Further Results on the SNR Exponent of Hybrid Digital Analog Space Time Codes

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***Abstract*— We consider the transmission of a real i.i.d. Gaussian source over a quasi-static MIMO fading channel. The relevant performance criterion is end-to-end average quadratic distortion D versus channel SNR, for given spectral efficiency η, defined as the ratio of the source bandwidth over the channel bandwidth. In the limit of high-resolution (vanishing distortion) for high-SNR, we define the distortion SNR exponent $a(\eta)$ for a family of source-channel coding schemes as the $a = -\log D / \log \mathsf{snr}$. In our recent paper [1], we have computed upper and lower bounds on $a(\eta)$ for the MIMO channel. Here, we first extend these to the parallel channel. Using these results, we propose a hybrid digital analog scheme that achieves the same exponent $a(\eta)$ as the schemes in [1] for the 2x2 MIMO channel for $\eta < 2M$.**

I. Space-time coding and end-to-end distortion

A. Background

Consider the standard MIMO block-fading channel [2] expressed by

$$\mathbf{y}_t = \sqrt{\frac{\rho}{M}}\mathbf{H}\mathbf{x}_t + \mathbf{w}_t, \quad t = 1, \ldots, T \tag{1}$$

where: T is the duration (in channel uses) of the transmitted block; $\mathbf{H} \in \mathbb{C}^{N\times M}$ is the channel matrix, assumed to be constant for all $t = 1, \ldots, T$ but random with i.i.d. elements $h_{i,j} \sim \mathcal{CN}(0, 1)$ (Rayleigh independent fading); $\mathbf{x}_t$ is the transmitted signal at time t; the transmitted codeword, $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_T]$, is normalized such that $\mathrm{tr}(\mathbb{E}[\mathbf{X}^{\mathsf{H}}\mathbf{X}]) \leq MT$; ρ denotes the Signal-to-Noise Ratio (SNR), defined as the average received signal energy per receiving antenna to the noise variance.

Consider a family of space-time coding schemes $\{\mathcal{C}_{r_c}(\rho)\}$ of rate $R_c = r_c \log \rho$, where $\mathcal{C}_{r_c}(\rho)$ denotes the scheme that operates at SNR ρ and the rate pre-log factor $r_c \geq 0$ is referred to as *multiplexing gain*. The SNR exponent of the family is defined as the limit

$$d(r_c) = -\frac{\log P_e(\rho)}{\log \rho} \tag{2}$$

The SNR exponent *of the channel*, $d^\star(r_c)$ is the supremum, over all possible coding families, of $d(r_c)$. In [3], $d^\star(r_c)$ was fully determined as the piecewise linear function joining the points $(r_c = j, d^\star(j) = (N - j)(M - j))$, for $r_c \in [0, M]$ ($d^\star(r_c) = 0$ for $r_c > M$). For any function the SNR $f(\rho)$, we define $\zeta(f) = -\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log \rho}$ and we use the notation $f(\rho) \doteq \rho^{-\zeta(f)}$ to denote that $f(\rho)$ decreases as $\rho^{-\zeta(f)}$ at high SNRs. This also means that $f(\rho) \doteq g(\rho)$, if $\zeta(f) = \zeta(g)$.

B. Problem statement

We wish to transmit an analog source of bandwidth W_s samples per second over the MIMO block-fading channel (1). To be precise, we consider an i.i.d. real Gaussian source $\sim \mathcal{N}(0, 1)$. A K-to-$(M \times T)$ source-channel encoder is a mapping $\mathcal{SC} : \mathbb{R}^K \to \mathbb{C}^{M\times T}$ that maps source blocks $\mathbf{s} \in \mathbb{R}^K$ onto channel codewords $\mathbf{X}$. The corresponding source-channel decoder is a mapping $\mathbb{C}^{M\times T} \to \mathbb{R}^K$ that maps the channel output $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_T]$ into an *approximation* $\widetilde{\mathbf{s}}$ of the source block. The average quadratic distortion is defined by

