Separate Source Coding of Correlated Gaussian Remote Sources

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***Abstract*— We consider the separate source coding problem of L correlated Gaussian observations $Y_i, i = 1, 2, \cdots, L$. We consider the case that $Y_i, i = 1, 2, \cdots, L$ satisfy $Y_i = X_i + N_i, i = 1, 2, \cdots, L$ where $X_i, i = 1, 2, \cdots, L$ are L correlated Gaussian random variables and $N_i, i = 1, 2, \cdots L$ are independent additive Gaussian noises also independent of $X_i, i = 1, 2, \cdots, L$. On this coding system the determination problem of the rate distortion region remains open. In our previous work, we derived explicit outer and inner bounds of the rate distortion region. In this paper we show that when Gaussian correlated sources satisfy some condition, the outer bound coincides with the inner bound.**

I. Introduction

Separate coding systems of correlated information sources are a form of communication system which is significant from both theoretical and practical point of view in multi-user source networks. The first fundamental theory in those coding systems was established by Slepian and Wolf [1]. They considered a separate source coding system of two correlated information sources. Those two sources are separately encoded and sent to a single destination, where the decoder reconstruct the original sources. In this system, Slepian and Wolf [1] determined the admissible rate region, the set that consists of a pair of transmission rates for which two sources can be decoded with an arbitrary small error probability.

In the above separate coding systems we can consider the case where the source outputs should be reconstructed with average distortions smaller than prescribed levels. Such a situation suggests the multiterminal rate distortion theory. The rate distortion theory for the separate coding system formulated by Slepian and Wolf has been studied by Wyner and Ziv [2], Wyner[3], Berger [4], Tung [5], Berger et. al [6], Kaspi and Berger [7], Berger and Yeung [8], and Oohama [9]. This problem, in general, remains an open problem and characterization of the rate distortion region has been unknown yet except for special cases.

As a practical situation of separate coding systems, we can consider the case where the separate encoders can not directly access the source outputs but can obtain their noisy observations. This situation was first studied by Yamamoto and Ito [10]. They call the investigated coding system the communication system with a remote source. Subsequently, the separate source system with a remote observation or with a incomplete source output was studied by Yamamoto [12] and Flynn and R. M. Gray [11].

Oohama [17] considered the separate source coding problem of L correlated Gaussian observations $Y_i, i = 1, 2 \cdots L$. We deal with the case that Y_i satisfy $Y_i = X_i + N_i, i = 1, 2, \cdots, L$, where $X_i, i = 1, 2, \cdots, L$ is a correlated Gaussian random vector and $N_i, i = 1, 2, \cdots, L$ are independent additive L Gaussian noises also independent of $X_i, i = 1, 2, \cdots, L$. In the above setup $Y_i, i = 1, 2, \cdots, L$ can be regarded as correlated Gaussian remote sources of $X_i, i = 1, 2, \cdots, L$, respectively. This coding system can also be considered as a vector version of the Gaussian CEO problem [13], [14], where $X_i, i = 1, 2, \cdots, L$ are identical. In this separate source coding system Oohama [17] derived explicit outer and inner bounds of the rate distortion region. The above coding problem was first posed and studied by Pandya et al. [15], and subsequently, Zhang and Wicker [16] investigated this problem. They derived inner and outer bounds of the rate distortion region. The results of Oohama [17] are sharper than theirs. In this paper, we show that the above inner and outer region coincides with each other when Gaussian correlated sources satisfy some condition.

II. Problem Formulation and Results

In this section we state the problem formulation. Throughout this paper all logarithms are taken to the base natural. Let $X_i, i = 1, 2, \cdots, L$ be correlated zero mean Gaussian random variables taking values in real lines $\mathcal{X}_i$. We write a L dimensional random vector as $X^L = (X_1, X_2, \cdots, X_L)$ and use similar notation of other random variables. We denote the covariance matrix of X^L by Σ_{X^L}. Let $\{(X_{1,t}, X_2, \cdots, X_{L,t})\}_{t=1}^{\infty}$ be a stationary memoryless multiple Gaussian source. For each $t = 1,2, \cdots$, $(X_{1,t}, X_{2,t}, \cdots, X_{L,t})$ obeys the same distribution as $(X_1, X_2, \cdots, X_L)$. Let a random vector consisting of n independent copies of the random variable X_i be denoted by $\boldsymbol{X}_i = X_{i,1} X_{i,2} \cdots X_{i,n}$. Furthermore, let $\boldsymbol{X}^L$ denote the random vector $(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_L)$.

