} - K_{Y^\nu,\Xi_{\nu}} K_{Y^\nu} Y^\nu),$$

$$\tilde{\kappa}_{\nu,\ell} = 0,$$

$$\tilde{\lambda}_\ell = 0.$$

Thus, for perfect feedback the parameters suggested in Section F.1 are LMMSE-estimation error parameters, which are optimal for perfect feedback in the sense discussed in Section 7.1.

By recursive computations, one can show that that $\alpha_{1,\ell} = \beta_{1,\ell}$ and $\alpha_{2,\ell} = \beta_{2,\ell}$ for $\ell \in \{1, \ldots, \eta - 1\}$, and thus (200)–(202) result in

$$\alpha_{1,\ell} = \alpha_{1,\ell-1} \frac{P_2(1 - \rho_{\ell-1}^2) + N}{P_1 + P_2 + 2\sqrt{P_1P_2(-1)^{\ell-1}\rho_{\ell-1}} + N},$$

$$\alpha_{2,\ell} = \alpha_{2,\ell-1} \frac{P_1(1 - \rho_{\ell-1}^2) + N}{P_1 + P_2 + 2\sqrt{P_1P_2(-1)^{\ell-1}\rho_{\ell-1}} + N},$$

$$\rho_\ell = \frac{\rho_{\ell-1}N - (-1)^{\ell-1}\sqrt{P_1P_2(1 - \rho_{\ell-1}^2)}}{\sqrt{P_1(1 - \rho_{\ell-1}^2) + N}} \frac{\sqrt{P_2(1 - \rho_{\ell-1}^2) + N}}.$$
for $\ell \in \{2, \ldots, \eta\}$.

Combining all these observations and (227) and (228) with the rate expressions (197)–(199) concludes the proof. \hfill \Box

### F.1 Description of Parameters

We only consider the noisy feedback setting. An analogous choice of the parameters for the partial feedback setting is obtained by similar modifications as in Remark 53.

We first describe how Inner Encoder 1 and Inner Encoder 2 map the fed symbols to the sequences of channel inputs $X_{1,1}, \ldots, X_{1,\eta}$ and $X_{2,1}, \ldots, X_{2,\eta}$. This, then determines $\tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2$. The matrix $\tilde{C}$ is chosen the LMMSE-estimation matrix.

The inner encoders use the same linear strategies as in Section E.1, with the only difference that here for every fed symbol, Inner Encoder 1 scales the first produced symbol by a constant $\sqrt{r}$, and similarly Inner Encoder 2 scales the first produced symbol by the same constant $\sqrt{r}$, where $r \in [0, 1]$ is defined as the solution to

$$\sqrt{r^2 P_1 P_2} = \rho^*(P_1, P_2, N). \tag{230}$$

Equation (230) has a unique solution in $[0, 1]$ because (230) is strictly increasing in $r \in [0, 1]$ and by

$$0 < \rho^*(P_1, P_2, N) < \sqrt{\frac{P_1 P_2}{(P_1 + N)(P_2 + N)}}. \tag{231}$$

Here, Equation (231) holds by the continuity of the expressions in (11), and because for $\rho = 0$ the right-hand side of (11) is strictly larger than its left-hand side, whereas for $\rho = \sqrt{\frac{P_1 P_2}{(P_1 + N)(P_2 + N)}}$ the left-hand side of (11) is strictly larger than its right-hand side.

For the detailed description of the inner encoders we again replace the fed symbols $\xi_1, \xi_2$ by the independent standard Gaussians $\Xi_1$ and $\Xi_2$. Then, Inner Encoder 1 produces

$$X_{1,1} = \sqrt{r P_1} \Xi_1,$$

$$X_{1,\ell} = \frac{P_1}{\beta_{1,\ell-1}} \left( \Xi_1 - \tilde{B}_{1,\ell-1} V_{1,\ell-1} \right), \quad \ell \in \{2, \ldots, \eta\},$$

and Inner Encoder 2 produces

$$X_{2,1} = \sqrt{r P_2} \Xi_2,$$

and

$$X_{2,\ell} = (-1)^{\ell-1} \sqrt{\frac{P_2}{\beta_{2,\ell-1}}} \left( \Xi_2 - \tilde{B}_{2,\ell-1} V_{2,\ell-1} \right), \quad \ell \in \{2, \ldots, \eta\},$$

where $\{\tilde{B}_{\nu,\ell}\}_{\ell=1}^{\eta-1}$, $\{M_{\nu}\}_{\ell=1}^{\eta-1}$, and $\{\tilde{b}_{\nu,\ell}\}_{\ell=1}^{\eta-1}$, for $\nu \in \{1, 2\}$, are defined in (212)–(215), and where $r$ is defined by (230).
The described encodings correspond to the following parameters in the concatenated scheme:

\[
\tilde{a}_1 \triangleq \left( \sqrt{T}P_1 \frac{P_1}{\beta_{1,1}} \ldots \frac{P_1}{\beta_{1,n-1}} \right)^T, \\
\tilde{a}_2 \triangleq \left( \sqrt{T}P_2 - \sqrt{T}P_2 \frac{P_2}{\beta_{2,1}} \ldots (-1)^{\eta-1} \frac{P_2}{\beta_{2,n-1}} \right)^T, 
\]

and

\[
\tilde{b}_1 \triangleq \left( 0 - \sqrt{T}b_1^{(0)} \ldots - \sqrt{T}b_1^{(0)} \right)^T, \\
\tilde{b}_2 \triangleq \left( 0 \sqrt{T}b_2^{(0)} \ldots (-1)^\eta \sqrt{T}b_2^{(0)} \right)^T, 
\]

where for \( \nu \in \{1, 2\} \) the parameters \( \{\tilde{\beta}_{\nu,\ell}\}_{\ell=1}^{\eta-1} \) are defined in (212) and (213), where \( \{\tilde{b}_{\nu,\ell}^{(0)}\}_{\ell=1}^{\eta-1} \) are defined by (225) and by (215), and where \( r \) is defined by (230).

As in Section E.1 the inner decoder produces the LMMSE-estimates of the input symbols \( (\tilde{X}_1, \tilde{X}_2) \) based on the output \( Y \), i.e.,

\[
\tilde{C} = \tilde{A}_b^T (\tilde{A}_b \tilde{A}_b^T + N\eta + \tilde{B}_b (K_{W_1W_2} \otimes I_\eta) \tilde{B}_b^T)^{-1}, 
\]

where \( \tilde{A}_b \triangleq (\tilde{a}_1 \ \tilde{a}_2) \) and \( \tilde{B}_b \triangleq (\tilde{B}_1 \ \tilde{B}_2) \).

## G Choice of Parameters III

In Section G.1, we present a specific choice of the parameters \( a_1, a_2, B_1, B_2, C_{SI} \) given \( \eta \in \mathbb{N} \) for noisy feedback with receiver side-information. We denote this choice by \( \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}_{SI} \).

As we shall see, the matrix \( \tilde{C}_{SI} \) is chosen the LMMSE-estimation matrix. Thus, the achievable region of our concatenated scheme with parameters \( \eta, \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}_{SI} \) is obtained by plugging the parameters \( \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2 \) into (188)–(190), i.e., into the recursive formulation of the constraints which determine \( R_{SI} (N; K_{W_1W_2}; \eta, \tilde{a}_1, \tilde{a}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}_{SI}) \). The resulting achievable region is presented in Corollary 58 ahead.