$$D(\rho) \triangleq \frac{1}{K}\mathbb{E}[|\mathbf{s} - \widetilde{\mathbf{s}}|^2] \tag{3}$$

where expectaction is with respect to $\mathbf{s}, \mathbf{H}$ and the channel noise. The *spectral efficiency* of the encoder is defined as $\eta = K/T$ source samples per channel use. Letting W_c denote the number of channel uses per second (channel bandwidth) and noticing that $K/W_s = T/W_c$, it follows that $\eta = W_s/W_c$.

Consider a family of source-channel coding schemes $\{\mathcal{SC}_\eta(\rho)\}$ of spectral efficiency η, where $\mathcal{SC}_\eta(\rho)$ denotes the scheme that operates at SNR ρ. We define the *distortion SNR exponent* of the family as the limit

$$a(\eta) = -\frac{\log D(\rho)}{\log \rho} = \zeta(D(\eta)) \tag{4}$$

The distortion SNR exponent *of the source-channel pair* $a^\star(\eta)$ is the supremum, over all possible coding families, of $a(\eta)$. In [1], we obtained upper bounds on $a^*(\eta)$ and proposed hybrid digital analog (HDA) schemes which resulted in achievable lower bounds on $a^*(\eta)$. The main results from [1] are summarized below and then the new results in this paper are discussed.

II. Main results from Previous Work [1]

For the $M \times N$ MIMO channel,

Theorem 1 [Exponent achievable by separation] The distortion SNR exponent

$$a_{\text{sep}}(\eta) = \frac{2(jd^\star(j-1)-(j-1)d^\star(j))}{2+\eta(d^\star(j-1)-d^\star(j))}, \quad \eta \in \left[\frac{2(j-1)}{d^\star(j-1)}, \frac{2j}{d^\star(j)}\right) \tag{5}$$

for $j = 1, \ldots, M$, is achievable by a *tandem source-channel coding scheme* discussed in Section III. □

Theorem 2 [Informed transmitter upper bound] The optimal distortion SNR exponent $a^\star(\eta)$ is upperbounded by

$$a_{\text{ub}}(\eta) = \sum_{i=1}^{M} \min\left\{\frac{2}{\eta}, 2i-1+|M-N|\right\}$$

□

Theorem 3 [Hybrid scheme lower bound] The hybrid digital-analog (HDA) space-time coding scheme of [1], discussed in IV-A achieves the following exponent:

1)

$$a_{\text{hybrid}}(\eta) = 1+\left(\frac{2}{\eta}-\frac{1}{M}\right)\frac{jd^\star(j-1)-(j-1)d^\star(j)-1}{\frac{2}{\eta}-\frac{1}{M}+d^\star(j-1)-d^\star(j)} \tag{6}$$

for

$$\eta \in \left[\frac{2(j-1)M}{Md^\star(j-1)-M+j-1}, \frac{2jM}{Md^\star(j)-M+j}\right), i \leq j \leq M-1$$

2)

$$a_{\text{hybrid}}(\eta) = 1+\left(\frac{2}{\eta}-\frac{1}{M}\right)\frac{M(N-M+1)-1}{\frac{2}{\eta}-\frac{1}{M}+N-M+1}, \tag{7}$$

for

$$\eta \in \left[\frac{2M(M-1)}{M(N-M)+M-1}, 2M\right]$$

3)

$$a_{\text{hybrid}}(\eta) = \frac{2M}{\eta}, \quad \eta \geq 2M \tag{8}$$

In this paper, we extend these results from [1] in the following ways - We first derive similar bounds as in Theorem 3 for the parallel channel and show that the HDA scheme in [1] is optimal for the parallel channel for $\eta \geq 2M$. This results is given in Theorem 4. Then, for $M = 2$, we show that the same bounds as in Theorem 4 can be achieved for $\eta \leq 2M$ using an alternate HDA scheme discussed in Section V which uses the idea of a matched tandem encoder from [4]. Using the results in Section V, in Section VI, we propose a new HDA space-time coding scheme which achieves the same achievable bound as in Theorem 3 for the 2x2 MIMO channel for $\eta \leq 2M$. Finally, we present simulation results which show that the predicted slope can be nearly achieved although the required SNRs are somewhat large.