We consider the separate coding system for L correlated sources, where L encoders can only access noisy version Y_i of X_i for $i = 1, 2, \cdots, L$, that is,

$$Y_i = X_i + N_i, i = 1, 2, \cdots, L \tag{1}$$

where $N_i, i = 1, 2, \cdots, L$ are zero mean independent Gaussian random variables with variance $\sigma_{N_i}^2$. We assume that X^L

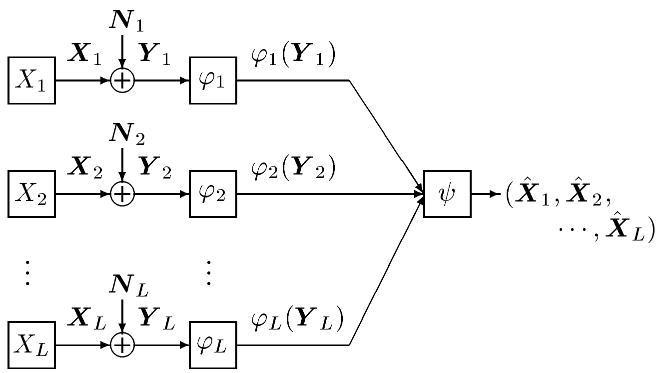

Fig. 1. Separate coding system for L correlated Gaussian remote sources

and N^L are independent. The separate coding system for L correlated Gaussian remote sources is shown in Fig. 1. The encoder functions φ_i , $i = 1, 2, \cdots, L$ are defined by

$$\varphi_i : \mathcal{X}_i^n \to \mathcal{M}_i = \{1, 2, \cdots, M_i\} \tag{2}$$

and satisfy rate constraints

$$\frac{1}{n} \log M_i \le R_i + \delta \tag{3}$$

where δ is an arbitrary prescribed positive number. The decoder function $\psi = (\psi_1, \psi_2, \cdots, \psi_L)$ is defined by

$$\psi_i : \mathcal{M}_1 \times \cdots \times \mathcal{M}_L \to \mathcal{X}_i^n , i = 1, 2, \cdots, L. \tag{4}$$

Denote by $\mathcal{F}_\delta^{(n)}(R_1, R_2, \cdots, R_L)$ the set that consists of all the $(L+1)$ tuple of encoder and decoder functions $(\varphi_1, \varphi_2, \cdots, \varphi_L, \psi)$ satisfying (2)-(4). For $\boldsymbol{X}^L = (\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_L)$ and its estimation

$$\begin{aligned}\hat{\boldsymbol{X}}^L &= (\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2, \cdots, \hat{\boldsymbol{X}}_L) \\ &= (\psi_1(\varphi_1(\boldsymbol{Y}_1)), \psi_2(\varphi_2(\boldsymbol{Y}_2)), \cdots, \psi_L(\varphi_L(\boldsymbol{Y}_L)),\end{aligned}$$

define the mean squared error by

$$\Delta(\boldsymbol{X}^L, \hat{\boldsymbol{X}}^L) \triangleq \frac{1}{n} \sum_{i=1}^{L} \mathrm{E}||\boldsymbol{X}_i - \hat{\boldsymbol{X}}_i||^2 .$$

where $||\cdot||$ stands for the Euclid norm of n dimensional vectors. For a given $D > 0$, the rate vector $R^L = (R_1, R_2, \cdots, R_L)$ is *admissible* if for any positive $\delta > 0$ and any n with $n \ge n_0(\delta)$, there exists $(\varphi_1, \varphi_2, \cdots, \varphi_L, \psi) \in \mathcal{F}_\delta^{(n)}(R_1, R_2 \cdots, R_L)$ such that $\Delta(\boldsymbol{X}, \hat{\boldsymbol{X}}) \le D + \delta$. Let $\mathcal{R}(D)$ denote the set of all admissible rate vectors. We call $\mathcal{R}_L(D)$ the rate distortion region. Let the rate distortion region in this system be denoted by $\mathcal{R}_L(D)$. Our main goal is to determine $\mathcal{R}_L(D)$ in an explicit form.