**Definition 57.** Let \( P_1, P_2, N > 0, K_{W_1W_2} \geq 0 \), and \( \eta \in \mathbb{N} \) be given. Define the region \( \tilde{R}_{\eta}(P_1, P_2, N, K_{W_1W_2}) \) as the set of all rate pairs \( (R_1, R_2) \) satisfying

\[
R_1 \leq \frac{1}{2\eta} \sum_{\ell=1}^{\eta} \log \left( 1 + \frac{P_1 \frac{a_1,\ell-1}{\beta_{1,\ell-1}} (1 - \rho_1^2)}{N} \right), \\
R_2 \leq \frac{1}{2\eta} \sum_{\ell=1}^{\eta} \log \left( 1 + \frac{P_2 \frac{a_2,\ell-1}{\beta_{2,\ell-1}} (1 - \rho_2^2)}{N} \right), \\
R_1 + R_2 \leq \frac{1}{2\eta} \sum_{\ell=1}^{\eta} \log \left( 1 + \frac{P_1 \frac{a_1,\ell-1}{\beta_{1,\ell-1}} + P_2 \frac{a_2,\ell-1}{\beta_{2,\ell-1}}}{N} + 2\sqrt{P_1 P_2} (1 + (-1)^\ell \rho_\ell-1) \sqrt{\frac{a_1,\ell-1 a_2,\ell-1}{\beta_{1,\ell-1} \beta_{2,\ell-1}}} \right), 
\]

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where \( \alpha_{1,0} = 1, \alpha_{2,0} = 1, \rho_0 = 0 \) and \( \{ \alpha_{1,\ell} \}_{\ell=1}^n, \{ \alpha_{2,\ell} \}_{\ell=1}^n, \) and \( \{ \rho_\ell \}_{\ell=1}^n \) are recursively given by

\[
\begin{align*}
\alpha_{1,\ell} &= \alpha_{1,\ell-1} \left( 1 + \frac{P_1 \alpha_{1,\ell-1} + 2 \sqrt{P_1 P_2} (1 + (-1)^{\ell-1} \rho_{\ell-1}) \sqrt{\frac{\alpha_{1,\ell-1} \alpha_{2,\ell-1}}{\beta_{1,\ell-1} \beta_{2,\ell-1}}} \right)^{-1}, \\
\alpha_{2,\ell} &= \alpha_{2,\ell-1} \left( 1 + \frac{P_2 \alpha_{2,\ell-1} + 2 \sqrt{P_1 P_2} (1 + (-1)^{\ell-1} \rho_{\ell-1}) \sqrt{\frac{\alpha_{1,\ell-1} \alpha_{2,\ell-1}}{\beta_{1,\ell-1} \beta_{2,\ell-1}}} \right)^{-1}, \\
\rho_\ell &= \frac{\rho_{\ell-1} N - \sqrt{P_1 P_2} \sqrt{\frac{\alpha_{1,\ell-1} \alpha_{2,\ell-1}}{\beta_{1,\ell-1} \beta_{2,\ell-1}}} (1 - \rho_{\ell-1}^2)}{P_1^{-1} \beta_{1,\ell-1} (1 - \rho_{\ell-1}^2) + N P_2^{-1} \beta_{2,\ell-1} (1 - \rho_{\ell-1}^2) + N}
\end{align*}
\]

and where \( \tilde{\beta}_{1,0} \triangleq 1, \tilde{\beta}_{2,0} \triangleq 1, \) and \( \{ \tilde{\beta}_{1,\ell} \}_{\ell=1}^{n-1} \) and \( \{ \tilde{\beta}_{2,\ell} \}_{\ell=1}^{n-1} \) are given as in (236) and (237).

**Corollary 58.** The capacity region \( C_{\text{NoisyFBSI}}(P_1, P_2, N, K_{W_1 W_2}) \) of the two-user Gaussian MAC with noisy feedback and receiver side-information includes all rate regions \( \tilde{R}_\eta(P_1, P_2, N, K_{W_1 W_2}) \) for positive integers \( \eta \), i.e.,

\[
C_{\text{NoisyFBSI}}(P_1, P_2, N, K_{W_1 W_2}) \supseteq \bigcup_{\eta \in \mathbb{N}} \tilde{R}_\eta(P_1, P_2, N, K_{W_1 W_2}).
\]

**G.1 Description of Parameters**

Let \( P_1, P_2, N > 0, K_{W_1 W_2} \geq 0, \) and \( \eta \in \mathbb{N} \) be given. We present our specific choice of the parameters \( \bar{a}_1, \bar{a}_2, B_1, B_2, \bar{C} \).

We first describe how Inner Encoder 1 and Inner Encoder 2 map the fed symbols to the channel inputs. This, then determines \( \bar{a}_1, \bar{a}_2, B_1, B_2, \bar{C} \). To simplify the description we replace the symbols \( \bar{X}_1 \) and \( \bar{X}_2 \) fed to the inner encoders by the independent standard Gaussians \( \Xi_1 \) and \( \Xi_2 \). We choose the inner encoders to produce

\[
\begin{align*}
X_{1,1} &= \sqrt{P_1} \Xi_1, \\
X_{1,\ell} &= \sqrt{ \frac{P_1}{\beta_{1,\ell-1}} } \left( \Xi_1 - \bar{b}_{1,\ell-1}^{(\text{SI})} V_{1,\ell-1} \right), \quad \ell \in \{2, \ldots, \eta\},
\end{align*}
\]

and

\[
\begin{align*}
X_{2,1} &= \sqrt{P_2} \Xi_2, \\
X_{2,\ell} &= (-1)^{\ell-1} \sqrt{ \frac{P_2}{\beta_{2,\ell-1}} } \left( \Xi_2 - \bar{b}_{2,\ell-1}^{(\text{SI})} M_{\ell-1} V_{2,\ell-1} \right), \quad \ell \in \{2, \ldots, \eta\},
\end{align*}
\]

where

\[
\begin{align*}
\bar{\beta}_{1,\ell} &\triangleq \text{Var} \left( \Xi_1 - \bar{b}_{1,\ell}^{(\text{SI})} V_{1,\ell} \right), \quad \ell \in \{1, \ldots, \eta - 1\}, \\
\bar{\beta}_{2,\ell} &\triangleq \text{Var} \left( \Xi_2 - \bar{b}_{2,\ell}^{(\text{SI})} M_{\ell} V_{2,\ell} \right), \quad \ell \in \{1, \ldots, \eta - 1\}, \\
M_{\ell} &\triangleq \text{diag} \left( 1, -1, 1, \ldots, (-1)^{\ell-1} \right), \quad \ell \in \{1, \ldots, \eta - 1\},
\end{align*}
\]
and where $\tilde{b}_{1,\ell}, \tilde{b}_{2,\ell} \in \mathbb{R}^\ell$ are given by
\[
\tilde{b}_{\nu,\ell} = K_{V_{\nu,\ell}^{-1}} \zeta, \quad \ell \in \{1, \ldots, \eta - 1\}, \; \nu \in \{1, 2\}.
\] (239)

Notice that the choice (239) implies that the $\ell$-th channel input produced by Inner Encoder 1 is a scaled version of the LMMSE-estimation error of $\Xi_1$ based on the past feedback outputs $V_{1,1}, \ldots, V_{1,\ell-1}$. Similarly, for Inner Encoder 2.