III. Tandem and Matched Tandem Source-Channel Coding Schemes

In a tandem source-channel coding scheme (or separated scheme), the source $\mathbf{s}$ is first encoded by a quantizer $\mathcal{Q}$, of rate R_s nats/sample which produces an index $u = \mathcal{Q}(\mathbf{s})$. The index u is encoded by a space-time code of rate R_c nats/use. At the receiver, the space-time decoder produces a decision $\hat{u}$ on the index u and the reconstruction point is given by $\widetilde{\mathbf{s}} = \mathcal{R}(\hat{u})$, where $\mathcal{R}$ denotes the reconstruction operation. Assuming the presence of an ideal error detection scheme, if $\hat{u} \neq u$, the reconstruction point is set to be the zero vector.

Let $D_{\mathcal{Q}}(R_s)$ denote the quantizer distortion-rate function and P_e denote the error probability of the space-time code. It can be shown (see [5]) that the end-to-end distortion achievable by a tandem scheme is upperbounded by

$$\begin{aligned} D_{\text{sep}}(R_s) &\leq D_{\text{noerror}} + D_{\text{error}} \\ &\doteq D_{\mathcal{Q}}(R_s) + \kappa P_e \end{aligned} \tag{9}$$

where κ is a constant that does not depend on the SNR. In [7] and [1], such a separation based scheme was considered for the MIMO channel. For long block lengths, it can be shown that the probability of error and probability of outage decay with SNR in the same exponential order and, $\zeta(D_{\text{error}}) = \zeta(D_{\text{outage}})$. Hence,

$$\begin{aligned} D_{\text{sep}}(R_s) &\leq D_{\text{nooutage}} + D_{\text{outage}} \\ &\doteq D_{\mathcal{Q}}(R_s) + \kappa P_{\text{outage}} \end{aligned} \tag{10}$$

and R_s and R_c were optimized so as to obtain, the largest $a(\eta)$, which is explicitly given in Theorem 1. The separated scheme is clearly suboptimal when the instantaneous channel capacity is not equal to R_c. Particularly, when space-time decoder produces an erroneous estimate, the distortion is assumed to be κ which is independent of the SNR, i.e. $\zeta(D_{\text{outage}}) = \zeta(P_{\text{outage}})$. In this paper, we construct HDA schemes for which $\zeta(D_{\text{outage}}) > \zeta(P_{\text{outage}})$. This is the intuition behind why the proposed schemes outperform the separated scheme.

A key ingredient in the design is that of a matched tandem encoder originally proposed by Mittal and Phamdo in [4]. For a single input channel with $T \geq K$, a matched tandem encoder is defined as one for which the first K components of the transmitted codeword X, $X_1^k = \delta\mathcal{R}(u)$, where δ is a scaling factor. That is, the quantization point appears systematically (up to a scaling) in the codeword. In [4], Mittal and Phamdo showed the existence of matched tandem encoders that are optimal source and channel codes for the Gaussian source and channel. The advantage of a matched tandem encoder is that when $\hat{u} \neq u$, a minimum mean squared error (MMSE) estimate of $\mathcal{R}(u)$ can be formed directly from the channel observations and used as $\tilde{\mathbf{s}}$.