Here, we state the previous results on the above problem. Pandya et al. [15] first posed and investigated this problem. They dealt with the case that $Y^L = AX^L + N^L$, where A is $L \times L$ a positive definite authentication matrix. They derived upper and lower bounds of the rate sum $\sum_{i=1}^L R_i$ for $R^L \in \mathcal{R}_L(D)$. Subsequently Zhang and Wicker [16] studied the same coding problem derived an explicit inner bound of $\mathcal{R}_L(D)$. The outer bounds of Oohama [17] which will be stated in the next section was tighter than that of Pandya et. al [15].

III. Inner and Outer Bounds

In this section we state the result of Oohama [17] on inner and outer bounds of $\mathcal{R}(D)$. To describe the result we define several functions and sets.

For $r_i \ge 0, i = 1, 2, \cdots, L$, let $N_i(r_i)$, $i = 1, 2, \cdots, L$ be L independent Gaussian random variables with mean 0 and variance $\sigma_{N_i}^2/(1-\mathrm{e}^{-2r_i})$. Let $\Sigma_{N^L(r^L)}$ be a covariance matrix for the random vector $N^L(r^L)$. Let $\Lambda = \{1, 2, \cdots, L\}$. For any subset $S \subset \Lambda$, we introduce the notation $r_S = (r_i)_{i \in S}$. In particular $r_\Lambda = r^L = (r_1, r_2, \cdots, r_L)$. Fix nonnegative vector r^L. Let $\alpha_i = \alpha_i(r^L), i = 1, 2, \cdots, L$ be L eigen values of the matrix $\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}$. For $S \subseteq \Lambda$, and $\theta > 0$, define

$$\Sigma_{N^L(r_{S^c})}^{-1} \triangleq \left. \Sigma_{N^L(r^L)}^{-1} \right|_{r_S = \mathbf{0}} ,$$

$$J_S(r^L, \theta) \triangleq \frac{1}{2} \log^+ \left[\frac{\prod_{i \in S} \mathrm{e}^{-2r_i}}{\left| \Sigma_{X^L}^{-1} + \Sigma_{N^L(r_{S^c})}^{-1} \right| \theta} \right] ,$$

where $S^c = \Lambda - S$ and $\log^+ x = \max\{\log x, 0\}$. Let $\mathcal{B}_L(D)$ be the set of all nonnegative vectors r^L that satisfy

$$\mathrm{tr} \left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1} \right)^{-1} \right] \le D .$$

Let $\partial \mathcal{B}_L(D)$ be the boundary of $\mathcal{B}_L(D)$, that is, the set of all nonnegative vectors r^L that satisfy

$$\mathrm{tr} \left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N(r^L)}^{-1} \right)^{-1} \right] = D .$$

Let ξ be nonnegative number that satisfy

$$\sum_{i=1}^{L} \left\{ [\xi - \alpha_i^{-1}]^+ + \alpha_i^{-1} \right\} = D.$$

Define

$$\theta(D, r^L) \triangleq \prod_{i=1}^{L} \left\{ [\xi - \alpha_i^{-1}]^+ + \alpha_i^{-1} \right\} .$$

Set

$$\begin{aligned}\mathcal{R}_L^*(\theta, r^L) &\triangleq \Big\{ R^L : \sum_{i \in S} R_i \ge J_S\left(r^L, \theta\right) \\ &\qquad\qquad \text{for any } S \subseteq \Lambda \, . \Big\} , \\ \mathcal{R}_L^{(out)}(D) &\triangleq \bigcup_{r^L \in \mathcal{B}_L(D)} \mathcal{R}_L^*(\theta(D, r^L), r^L) , \\ \mathcal{R}_L^{(in)}(D) &\triangleq \mathrm{conv} \left\{ \bigcup_{r^L \in \partial \mathcal{B}_L(D)} \mathcal{R}_L^*(\theta(D, r^L), r^L) \right\} ,\end{aligned}$$

where conv$\{A\}$ denotes a convex hull of the set A. Oohama [17] obtained the following result.