The described encodings correspond to the following parameters of the concatenated scheme:
\[
\tilde{a}_1 \triangleq \begin{pmatrix} \sqrt{P_1} & \sqrt{P_2} & \cdots & \sqrt{P_{\eta-1}} \end{pmatrix}^T,
\tilde{a}_2 \triangleq \begin{pmatrix} \sqrt{Q_1} & -\sqrt{P_2} & \cdots & (-1)^{\eta-1} \sqrt{P_{\eta-1}} \end{pmatrix}^T,
\]
and
\[
\tilde{b}_1 \triangleq \begin{pmatrix} 0 & -\sqrt{P_{\eta-1}} - \sqrt{P_1} \tilde{b}_{1,0}^{(0)} & \cdots & \sqrt{P_{\eta-1}} \tilde{b}_{1,\eta-1}^{(0)} \end{pmatrix}^T,
\tilde{b}_2 \triangleq \begin{pmatrix} 0 & \sqrt{P_{\eta-1}} \tilde{b}_{2,0}^{(0)} & \cdots & (-1)^{\eta-1} \sqrt{P_{\eta-1}} \tilde{b}_{2,\eta-1}^{(0)} \end{pmatrix}^T,
\]
where the vectors $\{\tilde{b}_{1,\ell}^{(0)}\}_{\ell=1}^{\eta-1}$ and $\{\tilde{b}_{2,\ell}^{(0)}\}_{\ell=1}^{\eta-1}$ are defined as the $\eta$-dimensional vector obtained by stacking the vector $\tilde{b}_{\nu,\ell}$ on top of an $(\eta - \ell)$-dimensional zero-vector, i.e.,
\[
\tilde{b}_{\nu,\ell}^{(0)} \triangleq \begin{pmatrix} \tilde{b}_{\nu,\ell} & 0 \end{pmatrix}, \quad \ell \in \{1, \ldots, \eta - 1\}, \; \nu \in \{1, 2\}.
\]

The matrix $\tilde{C}_{SL}$ is chosen as the LMMSE-estimation matrix with side-information, i.e.,
\[
\tilde{C}_{SL} = \tilde{A}_b^T (\tilde{A}_b \tilde{A}_b^T + N_1)^{-1},
\]
where $\tilde{A} \triangleq (\tilde{a}_1 \; \tilde{a}_2)$.

\section{Rate-Splitting with Carleial’s Cover-Leung Scheme}

In this section we describe the rate-splitting scheme in Section 5.2 in more detail. We consider the version of the scheme where after each Block $b \in \{1, \ldots, B\}$ Transmitter 1 first decodes Message $M_{2,CS,b}$ before decoding $M_{2,CL,b}$. Similarly, for Transmitter 2.

We first describe the encodings. We start with the encodings in Block $b$, for a fixed $b \in \{1, \ldots, B\}$, where we assume that from decoding steps in the previous block $(b - 1)$ both transmitters are cognizant of the pair $(M_{1,CL,b-1}, M_{2,CL,b-1})$. Given $M_{\nu,CL,b} = m_{\nu,CL,b}$, $M_{1,CL,b-1} = m_{1,CL,b-1}$, and $M_{2,CL,b-1} = m_{2,CL,b-1}$, Transmitter $\nu$, for $\nu \in \{1, 2\}$, picks the codewords $u_{\nu,b}(m_{\nu,CL,b}) \triangleq (u_{\nu,b,1}, \ldots, u_{\nu,b,\eta_n})$, $\omega_{1,b}(m_{1,CL,b-1}) \triangleq (\omega_{1,b,1}, \ldots, \omega_{1,b,\eta_n})$, and $\omega_{2,b}(m_{2,CL,b-1}) \triangleq (\omega_{2,b,1}, \ldots, \omega_{2,b,\eta_n})$ from the corresponding codebooks, which have independently been generated by randomly drawing each entry according to an IID zero-mean unit-variance Gaussian distribution$^{12}$. Fix correlation coefficients $\rho_1, \rho_2 \in [0, 1]$, which are

$^{12}$To satisfy the power constraints the Gaussian distribution should be of variance slightly less than 1. However, this is a technicality which we ignore.
constant over all blocks $b \in \{1, \ldots, B\}$. Transmitter $\nu$ computes the following linear combinations for $i \in \{1, \ldots, n\}$ and $\nu \in \{1, 2\}$:

$$
\sqrt{(1 - \rho^2_\nu)} P_\nu u_{\nu,b,i} + \sqrt{\frac{1}{2} \rho^2_\nu P_\nu} (\omega_{1,b,i} + \omega_{2,b,i}),
$$

where

$$
u,b,i \triangleq (u_{\nu,b,(i-1)\eta+1}, \ldots, u_{\nu,b,\eta})^T, \quad i \in \{1, \ldots, n\}, \nu \in \{1, 2\},$$

$$\omega_{\nu,b,i} \triangleq (\omega_{\nu,b,(i-1)\eta+1}, \ldots, \omega_{\nu,b,\eta})^T, \quad i \in \{1, \ldots, n\}, \nu \in \{1, 2\}.$$

Moreover, given $M_{\nu,CS,b} = m_{\nu,CS,b}$, Transmitter $\nu$ feeds $m_{\nu,CS,b}$ to Outer Encoder $\nu$, which picks the codeword $\xi_\nu(m_{\nu,CS,b}) \triangleq (\xi_{\nu,h,1}, \ldots, \xi_{\nu,h,n})^T$ corresponding to $m_{\nu,CS,b}$ and feeds it to Inner Encoder $\nu$. Denoting the parameters of Inner Encoder $\nu$ by $a_\nu$ and $B_\nu$, Inner Encoder $\nu$ produces the $\eta$-dimensional vectors

$$a_\nu \xi_{\nu,b,i} + B_\nu V_{\nu,b,i}, \quad i \in \{1, \ldots, n\}, \nu \in \{1, 2\},$$

where for $i \in \{1, \ldots, n\}$ and $\nu \in \{1, 2\}$

$$V_{\nu,b,i} \triangleq (V_{\nu,(b-1)\eta+1}, \ldots, V_{\nu,(b-1)\eta+i})^T.$$

The signal transmitted by Transmitter $\nu$ is then described by the sum of the vectors in (240) and (241) as follows. For $i \in \{1, \ldots, n\}$ and $\nu \in \{1, 2\}$

$$X_{\nu,b,i} = \sqrt{(1 - \rho^2_\nu)} P_\nu'u_{\nu,b,i} + \sqrt{\frac{1}{2} \rho^2_\nu P_\nu'} (\omega_{1,b,i} + \omega_{2,b,i}) + a_\nu \xi_{\nu,b,i} + B_\nu V_{\nu,b,i},$$

where

$$X_{\nu,b,i} \triangleq (X_{\nu,(b-1)\eta+1}, \ldots, X_{\nu,(b-1)\eta+i})^T.$$

Notice that if $a_1, a_2, B_1$, and $B_2$ satisfy the power constraints (21) and (22) for powers $(P_1 - P'_1)$ and $(P_2 - P'_2)$, noise variance $(N + P'_1 + P'_2 + 2\sqrt{P'_1 P'_2 \rho_1 \rho_2})$, and feedback-noise covariance matrix $K_{W_1W_2}$ and if the outer code’s codewords are zero-mean and average block-power constrained to 1, then the channel input sequences satisfy the power constraints with arbitrary high probability.