IV. PARALLEL CHANNEL

Consider the same set up as in Section I-A but with a set of M parallel Gaussian channels as the communication channel. The received vector at time t can be modelled as

$$\mathbf{y}_t = \sqrt{\frac{\rho}{M}}\mathbf{H}\mathbf{x}_t + \mathbf{w}_t, \quad t = 1, \ldots, T \qquad (11)$$

where the diagonal elements of $\mathbf{H}$, $h_{ii} \sim \mathcal{CN}(0,1)$ and the off diagonal elements are zeros. We use $g_i = |h_{ii}|^2$ to denote the squared magnitude of the channel gain. The exponent achievable by separation, $a_{\text{sep,parallel}}(\eta)$ and the informed transmitter upper bound $a_{\text{ub,parallel}}(\eta)$ can be computed as in [1] and seen to be $a_{\text{ub,parallel}}(\eta) = 2M/\eta$, for $\eta \geq 2M$.

An achievable lower bound for the parallel channel is given by

Theorem 4 [Hybrid scheme lower bound] The hybrid digital-analog (HDA) space-time coding scheme of [1] achieves the following exponent :

$$a_{\text{hybrid,parallel}}(\eta) = \frac{2M/\eta}{\frac{2}{\eta} - \frac{1}{M} + 1}, \text{for} \quad \eta < 2M \qquad (12)$$

$$= \frac{2M}{\eta}, \quad \text{for} \quad \eta \geq 2M \qquad (13)$$

□

The proof for this also follows closely the approach in [1]. As in [3], we use the usual change of variable $\alpha_i = -\log g_i / \log \rho$. For the parallel channel case, since g_i's are exponentially distributed and independent, the distribution of $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_M]$ is given by

$$P_{\boldsymbol{\alpha}}(\alpha_1, \ldots, \alpha_M) = (\log \rho)^M \prod_{i=1}^{M} \rho^{-\alpha_i} \exp\left[-\sum \rho^{-\alpha_i}\right].$$

Also, note that the optimal diversity-multiplexing tradeoff is given by $d_p^*(r_c) = M - r_c$. Using these we can separately consider the two HDA schemes for $\eta \leq 2M$ and $\eta \geq 2M$. Since the scheme and analysis are identical to that in [1], the details are omitted. The description of the scheme and outline are presented to make the proofs in the following sections easier to understand.

A. Case $\eta \leq 2M$

The HDA scheme for this case is given as follows. The source block $\mathbf{s}$ is input to a K-to-$(M \times T_d)$ tandem encoder. The corresponding codeword is denoted by $\mathbf{X}^{(d)}$. Let $\widehat{\mathbf{s}}$ and $\mathbf{e} = \mathbf{s} - \widehat{\mathbf{s}}$ denote the quantized source block and the quantization error vector, respectively. Then, $\mathbf{e}$ is multiplexed on to the M parallel channels. Finally, the transmitted codeword is given by the *concatenation* $\mathbf{X} = [\mathbf{X}^{(d)}, \mathbf{X}^{(a)}]$. The resulting block length is $T = T_d + K/(2M)$.

The receiver first tries to decode $\mathbf{X}^{(d)}$. If decoding is successful, the source block approximation is obtained as

$$\widetilde{\mathbf{s}} = \widehat{\mathbf{s}} + \widetilde{\mathbf{e}}$$

where $\widetilde{\mathbf{e}}$ is the linear MMSE estimate of $\mathbf{e}$ from the channel output corresponding to $\mathbf{X}^{(a)}$. If decoding is unsuccessful, the output is trivially $\widetilde{\mathbf{s}} = \mathbf{0}$.