Theorem 1: (Oohama [17])

$$\mathcal{R}_L^{(in)}(D) \subseteq \mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(out)}(D)\,.$$

The definitions of $\mathcal{R}_L^{(out)}(D)$ and $\mathcal{R}_L^{(in)}(D)$ are slightly different from original ones in [17]. There is no essential difference between the present ones and original ones. The present forms of $\mathcal{R}_L^{(out)}(D)$ and $\mathcal{R}_L^{(in)}(D)$ are useful for examining their mathematical properties.

IV. Computation of Inner and Outer Bounds

In this section we compute our inner and outer bounds to derive sufficient conditions that the outer bound coincides the inner bound.

A. The Case of Two Remote Sources

In this subsection we compare our inner and outer bounds in the case $L=2$. Considering a proper scale transformation to Y_1 or Y_2, we may assume without loss of generality that $\sigma_{N_1}^2 = \sigma_{N_2}^2 = \sigma_N$. In the following we deal with the above case of identical noise variances.

For simplicity of notation, we set

$$a_i \triangleq \frac{1}{(1-\rho^2)\sigma_{X_i}^2}, i=1,2\,.$$

and assume $a_1 \geq a_2$. Furthermore, set

$$a \triangleq \frac{a_1+a_2}{2}\,, b \triangleq \frac{\rho}{1-\rho^2}\frac{1}{\sigma_{X_1}\sigma_{X_2}}\,, c \triangleq \frac{1}{\sigma_{N_1}^2} = \frac{1}{\sigma_{N_2}^2}\,.$$

Set

$$u_i \triangleq Da_i + Dc(1-\mathrm{e}^{-2r_i}), i=1,2$$

and transform the variable (r_1,r_2) into (u_1,u_2). The range of $u_i, i=,1,2$ are

$$Da_i \leq u_i \leq D(a_i+c)\,, i=1,2\,. \tag{5}$$

Since

$$D(\Sigma_{X^2}^{-1} + \Sigma_{N^2(r_1,r_2)}^{-1}) = \begin{bmatrix} u_1 & -Db \\ -Db & u_2 \end{bmatrix}$$

The condition

$$\mathrm{tr}\left[\left(\Sigma_{X^2}^{-1} + \Sigma_{N^2(r_1,r_2)}^{-1}\right)^{-1}\right] \leq D\,,$$

is given by

$$(u_1-1)(u_2-1) \geq 1+(Db)^2\,. \tag{6}$$

We consider the case

$$D\left(a+\tfrac{c}{2}\right) < 1+\sqrt{1+(Db)^2}\,, \tag{7}$$

$$D(a_1+c)D(a_2+c) - 2D\left(a+\tfrac{c}{2}\right) > (Db)^2\,. \tag{8}$$

In this case, for each fixed u_2 with

$$1+\frac{(Db)^2}{D(a_1+c)-1} < u_2 < D(a_2+c)\,,$$

we have

$$1+\frac{1+(Db)^2}{u_2-1} \leq u_1 < D(a_1+c)\,. \tag{9}$$

The equality in the first inequality of (9) implies $(u_1,u_2) \in \partial\mathcal{B}_2(D)$. Similarly, for each fixed u_1 with

$$1+\frac{(Db)^2}{D(a_2+c)-1} < u_1 < D(a_1+c)\,,$$

we have

$$1+\frac{1+(Db)^2}{u_1-1} \leq u_2 < D(a_2+c)\,. \tag{10}$$

The equality in the first inequality of (10) implies $(u_1,u_2) \in \partial\mathcal{B}_2(D)$. Define the function $g(z)$ by $g(z) \triangleq \sqrt{1+z^2}+z\,.$ Our result is the following.