In Block $(B + 1)$ the two transmitters only send information about the pair $(M_{1,CL,B}, M_{2,CL,B})$. Given $M_{1,CL,B} = m_{1,CL,B}$ and $M_{2,CL,B} = m_{2,CL,B}$, both transmitters pick the codewords $\omega_{1,B+1}(m_{1,CL,B}) \triangleq (\omega_{1,B+1,1}, \ldots, \omega_{1,B+1,m})^T$ and $\omega_{2,B+1}(m_{2,CL,B}) \triangleq (\omega_{2,B+1,1}, \ldots, \omega_{2,B+1,m})^T$ from the corresponding codebooks and form a linear combination of power $P'_\nu$. Thus, defining

$$X_{\nu,B+1} \triangleq (X_{\nu,Bm+1}, \ldots, X_{\nu,(B+1)m})^T, \quad \nu \in \{1, 2\},$$

$$\omega_{\nu,B+1} \triangleq (\omega_{\nu,B+1,1}, \ldots, \omega_{\nu,B+1,m})^T, \quad \nu \in \{1, 2\},$$

the signal transmitted by Transmitter $\nu$ can be described as

$$X_{\nu,B+1} = \sqrt{\frac{1}{2} \rho^2_\nu P_\nu'} (\omega_{1,B+1} + \omega_{2,B+1}), \quad \nu \in \{1, 2\}.\quad (243)$$
Next, we describe the decodings. We start with the decoding at Transmitter 2: the decoding at Transmitter 1 is performed similarly and therefore omitted; and the decodings at the receiver are described later on.

Recall that after a fixed block, for \( b \in \{1, \ldots, B\} \), Transmitter 2 first decodes Message \( M_{1,CS,b} \), followed by decoding Message \( M_{1,CL,b} \). After Block \( b \), for \( b \in \{1, \ldots, B\} \), Transmitter 2 observed \( \{V_{2,b,1}, \ldots, V_{2,b,n}\} \), and additionally is cognizant of the realizations of \( \{U_{2,b,1}, \ldots, U_{2,b,n}\}, \{\Omega_{1,b,1}, \ldots, \Omega_{1,b,n}\}, \{\Omega_{2,b,1}, \ldots, \Omega_{2,b,n}\}, \text{and } \{\Xi_{2,b,1}, \ldots, \Xi_{2,b,n}\} \). It can thus compute for \( i \in \{1, \ldots, n\} \):

\[
\hat{V}_{2,b,i} \triangleq V_{2,b,i} - \sqrt{(1 - \rho_2^2)P_2} U_{2,b,i} - \left( \sqrt{\frac{1}{2} \rho_1^2 P_1' + \frac{1}{2} \rho_2^2 P_2'} \right) (\Omega_{1,b,i} + \Omega_{2,b,i}) - a_2 \Xi_{2,b,i} - (B_1 + B_2) V_{2,b,i} = a_1 \cdot \Xi_{1,b,i} + \sqrt{(1 - \rho_1^2)P_1'} U_{1,b,i} + Z_{b,i} + W_{2,b,i} + B_1 (W_{1,b,i} - W_{2,b,i}),
\]

where for \( i \in \{1, \ldots, n\} \) and \( \nu \in \{1, 2\} \):

\[
Z_{b,i} \triangleq (Z_{b-1}p_1 + (i-1)n + 1, \ldots, Z_{b-1}p_1 + in)\top, \quad W_{\nu,b,i} \triangleq \left( W_{\nu,(b-1)p_1 + (i-1)n + 1}, \ldots, W_{\nu,(b-1)p_1 + in}\right)\top.
\]

Since the sequence \( \{\hat{V}_{2,b,1}, \ldots, \hat{V}_{2,b,n}\} \) is independent of the additional information \( \{U_{2,b,1}, \ldots, U_{2,b,n}\}, \{\Omega_{1,b,1}, \ldots, \Omega_{1,b,n}\}, \{\Omega_{2,b,1}, \ldots, \Omega_{2,b,n}\}, \text{and } \{\Xi_{2,b,1}, \ldots, \Xi_{2,b,n}\} \), Transmitter 2 can optimally decode Message \( M_{1,CS,b} \) based on \( \{\hat{V}_{2,b,1}, \ldots, \hat{V}_{2,b,n}\} \) only. To this end, it does not apply the inner and outer decoder of the concatenated scheme, but directly applies an optimal decoder for a Gaussian single-input antenna/\( \eta \)-output antenna channel with temporally-white noise sequences which are correlated across antennas.

Let \( \hat{M}_{1,CS}^{(Tx2)} \) denote Transmitter 2’s guess of Message \( M_{1,CS} \) and let \( \left( \hat{\Xi}_{1,b,1}^{(Tx2)}, \ldots, \hat{\Xi}_{1,b,n}^{(Tx2)} \right)\top \) be the corresponding codeword of the outer code. Transmitter 2 attempts to subtract the influence of the sequence produced by encoding \( M_{1,CS,b} \) and computes

\[
\tilde{V}_{2,b,i}^{(2)} \triangleq \hat{V}_{2,b,i} - a_1 \hat{\Xi}_{1,b,i}^{(Tx2)}, \quad i \in \{1, \ldots, n\},
\]

which, if Transmitter 2 successfully decoded \( M_{1,CS,b} \), equals

\[
\sqrt{(1 - \rho_1^2)P_1'} U_{1,b,i} + Z_{b,i} + W_{2,b,i} + B_1 (W_{1,b,i} - W_{2,b,i}), \quad i \in \{1, \ldots, n\}.
\]

Transmitter 2 then decodes Message \( M_{1,CL,b} \) based on the sequences \( \{\tilde{V}_{2,b,1}, \ldots, \tilde{V}_{2,b,n}\} \) using an optimal decoder for a Gaussian \( \eta \)-input antenna/\( \eta \)-output antenna channel with temporally-white noise sequences correlated across antennas.