For sufficiently large K, decoding errors can be detected with arbitrary large probability [6] and the error probability can be replaced (as far as the exponential order is concerned) by the information outage probability $P(\mathcal{A})$, where the information outage event at coding rate R_c is defined by [3]

$$\mathcal{A} = \left\{\mathbf{H} \in \mathbb{C}^{N \times M} \ : \ \log\det\left(\mathbf{I} + \frac{\rho}{M}\mathbf{H}\mathbf{H}^{\mathsf{H}}\right) \leq R_c\right\} \qquad (14)$$

Letting again $R_s = r_s \log \rho$ be the quantizer rate and $R_c = r_c \log \rho$ be the space-time coding rate, we have $r_c = \frac{K}{T_d} r_s$ and $\eta = \frac{K}{T_d + K/(2M)}$ which yields

$$r_s = \left(\frac{1}{\eta} - \frac{1}{2M}\right) r_c \qquad (15)$$

Let $\mathsf{mmse}(\mathbf{H})$ denote the MMSE in estimating $\mathbf{e}$. Then, we can write the overall average distortion of the hybrid scheme as

$$D_{\text{hybrid}}(\rho) \doteq \int_{\mathcal{A}^c} \mathsf{mmse}(\mathbf{H}) dF(\mathbf{H}) + \kappa P(\mathcal{A}) \quad (16)$$

Assuming (asymptotically) optimal quantization, that achieves distortion ρ^{-2r_s}, it is not difficult to show that

$$\mathsf{mmse}(\mathbf{H}) = \frac{\rho^{-2r_s}}{M} \sum_{i=1}^{M} \frac{1}{1 + \frac{\rho}{M} g_i} \qquad (17)$$

We can show that the exponents in the first and second terms of (16) are $d_p^*(r_c) = M - r_c$ and $(2r_s + 1)$, is the optimal diversity order for a multiplexing rate of r_c for the parallel channel. Equating the two exponent gives the result in (12).

B. Case $\eta > 2M$

The HDA scheme in this case is given as follows. The source block $\mathbf{s}$ is split into two subblocks, $\mathbf{s}^{(d)}$ of length $K(1 - 2M/\eta)$ and $\mathbf{s}^{(a)}$ of length $K(2M/\eta)$. The subblock $\mathbf{s}^{(a)}$ is multiplexed on to the M parallel channels. The subblock $\mathbf{s}^{(d)}$ is mapped onto the codeword $\mathbf{X}^{(d)} \in \mathbb{C}^{M \times T}$ by a tandem encoder. Finally, the transmitted codeword is obtained by superposition of $\mathbf{X}^{(a)}$ and $\mathbf{X}^{(d)}$ as

$$\mathbf{X} = \sqrt{\beta}\mathbf{X}^{(a)} + \sqrt{1-\beta}\mathbf{X}^{(d)} \qquad (18)$$

where $\beta \in [0,1]$ is a power allocation factor.

The receiver first tries to decode $\mathbf{X}^{(d)}$ by treating $\mathbf{X}^{(a)}$ as independent noise. If decoding is successful, then $\mathbf{X}^{(d)}$ is subtracted from the received signal and $\mathbf{s}^{(a)}$ is estimated by using a linear MMSE estimator from the clean signal. The reconstructed source block in this case is $\widetilde{\mathbf{s}} = [\widehat{\mathbf{s}}^{(d)}, \widetilde{\mathbf{s}}^{(a)}]$, where $\widehat{\mathbf{s}}^{(d)}$ denotes the quantized version of $\mathbf{s}^{(d)}$ and $\widetilde{\mathbf{s}}^{(a)}$ denotes the MMSE estimate of $\mathbf{s}^{(a)}$. If decoding of the digital part is not successful, the receiver produces an MMSE estimate $\widetilde{\mathbf{s}}^{(a)}$ of $\mathbf{s}^{(a)}$ by treating $\mathbf{X}^{(d)}$ as independent noise, and the reconstructed source block in this case is $\widetilde{\mathbf{s}} = [\mathbf{0}, \widetilde{\mathbf{s}}^{(a)}]$.

The analysis of this scheme proceeds along the same lines as in [1] and by letting $\beta = \rho^{-(\eta-2M)r_s}$ we get the result in (13).