Theorem 2: Suppose that a_1,a_2,a,b,c,D are chosen so as to satisfy (7) and (8). If a_1,a_2,a,b,c,D further satisfy

$$g\left(\sqrt{D^2\left(\frac{a_1-a_2}{2}+\frac{c}{2}\right)^2+(Db)^2}\right) \leq \frac{D(a_i+c)+1}{D(a_i+c)-1} \quad \text{for } i=1,2, \tag{11}$$

then, we have $\mathcal{R}_2^{(in)}(D) = \mathcal{R}_2(D) = \mathcal{R}_2^{(out)}(D)$.

Next, we derive the rate distortion region explicitly in the case $a_1=a_2=a$. To describe our result, set

$$\theta^*(D,r_1,r_2) \triangleq \frac{D^2}{2D(a+c)-Dc(\mathrm{e}^{-2r_1}+\mathrm{e}^{-2r_2})}\,.$$

Furthermore, set

$$\begin{aligned}
&\hat{\mathcal{R}}_1(D,r_1,r_2)\\
&\triangleq \{(R_1,R_2) : R_1 \geq J_{\{1\}}(r_1,s_2,\theta^*(D,r_1,s_2)),\\
&\qquad R_2 \geq J_{\{2\}}(r_1,r_2,\theta^*(D,r_1,r_2)),\\
&\qquad R_1+R_2 \geq J_{\{1,2\}}(r_1,s_2,\theta^*(D,r_1,s_2)),\\
&\qquad \text{for some } s_2 \geq r_2.\}\,.\\
&\hat{\mathcal{R}}_2(D,r_1,r_2)\\
&\triangleq \{(R_1,R_2) : R_1 \geq J_{\{1\}}(r_1,r_2,\theta^*(D,r_1,r_2)),\\
&\qquad R_2 \geq J_{\{2\}}(s_1,r_2,\theta^*(D,s_1,r_2)),\\
&\qquad R_1+R_2 \geq J_{\{1,2\}}(s_1,r_2,\theta^*(D,s_1,r_2)),\\
&\qquad \text{for some } s_1 \geq r_1.\}\,.
\end{aligned}$$

and

$$\hat{\mathcal{R}}(D,r_1,r_2) \triangleq \hat{\mathcal{R}}_1(D,r_1,r_2) \cup \hat{\mathcal{R}}_2(D,r_1,r_2)\,.$$

Set

$$r^* \triangleq \frac{1}{2}\log\left[\frac{Dc}{D(a+c)-\sqrt{1+(Db)^2}-1}\right]\,.$$

Then, we have the following.

Theorem 3: Suppose that $a_1=a_2=a,b,c,D$ satisfy (7) and (8). If a,b,c,D further satisfy the condition (11) for $a_1=a_2$, that is,

$$g\left(D^2\left(\frac{c^2}{4}+b^2\right)\right) \leq \frac{D(a+c)+1}{D(a+c)-1}\,, \tag{12}$$

then, we have $\mathcal{R}_2(D) = \hat{\mathcal{R}}(D,r^*,r^*)\,.$

B. The Case of L Remote Sources

Let $a_{ij}, i,j \in \Lambda$ be (i,j) elements of $\Sigma_{X^L}^{-1}$ and set

$$u_i \stackrel{\triangle}{=} Da_{ii} + \frac{D}{\sigma_{N_i}^2}(1-\mathrm{e}^{2r_i})\,, i \in \Lambda\,. \quad (13)$$