As a last element, we describe the decodings at the receiver. After each block \( b \in \{1, \ldots, B\} \) the receiver first decodes Messages \( \{M_{1,CS,b}, M_{2,CS,b}\} \) while treating the sequences produced by encoding \( M_{1,CL,b-1}, M_{2,CL,b-2}, M_{1,CL,b}, \text{and } M_{2,CL,b} \) as additional noise. For this decoding step the receiver uses inner and outer decoders of the concatenated scheme. Let \( \left( \hat{M}_{1,CS,b}, \hat{M}_{2,CS,b} \right) \) denote the receiver’s guess of the pair \( (M_{1,CS,b}, M_{2,CS,b}) \) and let \( \left( \hat{\Xi}_{1,b,1}^{(Rx)}, \ldots, \hat{\Xi}_{1,b,n}^{(Rx)} \right)\top \) and \( \left( \hat{\Xi}_{2,b,1}^{(Rx)}, \ldots, \hat{\Xi}_{2,b,n}^{(Rx)} \right)\top \) be the corresponding codewords of the outer
code. The receiver attempts to cancel the influence of the sequences produced by encoding the \((M_{1,\text{CS},b}, M_{2,\text{CS},b})\) and computes
\[
\widehat{Y}_{b,i} \triangleq Y_{b,i} - a_1 \hat{\Xi}_{1,b,i} - a_2 \hat{\Xi}_{2,b,i} - B_1 Y_{b,i} - B_2 Y_{b,i}, \quad i \in \{1, \ldots, n\},
\]  
where
\[
Y_{b,i} \triangleq (Y_{(b-1)\eta n + (i-1)\eta + 1}, \ldots, Y_{(b-1)\eta n + \eta})^T, \quad i \in \{1, \ldots, n\},
\]  
and where, in case the first decoding step was successful, (244) corresponds to
\[
\sqrt{(1 - \rho_1^2)} P_1 U_{1,b,i} + \sqrt{(1 - \rho_2^2)} P_2 U_{2,b,i} \\
+ \left( \frac{1}{2} \rho_1^2 P_1' + \frac{1}{2} \rho_2^2 P_2' \right) (\Omega_{1,b,i} + \Omega_{2,b,i}) + B_1 W_{1,b,i} + B_2 W_{2,b,i} + Z_{b,i}, \quad i \in \{1, \ldots, n\}.
\]  
Moreover, from the decoding steps performed after the reception of the previous block, the receiver is assumed to be cognizant of messages \(M_{1,\text{CL},b-2}, M_{2,\text{CL},b-2}, M_{1,\text{CS},b-1},\) and \(M_{2,\text{CS},b-1}\). It can thus compute for \(i \in \{1, \ldots, n\}\):
\[
\widehat{Y}_{b-1,i}^{(2)} \triangleq Y_{b-1,i} - \left( \sqrt{\frac{1}{2} \rho_1^2 P_1'} + \sqrt{\frac{1}{2} \rho_2^2 P_2'} \right) (\Omega_{1,b-1,i} + \Omega_{2,b-1,i}) \\
- a_1 \Xi_{1,b-1,i} - a_2 \Xi_{2,b-1,i} - B_1 Y_{b,i} - B_2 Y_{b,i}, \\
= \sqrt{(1 - \rho_1^2)} P_1 U_{1,b-1,i} + \sqrt{(1 - \rho_2^2)} P_2 U_{2,b-1,i} + B_1 W_{1,b-1,i} + B_2 W_{2,b-1,i} + Z_{b-1,i}.
\]  
The receiver finally, decodes Messages \((M_{1,\text{CL},b-1}, M_{2,\text{CL},b-1})\) based on the sequence \(\left\{ (\widehat{Y}_{b,i}, \widehat{Y}_{b-1,i}) \right\}_{i=1}^{n}\) using an optimal decoder for a \(2\eta\)-input antenna/\(2\eta\)-output antenna Gaussian MAC with temporally-white noise that is correlated across antennas.

Finally, after Block \((B + 1)\) the receiver decodes Messages \((M_{1,\text{CL},b}, M_{2,\text{CL},b})\) based on \(\widehat{Y}_{B,1}, \ldots, \widehat{Y}_{B,n}^{(2)}\) and based on the sequence \((Y_{B\eta n + 1}, \ldots, Y_{(B+1)\eta n})\). To this end, it again uses an optimal decoder for a \(2\eta\)-input antenna/\(2\eta\)-output antenna Gaussian MAC with temporally-white noise that is correlated across antennas.

### H.1 Noisy and Perfect Partial Feedback

The proposed extension applies also to settings with noisy or perfect partial feedback to Transmitter 2, if \(B_1\) is set to the all-zero matrix and if Carleial’s scheme for partial feedback is applied. Thus, our scheme should be modified so that there are no codings taking place at Transmitter 1 and so that in (242) and (243) the term \(\sqrt{\frac{1}{2} \rho_1^2 P_1'} (\omega_{1,b,i} + \omega_{2,b,i})\) is replaced by \(\sqrt{\rho_1^2 P_1'} \omega_{1,b,i}\).

Notice that in a setting with perfect partial feedback to Transmitter 2 the components of the noise vectors corrupting \(\{\tilde{V}_{2,b,i}\}\) are uncorrelated, similarly for \(\{\tilde{V}_{2,b,i} - \sqrt{(1 - \rho_1^2 P_1')} u_{1,b-1,i}\}\) and for \(\tilde{Y}_{b,i}\) and \(\tilde{Y}_{b,i}^{(2)}\). Thus, optimal decoders for Gaussian multi-input antenna/multi-output antenna channels with uncorrelated white noise sequences can be used to decode \(M_{1,\text{CL},b}\) at Transmitter 2 and to decode \((M_{1,\text{CL},b}, M_{2,\text{CL},b})\) at the receiver. Moreover, the observation \(\{Y_{b,i}\}\) at the receiver is a degraded version of
the observation $\{\tilde{V}_{1,b,i}\}$ at Transmitter 2. Thus, since the receiver decodes $(M_{1,CS,b}, M_{2,CS,b})$ based on $\{Y_{b,i}\}$, in settings with perfect partial feedback there is no loss in optimality in the presented rate-splitting scheme if based on $\{\tilde{V}_{1,b,i}\}$ Transmitter 2 first decodes message $M_{1,CS,b}$ before decoding $M_{1,ICL,b}$. In particular, the set of achievable rates of the concatenated scheme is solely constrained by the decoding at the receiver.

I Interleaving and Rate-Splitting with Carleial’s Cover-Leung Scheme

We describe the scheme in Section 5.3 in more detail. We start with the encodings and first consider the encodings in the $\ell$-th subblock of Block $b$, for a fixed $b \in \{1, \ldots, B\}$ and $\ell \in \{1, \ldots, \eta\}$. Define $\tilde{b} = (b - 1)\eta + \ell$. We assume that from decoding steps after previous subblocks $((b - 2)\eta + 1), \ldots, (b - 1)$, both transmitters are cognizant of the pairs $\{(M_{1,ICL,(b-2)\eta+1}, M_{2,ICL,(b-2)\eta+1}), \ldots, (M_{1,ICL,\tilde{b}-1}, M_{2,ICL,\tilde{b}-1})\}$.