V. Matched Tandem Encoding and Repetition Coding

We now analyze a scheme that does not use an optimal code for the parallel channel, but uses a matched tandem encoder followed by a repetition code. The interesting result is that for $M = 2$ and for $\eta \leq 2M$, the same exponents as in (6) can be obtained. We first consider the case an arbitrary M and $\eta < 2M$. The source sequence $\mathbf{s}$ is first encoded using a matched tandem encoder to produce $\mathbf{x} \in C^{1\times T}$, where the source coding rate is $r_s \log\rho$ and transmission rate is $r_c \log\rho$. The same $\mathbf{x}$ is repeated and transmitted over the M parallel channels. In addition to this, the quantization error is multiplexed over the M parallel channels as before.

If the channel is not in outage, then $\mathbf{x}$ is decoded and the quantization point is recovered. An MMSE estimate of the quantization error is formed. When the channel is in outage, an MMSE estimate of $\mathcal{R}(u)$, given by $\tilde{\mathcal{R}}(u)$ is formed and used as $\tilde{\mathbf{s}}$. The overall distortion will then be

$$E[|\mathbf{s}-\hat{\mathcal{R}}(u)|^2] = E[|\mathbf{s}-\mathcal{R}(u)|^2] + E[|\mathcal{R}(u)-\hat{\mathcal{R}}(u)|^2].$$

This follows from the orthogonality of the quantization error and the source for an optimal quantizer. The first term is the quantization error D_Q and let us denote the second term by D_{mmse}. Let $\mathcal{A}$ denote the outage region. That is,

$$\mathcal{A} = \{g_1, \ldots, g_M : \log\left(1 + \left(\sum g_i\right)\rho\right) < r_c \log\rho\}$$

Using the high SNR approximation, this region is given by

$$\mathcal{A} = \{g_1, \ldots, g_M : \sum g_i \leq \rho^{-(1-r_c)^+}\}$$

Since $\sum g_i$ is a Chi-squared random variable with $2M$ degrees of freedom it follows that

$$P_{\text{outage}} = P(\mathcal{A}) = \int_{\mathcal{A}} 1\; d\mathbf{g} \doteq \rho^{-M(1-r_c)^+}$$

The overall distortion D is then given by,

$$D = \int_{\mathcal{A}} D_Q d\mathbf{g} + \int_{\mathcal{A}} \frac{\kappa_1}{1+\sum g_i} d\mathbf{g} + \int_{\mathcal{A}^c} D_Q \frac{1}{1+\sum g_i} d\mathbf{g}$$

where κ_1 is a constant independent of ρ. Let the three terms decrease with ρ as ρ^{-a_1}, ρ^{-a_2} and ρ^{-a_3}. Since $D_Q = \rho^{-2r_s}$, $a_1 = 2r_s + M(1-r_c)^+$, $a_3 = 2r_s + 1$. To determine a_2, notice that

$$\int_{\mathcal{A}} \frac{\kappa_1}{1+\rho^{-(1-r_c)^+}\rho} d\mathbf{g} \leq \int_{\mathcal{A}} \frac{\kappa_1}{1+\sum g_i \rho} d\mathbf{g} \leq \int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} d\mathbf{g}$$

The lower bound follows from the fact that for $\mathbf{g} \in \mathcal{A}$, $\sum_i g_i \leq \rho^{-(1-r_c)^+}$ and the upper bound is true since g_i's are all positive. Using the high SNR approximation that $1 + \rho^{1-(1-r_c)^+} \doteq \rho^{1-(1-r_c)^+}$, the negative exponent of the lower bound can be readily seen to be $1 + (M-1)(1-r_c)^+$. The upper bound can be written as

$$\int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} d\mathbf{g} = \int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} dg_1 \int_{\mathcal{A}} 1\; dg_2 \ldots dg_M.$$

We will now show that $\int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} dg_1 \doteq \rho^{-1}$. We know from results in [1], that for the 1x1 channel,

$$\int_0^\infty \frac{\kappa_1}{1+g_1\rho} dg_1 \doteq \rho^{-1}. \tag{19}$$

Since,

$$\int_0^\infty \frac{\kappa_1}{1+g_1\rho} dg_1 = \int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} dg_1 + \int_{\mathcal{A}^c} \frac{\kappa_1}{1+g_1\rho} dg_1,$$

then it must be that the negative rate of decay of each of the above terms must be at least 1. However, an upper bound on the rate of decay can be obtained by noting that for $\mathbf{g} \in \mathcal{A}$,

$$\int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} dg_1 \leq \int_{\mathcal{A}} \frac{\kappa_1}{\rho^{-(1-r_c)^+}\rho} dg_1 \doteq \rho^{-1}$$

Hence,

$$\int_{\mathcal{A}} \frac{\kappa_1}{1+g_1\rho} dg_1 \doteq \rho^{-1}.$$

Since $\sum_{i=2}^M g_i$ is Chi-squared distributed with $2(M-1)$ degrees of freedom, $a_2 = 1 + (M-1)(1-r_c)^+)$. Since $r_s = \left(\frac{1}{\eta} - \frac{1}{2M}\right) r_c$, it can be seen that $a_1 \geq a_2$, for $\eta \leq 2M$. Hence the overall exponent $a_{\text{hybrid}}(\eta) = \min(a_2, a_3)$. The optimal value of r_c is obtained by setting $a_2 = a_3$ which results in the exponent being equal to

$$a_{\text{hybrid}}(\eta) = \frac{M\left(\frac{2}{\eta} - \frac{1}{M}\right) + M - 1}{\left(\frac{2}{\eta} - \frac{1}{M}\right) + M - 1}. \tag{20}$$

For $M = 2$, it can be seen that this is the same as that in (6) for all $\eta < 2M$. For $M > 2$, the exponent is worse than that in (6).

VI. Extension to the 2x2 MIMO Channel

In this section, we show how to obtain the same exponent as in (6) for $M = N = 2$ and $\eta \leq 2M$ using a different scheme than in [1]. The source symbols $\mathbf{s}$ are first encoded by a matched tandem encoder in to the codeword $\mathbf{x}$ such that the source coding rate $r_s \log\rho$ and transmission rate $r_c \log\rho$. $\mathbf{x}$ is then encoded using an Alamouti code which constitutes the digital part. In addition, the quantization error is transmitted using analog space-time coding as in the scheme of [1].

At the receiver, the digital part is decoded (if the channel is not in outage) and an MMSE estimate of the error signal is formed. If the digital part cannot be decoded, since a matched tandem encoder is used, an MMSE estimate of $\mathcal{R}(u)$ is obtained and used as the reconstruction of the source. We now show that the exponent of this scheme is the same as that in (6).

Let the 2×2 channel be $H = \begin{bmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{bmatrix}$ and let $q_1 = |h_{1,1}|^2 + |h_{2,1}|^2$ and $q_2 = |h_{1,2}|^2 + |h_{2,2}|^2$. First, note that the 2x2 channel with the Alamouti code can be decomposed in to two parallel channels with equivalent gains q_1 and q_2 where the input to the two channels are the same (or, the output of a repetition code). The situation here is similar to that analyzed in the previous section with $M = 2$. The difference is that q_1 and q_2 are independent Chi-squared random variables with 4 degrees of freedom instead of 2 degrees of freedom. Nevertheless, the analysis in the previous section can still be used as the analysis depended only on the distribution of $\sum g_i$. Note that the distribution of $\sum_{i=1}^{2} q_i$ for the MIMO channel is identical to that of $\sum_{i=1}^{4} g_i$ for the parallel channel. Further, it can also be shown that

$$\int_0^\infty \frac{1}{1+q_1\rho}\, dq_1 \doteq \rho^{-1} \tag{21}$$

where q_1 is a Chi-squared random variable with 4 degrees of freedom. Hence, using the same idea as in the previous section, it can be shown that