The encodings in Subblock $\tilde{b}$ consist of four steps. In the first step Transmitter 1 produces an $n$-length vector to encode messages $M_{1,ICL,\tilde{b}}, M_{1,ICL,\tilde{b}-\eta},$ and $M_{2,ICL,\tilde{b}-\eta}$ as follows. Given $M_{1,ICL,\tilde{b}} = m_{1,ICL,\tilde{b}}, M_{1,ICL,\tilde{b}-\eta} = m_{1,ICL,\tilde{b}-\eta},$ and $M_{2,ICL,\tilde{b}-\eta} = m_{2,ICL,\tilde{b}-\eta},$ Transmitter 1 first picks codewords $u_{1,\tilde{b}}(M_{1,ICL,\tilde{b}}), \omega_{1,\tilde{b}}(M_{1,ICL,\tilde{b}-\eta})$, and $\omega_{2,\tilde{b}}(M_{2,ICL,\tilde{b}-\eta})$ from the corresponding codebooks, which have independently been generated by randomly drawing each entry according to an IID zero-mean unit-variance Gaussian distribution\(^{13}\). Transmitter 1 then completes the first step by computing the following linear combination

$$\sqrt{(1 - \rho_1^2)P_1'} u_{1,\tilde{b}} + \sqrt{\frac{1}{2} \rho_1^2 P_1'} (\omega_1, \bar{V}_{1,\tilde{b}}) = \sqrt{(1 - \rho_2^2)P_2'} u_{2,\tilde{b}} + \sqrt{\frac{1}{2} \rho_2^2 P_2'} (\omega_2, \bar{V}_{2,\tilde{b}}),$$

(245)

where $\rho_1 \in [0, 1]$ is a fixed chosen parameter of the scheme, which does not depend on $\tilde{b}$. Similarly, for Transmitter 2.

In the second step, Transmitter 1 computes the “cleaned” feedback vectors $\tilde{V}_{\nu,((b-1)\eta+1), \ldots, \tilde{V}_{\nu,((b-1)\eta+\ell-1)}$, where $\tilde{V}_{\nu,\tilde{b}}$ for $\tilde{b}' \in \{(b - 1)\eta + 1, \ldots, (b - 1)\eta + \ell - 1\}$ is defined as:

$$\tilde{V}_{1,\tilde{b}} \triangleq V_{1,\tilde{b}} - \sqrt{(1 - \rho_1^2)P_1} U_{1,\tilde{b}} - \sqrt{(1 - \rho_2^2)P_2} U_{2,\tilde{b}} - \left(\sqrt{\frac{1}{2} \rho_1^2 P_1} + \sqrt{\frac{1}{2} \rho_2^2 P_2}\right) (\Omega_{1,\tilde{b}} + \Omega_{2,\tilde{b}}).$$

(246)

where $V_{\nu,\tilde{b}} \triangleq (V_{\nu,((b-1)\eta+1), \ldots, V_{\nu,((b-1)\eta+\ell-1)}$. Similarly, for Transmitter 2. Notice that for $\tilde{b}' \in \{(b - 1)\eta + 1, \ldots, (b - 1)\eta + \ell - 1\}$ the “cleaned” feedback vectors satisfy

$$\tilde{V}_{1,\tilde{b}} - \bar{W}_{1,\tilde{b}} = \tilde{V}_{2,\tilde{b}} - \bar{W}_{2,\tilde{b}},$$

where for $\tilde{b}' \in \{(b - 1)\eta, \ldots, (b - 1)\eta + \ell - 1\}$ and $\nu \in \{1, 2\}$:

$$\bar{W}_{\nu,\tilde{b}} \triangleq (W_{\nu,((b-1)\eta+1), \ldots, W_{\nu,((b-1)\eta+\ell-1)}.$$

\(^{13}\)To satisfy the power constraints the Gaussian distribution should be of variance slightly less than 1. However, this is a technicality which we ignore.
Thus, they correspond to the feedback vectors of a “cleaned” channel where the channel outputs are described by the vectors \((\bar{\mathbf{V}}_{1,b} - \mathbf{W}_{1,b})\).

In the third step, Transmitter 1 produces an \(n\)-length vector to encode Message \(M_{1,ICS,b}\) using the “cleaned” feedback vectors in (246) as explained shortly. Assume that at the beginning of Block \(b\) Transmitter 1 fed Message \(M_{1,ICS,b}\) to its outer encoder and that the outer encoder produced the codeword \(\mathbf{\xi}_{1,b}\). Let

\[
\mathbf{a}_1 \triangleq (a_{1,1}, \ldots, a_{1,n})^\top, \\
\mathbf{B}_1 \triangleq \begin{pmatrix}
    b_{1,1,1} & \cdots & b_{1,1,n} \\
    \vdots & \ddots & \vdots \\
    b_{1,n,1} & \cdots & b_{1,n,n}
\end{pmatrix},
\]

denote the parameters of Transmitter 1’s modified inner encoder. The modified inner encoder then produces the \(n\)-length vector

\[
a_{1,\ell} \mathbf{\xi}_{1,b} + \sum_{j=1}^{\ell-1} b_{1,\ell,j} \bar{\mathbf{V}}_{1,(b-1)\eta+j},
\]

which is also the \(n\)-length vector that Transmitter 1 produces in this third step. Similarly, for Transmitter 2.

In the forth and last step, Transmitter 1 sums the \(n\)-length vectors in (245) and (247), and sends the resulting symbols over the channel. Similarly, for Transmitter 2.

Thus, the signal transmitted by Transmitter \(\nu\) in Subblock \(b\) can be described as follows:

\[
\mathbf{X}_{\nu,b} = \sqrt{(1 - \rho_\nu^2)P_\nu} \mathbf{u}_{\nu,b} + \sqrt{\frac{1}{2} \rho_\nu^2 P_\nu} (\mathbf{\omega}_{1,b} + \mathbf{\omega}_{2,b}) + \mathbf{a}_{\nu,\ell} \mathbf{\xi}_{\nu,b} + \sum_{j=1}^{\ell-1} b_{\nu,\ell,j} \bar{\mathbf{V}}_{\nu,(b-1)\eta+j},
\]

where \(\mathbf{X}_{\nu,b} \triangleq (X_{\nu,(b-1)n+1}, \ldots, X_{\nu,bn})^\top\).

Notice that if the parameters \((\mathbf{a}_1, \mathbf{a}_2, \mathbf{B}_1, \mathbf{B}_2)\) satisfy the power constraints (21) and (22) for transmit powers \((P_1 - P_1')\) and \((P_2 - P_2')\), noise variance \(N\), and feedback-noise covariance matrix \(\mathbf{K}_{W_1W_2}\), then the input sequences satisfy the power constraints (3) with arbitrary high probability.

We next consider the encodings in the last Block \((B+1)\), where the two transmitters send information about the pairs \(\{(M_{1,\text{CL},(B-1)\eta+1}, M_{2,\text{CL},(B-1)\eta+1}), \ldots, (M_{1,\text{CL},B\eta}, M_{2,\text{CL},B\eta})\}\). We consider a fixed subblock \(b \in \{B\eta + 1, \ldots, (B+1)\eta\}\). The transmitters send their channel inputs in this last block \((B+1)\) as follows. Given \(M_{1,\text{CL},b-\eta} = m_{1,\text{CL},b-\eta}\) and \(M_{2,\text{CL},b-\eta} = m_{2,\text{CL},b-\eta}\), both transmitters choose the codewords \(\mathbf{\omega}_{1,b}(M_{1,\text{CL},b-\eta})\), and \(\mathbf{\omega}_{2,b}(M_{2,\text{CL},b-\eta})\) from the corresponding codebooks and send a linear combination of the chosen codewords over the channel. Thus, the signal transmitted by Transmitter \(\nu\) in Subblock \(b\) can be described as

\[
\mathbf{X}_{\nu,b} = \sqrt{\frac{1}{2} \rho_\nu^2 P_\nu} (\mathbf{\omega}_{1,b} + \mathbf{\omega}_{2,b}), \quad 1 \in \{1, 2\},
\]

where

\[
\mathbf{X}_{\nu,b} \triangleq (X_{\nu,(b-1)n+1}, \ldots, X_{\nu,bn})^\top.
\]
We next describe the decoding at Transmitter 2; the decoding at Transmitter 1 is performed similarly and therefore omitted; and the decoding at the receiver will be described later on.

After each subblock \( \tilde{b} \in \{1, \ldots, B \eta \} \) Transmitter 2 decodes Message \( M_{1, \text{ICL}, \tilde{b}} \). We consider a fixed Subblock \( \tilde{b} \in \{1, \ldots, B \eta \} \) and define \( b \in \{1, \ldots, B \} \) and \( \ell \in \{1, \ldots, \eta \} \) so that \( \tilde{b} = (b - 1) \eta + \ell \). Before describing the decoding of Message \( M_{1, \text{ICL}, \tilde{b}} \) at the end of this paragraph, we notice the following. After Subblock \( \tilde{b} \), Transmitter 2 observed the feedback vectors \( \tilde{V}_{2,(b-1)\eta+\ell}, \ldots, \tilde{V}_{2,(b-1)\eta+\ell+\ell} \) and is additionally cognizant of Messages \( M_{2, \text{ICS}, b}, \{M_{2, \text{ICL}, (b-1)\eta+1}, \ldots, M_{2, \text{ICL}, (b-1)\eta+\ell} \} \), and (assuming its previous decoding steps were successful) of Messages \( \{M_{1, \text{ICL}, (b-1)\eta+1}, \ldots, M_{1, \text{ICL}, (b-1)\eta+\ell-1} \} \). It can therefore reconstruct the sequences produced to encode these messages. Moreover, Transmitter 2 can estimate Transmitter 1’s feedback outputs \( \tilde{V}_{1,(b-1)\eta+1}, \ldots, \tilde{V}_{1,(b-1)\eta+\ell} \), (even though it cannot reconstruct them because it is incognizant of the feedback noises). By subtracting the reconstructed sequences and the estimated sequence from its feedback outputs Transmitter 2 can thus compute the \( n \)-dimensional vectors \( \tilde{\mathbf{N}}_{2,(b-1)\eta+1}, \ldots, \tilde{\mathbf{N}}_{2,(b-1)\eta+\ell-1} \) and \( \tilde{\mathbf{V}}_{2,(b-1)\eta+\ell} \), which are defined as:

\[
\tilde{\mathbf{V}}_{2,(b-1)\eta+\ell} \triangleq \frac{1}{\sqrt{1 - \rho_2^2}} \mathbf{U}_{2,(b-1)\eta+\ell} - \left( \frac{1}{2} \rho_2^2 \mathbf{P}_1' + \frac{1}{2} \rho_2^2 \mathbf{P}_2' \right) \left( \mathbf{\Omega}_{1,(b-1)\eta+\ell} + \mathbf{\Omega}_{2,(b-1)\eta+\ell} \right) - a_{2,\ell} \mathbf{\Xi}_{2,\tilde{b}} - \sum_{j=1}^{\ell-1} (b_{1,\ell,j} + b_{2,\ell,j}) \tilde{\mathbf{V}}_{2,(b-1)\eta+j} + \sqrt{1 - \rho_1^2} \mathbf{P}_1' \mathbf{U}_{1,(b-1)\eta+\ell} + a_{1,\ell} \mathbf{\Xi}_{1,\tilde{b}} + \sum_{j=1}^{\ell-1} b_{1,\ell,j} (\mathbf{W}_{1,(b-1)\eta+j} - \mathbf{W}_{2,(b-1)\eta+j}) + \mathbf{Z}_{(b-1)\eta+\ell} + \mathbf{W}_{2,(b-1)\eta+\ell},
\]

where \( \mathbf{Z}_{(b-1)\eta+\ell} \triangleq (Z_{1,(b-1)\eta+\ell+1}, \ldots, Z_{\eta,(b-1)\eta+\ell})' \); and for \( \tilde{b}' = (b - 1) + \ell' \) and \( \ell' \in \{1, \ldots, \ell - 1\} \):

\[
\tilde{\mathbf{N}}_{2,\tilde{b}'} \triangleq \mathbf{V}_{2,\tilde{b}'} - \left( \frac{1}{2} \rho_2^2 \mathbf{P}_1' + \frac{1}{2} \rho_2^2 \mathbf{P}_2' \right) \left( \mathbf{\Omega}_{1,\tilde{b}'} + \mathbf{\Omega}_{2,\tilde{b}'} \right) - a_{2,\ell'} \mathbf{\Xi}_{2,\tilde{b}} - \sum_{j=1}^{\ell'-1} (b_{1,\ell',j} + b_{2,\ell',j}) \tilde{\mathbf{V}}_{2,(b-1)\eta+j} + \sqrt{1 - \rho_1^2} \mathbf{P}_1' \mathbf{U}_{1,\tilde{b}'} + a_{1,\ell'} \mathbf{\Xi}_{1,\tilde{b}'} + \sum_{j=1}^{\ell'-1} b_{1,\ell',j} (\mathbf{W}_{1,(b-1)\eta+j} - \mathbf{W}_{2,(b-1)\eta+j}) + \mathbf{Z}_{\tilde{b}'} + \mathbf{W}_{2,\tilde{b}'};
\]

where \( \mathbf{Z}_{\tilde{b}'} \triangleq (Z_{(b-1)\eta+1}, \ldots, Z_{\eta\tilde{b}'})' \). Transmitter 2 finally decodes Message \( M_{1, \text{ICL}, \tilde{b}} \) based on \( \tilde{\mathbf{N}}_{2,(b-1)\eta+1}, \ldots, \tilde{\mathbf{N}}_{2,(b-1)\eta+\ell-1} \), and \( \tilde{\mathbf{V}}_{2,(b-1)\eta+\ell} \) using an optimal decoder for a single-input antenna/multi-output antenna Gaussian channel with correlated but temporally-white noise sequences.