$$\int_{\mathcal{A}} \frac{1}{1+q_1\rho}\, dq_1 \doteq \rho^{-1} \tag{22}$$

Writing the overall distortion as $D = D_{\text{outage}} + D_{\text{nooutage}}$, it can be shown that $D_{\text{outage}} \doteq \rho^{-(4-3r_c)}$ and $D_{\text{nooutage}} \doteq \rho^{-(2r_s+1)}$. Equating the two exponents and noting that $r_s = \left(\frac{1}{\eta} - \frac{1}{2M}\right)$, we get

$$a_{\text{hybrid}}(\eta) = \frac{\frac{8}{\eta}+1}{\frac{2}{\eta}+\frac{5}{2}} \tag{23}$$

which coincides with (6) for $M = 2$.

VII. Discussion

The main advantage of this scheme is that a space-time code that achieves the optimal diversity multiplexing tradeoff and the associated decoding complexity is not required. However, a matched tandem joint source channel code that achieves the optimal slope of the quantization error and outage probabilities is required. Using the fact that scalar quantization provides the optimal slope at high SNRs [8], a possible way to construct a matched tandem encoder would be to first use scalar quantization to quantize the ith source symbol S_i in to u_i where $\mathcal{R}(u_i) \in \rho^{r_s}$-PAM. Let $[u_{i,1}, \ldots, u_{i,p}, \ldots, u_{i,L}]$, where $L = r_s \log \rho$. The bit sequence $\{u_{i,j}\}$, for all i, j can be encoded using either a bit interleaved coded modulation scheme or a multilevel codng scheme using a rate $2K/T$ low density parity check (LDPC) code and mapped on to a total of T QAM symbols. The encoding and demodulation of the Alamouti code is computationally easy. The main complexity is now in the soft demodulator that produces $p(u_{i,j}) = 1|\mathbf{r})$, where $\mathbf{r}$ is the output of the Alamouti demodulator, which is that of demodulation of QAM signals transmitted over a SISO Gaussian channel. This could be smaller than that of demodulation for a space-time code. The performance of such a scheme remains to be studied in detail.

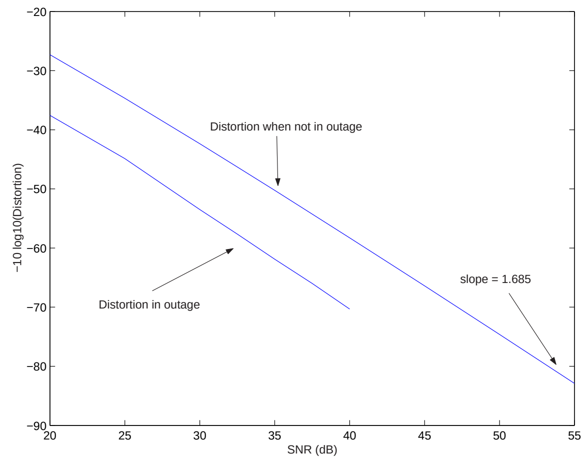

Fig. 1. Simulation results $M = 2, N = 2$

VIII. Simulation Results

The above HDA scheme was simulated for a 2x2 MIMO channel and $\eta = 4/3$, for which the optimal values are $r_s = 0.375$, $r_c = 0.75$, and $a_{hybrid}(\eta) = 1.75$. Figure 1 shows a plot of MSE in dB versus signal to noise ratio. An optimal matched tandem encoder is assumed and, hence, the channel is assumed to be in outage if $\log(1 + (q_1 + q_2)\rho) < 0.75 \log \rho$. The distortion when in outage and when not in outage are plotted separately. It can be seen from the figures that the slope steadily increases and around a channel SNR of 55 dB, reaches 1.685. A higher SNR would be required to reach the optimal value of 1.75. Note this is significantly better than what is achievable through a separation based scheme.

References

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