We next describe the decoding at the receiver. We first consider the decoding of the pair \( (M_{1, \text{ICL}, \tilde{b}}, M_{2, \text{ICL}, \tilde{b}}) \) after a fixed subblock \( \tilde{b} \in \{\eta + 1, \ldots, (B + 1)\eta\} \). Define
where \( Y \) and \( \bar{Y} \) are defined as above. Through the "cleaned" outputs \( \bar{Y} \), the receiver can reconstruct the sequences produced to encode these messages and subtract them from the output signal. Thus, the receiver can compute for \( b' \in \{b-1, b\} \) and \( \ell' \in \{1, \ldots, \ell - 1\} \) the "cleaned" output vector

\[
\bar{Y}_{(b')\eta + \ell'} \triangleq (Y_{(b')\eta + \ell'} - \bar{Y}_{(b')\eta + \ell'}) = Y_{(b')\eta + \ell'} - \sqrt{(1 - \rho^2_1)P_1^\prime \mathbf{U}_1,(b')\eta + \ell'} - \sqrt{(1 - \rho^2_2)P_2^\prime \mathbf{U}_2,(b')\eta + \ell'}
\]

\[
- \left( \sqrt{\frac{1}{2}\rho^2_1 + \frac{1}{2}\rho^2_2} \right) \left( \mathbf{\Omega}_1,(b')\eta + \ell' + \mathbf{\Omega}_2,(b')\eta + \ell' \right) \]

\[
= a_1,\ell \bar{\Xi}_{1,b} + a_2,\ell \bar{\Xi}_{2,b} + \sum_{j=1}^{\ell' - 1} (b_1,\ell' j \bar{\bar{V}}_{1,(b')\eta + j} + b_2,\ell' j \bar{\bar{V}}_{2,(b')\eta + j}) + \mathbf{Z}_{(b')\eta + \ell'},
\]

where \( Y_{(b')\eta + \ell'} \triangleq (Y_{(b')\eta + \ell'} - \bar{Y}_{(b')\eta + \ell'})^{\top} \), and it can compute

\[
\bar{Y}_{(b)\eta + \ell} \triangleq \left( Y_{(b)\eta + \ell} - \bar{Y}_{(b)\eta + \ell} \right) = \sqrt{(1 - \rho^2_1)P_1^\prime \mathbf{U}_1,(b)\eta + \ell} + \sqrt{(1 - \rho^2_2)P_2^\prime \mathbf{U}_2,(b)\eta + \ell} + a_1,\ell \bar{\Xi}_{1,b} + a_2,\ell \bar{\Xi}_{2,b} + \sum_{j=1}^{\ell - 1} (b_1,\ell j \bar{\bar{V}}_{1,(b)\eta + j} + b_2,\ell j \bar{\bar{V}}_{2,(b)\eta + j}) + \mathbf{Z}_{(b)\eta + \ell},
\]

where \( Y_{(b)\eta + \ell} \triangleq (Y_{(b)\eta + \ell} - \bar{Y}_{(b)\eta + \ell})^{\top} \). Notice that the "cleaned" output vector \( \bar{Y}_{(b')\eta + \ell'} \) equals the difference \( \left( \bar{V}_{1,(b')\eta + \ell'} - \mathbf{W}_{1,(b')\eta + \ell'} \right) \). Notice further, that even though the "cleaned" outputs \( \bar{Y}_{(b)\eta + \ell} \) and \( \bar{Y}_{(b')\eta + \ell'} \) do not depend on the pair \( (M_{1,CL,b}, M_{2,CL,b}) \), they are correlated with the noise sequences corrupting \( \bar{Y}_{(b)\eta + \ell} \) and \( \bar{Y}_{(b')\eta + \ell'} \) and should be taken into account by the receiver when decoding \( (M_{1,CL,b}, M_{2,CL,b}) \). Thus, the receiver should decode the pair \( (M_{1,CL,b}, M_{2,CL,b}) \) based on the vectors \( \bar{Y}_{(b)\eta + \ell}, \ldots, \bar{Y}_{(b)\eta + \ell - 1}, \bar{Y}_{(b')\eta + \ell}, \ldots, \bar{Y}_{(b')\eta + \ell - 1}, \bar{Y}_{(b')\eta + \ell} \). To this end, the receiver first partly "decorrelates" the vectors by computing

\[
\bar{Y}_{(b')\eta + \ell} \triangleq \left( Y_{(b')\eta + \ell} - \sum_{j=1}^{\ell - 1} (b_1,\ell j + b_2,\ell j) \bar{Y}_{(b')\eta + j} \right),
\]

\[
= \sqrt{(1 - \rho^2_1)P_1^\prime \mathbf{U}_1,(b')\eta + \ell} + \sqrt{(1 - \rho^2_2)P_2^\prime \mathbf{U}_2,(b')\eta + \ell}
\]

\[
+ \left( \sqrt{\frac{1}{2}\rho^2_1 + \frac{1}{2}\rho^2_2} \right) \left( \mathbf{\Omega}_1,(b')\eta + \ell + \mathbf{\Omega}_2,(b')\eta + \ell \right) + a_1,\ell \bar{\Xi}_{1,b} + a_2,\ell \bar{\Xi}_{2,b}
\]

\[
+ \sum_{j=1}^{\ell - 1} (b_1,\ell j \mathbf{W}_{1,(b')\eta + j} + b_2,\ell j \mathbf{W}_{2,(b')\eta + j}) + \mathbf{Z}_{(b')\eta + \ell},
\]
The receiver then decodes the pair of messages \((M_{1,\text{ICL},b}, M_{2,\text{ICL},b})\) based on \(\hat{Y}_{(b-2)\eta+\ell}^{(2)}\) and \(\hat{Y}_{(b)\eta+\ell}^{(3)}\) using an optimal decoder for a 2-input/2-output antenna Gaussian MAC with temporally-white noise sequences correlated across antennas.

After the decoding of Messages \(\{(M_{1,\text{ICL},b}, M_{2,\text{ICL},b})\}_{b=1}^{B}\) the receiver decodes Messages \(\{(M_{1,\text{ICS},b}, M_{2,\text{ICS},b})\}_{b=1}^{B}\). To this end, it first reverses the interleaving introduced by the modified inner encoders on the “cleaned” output vectors \(\hat{Y}_{1}, \ldots, \hat{Y}_{B}\). That is, for \(b \in \{1, \ldots, B\}\), it constructs the \(\eta m\)-dimensional vector

\[
\mathbf{Y}_{\text{DeInt},b} \triangleq (\hat{Y}_{(b-1)\eta+1,1}, \ldots, \hat{Y}_{b_1}, \hat{Y}_{(b-1)\eta+1,2}, \ldots, \hat{Y}_{b_n}, \hat{Y}_{(b-1)\eta+1,2}, \ldots, \hat{Y}_{b_1})^T,
\]

where \(\hat{Y}_{b_i}\) denotes the \(i\)-th entry of vector \(\hat{Y}_{b}\). It then decodes Messages \((M_{1,\text{ICS},b}, M_{2,\text{ICS},b})\) applying inner and outer decoder of the concatenated scheme to the vector \(\mathbf{Y}_{\text{DeInt},b}\).

### I.1 Noisy and Perfect Partial Feedback

The proposed extension can also be applied in settings with noisy or perfect partial feedback, if \(B_1\) is set to the all-zero matrix and if Carleial’s scheme for noisy or perfect feedback is applied. Accordingly, our scheme should be modified so that there is no decoding taking place at Transmitter 1. Therefore, in (248) and (249) the term \(\sqrt{\frac{1}{2} \rho_{\nu}^2 P_{s}(\omega_{1,b} + \omega_{2,b})}\) should be replaced by \(\sqrt{\rho_{\nu}^2 P_{s}(\omega_{1,b})}\) for \(\nu \in \{1, 2\}\).

Notice that—as in the second extension—for perfect partial feedback the various vectors computed for the decodings at Transmitter 2 and for the decodings at the receiver have uncorrelated noise components. Therefore, without loss in optimality, Transmitter 2 and the receiver can use optimal decoders for Gaussian multi-input antenna/multi-output antenna channels with uncorrelated white noise sequences.
References