To simplify our analysis we restrict our argument to some class of Gaussian sources where Σ_{X^L} and Σ_{N^L} have identical diagonal and nondiagonal elements, that is, for $i \in \Lambda$,

$$\mathrm{Var}[N_i] = \sigma_{N_i}^2 = \sigma_N^2\,, \mathrm{Var}[X_i] = \sigma_{X_i}^2 = \sigma^2\,,$$

and for $i,j \in \Lambda, i \neq j$,

$$\mathrm{Cov}[X_i, X_j] = \rho\sigma_{X_i}\sigma_{X_j} = \rho\sigma^2\,.$$

In this identical variance case, (i,j) elements a_{ij} of $\Sigma_{X^L}^{-1}$ is given by

$$a_{ij} = \begin{cases} \frac{1+(L-2)\rho}{(1-\rho)(1+(L-1)\rho)} \cdot \frac{1}{\sigma^2} & \text{if } i = j \\ \frac{-\rho}{(1-\rho)(1+(L-1)\rho)} \cdot \frac{1}{\sigma^2} & \text{if } i \neq j \end{cases}$$

For simplicity of notations we set

$$a \stackrel{\triangle}{=} a_{ii}, b \stackrel{\triangle}{=} -a_{ij}, c \stackrel{\triangle}{=} \frac{1}{\sigma_N^2}. \quad (14)$$

By the transformation (13), each r^L is mapped onto an element $u^L = (u_1, u_2, \cdots, u^L)$ of the set of direct L product of semi open intervals given by $\mathcal{D} = [D(a), D(a+c))^L$. We first derive an explicit form of the set $\mathcal{B}_L(D)$ with respect to $u^L = (u_1, u_2, \cdots, u_L)$. To this end we use the following formula

$$\begin{vmatrix} z_1 & \delta & \ldots & \delta \\ \delta & z_2 & \ldots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ \delta & \delta & \ldots & z_L \end{vmatrix} = \prod_{i=1}^{L}(z_i - \delta)\left\{1 + \delta\sum_{i=1}^{L}\frac{1}{z_i-\delta}\right\}. \quad (15)$$

Using (15), the condition

$$\mathrm{tr}\left[\left(\Sigma_{X^L}^{-1} + \Sigma_{N^L(u^L)}^{-1}\right)^{-1}\right] \leq D\,,$$

is explicitly given by the following:

$$(Db)\sum_{i\neq j}\frac{1}{(u_i+Db)(u_j+Db)}$$
$$-(1+Db)\sum_{i=1}^{L}\frac{1}{(u_i+Db)(u_j+Db)} + 1 \geq 0\,.$$

Set

$$\kappa_1 \stackrel{\triangle}{=} \frac{1}{2}\cdot\frac{1}{L-1}\cdot\frac{1+Db}{Db}\,,$$
$$\kappa_2 \stackrel{\triangle}{=} \frac{L}{4(L-1)}\cdot\left(\frac{1+Db}{Db}\right)^2 - (Db)^{-1}\,.$$

Then, the above condition is rewritten as

$$\sum_{i\neq j}\left(\kappa_1 - \frac{1}{u_i+Db}\right)\left(\kappa_1 - \frac{1}{u_j+Db}\right) \geq \kappa_2\,.$$

Let L_0 be the solution of the following equation

$$L(L-1)\left(\kappa_1 - \frac{1}{L_0+Db}\right)^2 = \kappa_2\,.$$

L_0 is given by

$$L_0 = \frac{L}{2}\left[1 + Db + \sqrt{(1-Db)^2 + \frac{4Db}{L}}\right] - Db\,.$$

We choose a, b, c, D, so that

$$L(L-1)\left(\kappa_1 - \frac{1}{D(a+b+c)}\right)^2 > \kappa_2 \quad (16)$$

$$(L-1)\left(\kappa_1 - \frac{1}{D(a+b)}\right)\left(\kappa_1 - \frac{1}{D(a+b+c)}\right)$$
$$+(L-1)^2\left(\kappa_1 - \frac{1}{D(a+b+c)}\right)^2 < \kappa_2\,. \quad (17)$$

The condition (16) is equivalent to $D(a+c) > L_0$. We set $D(a+c) = L+\varepsilon$. Computing the difference $L_0 - L$, the condition $D(a+c) > L_0$ is given by

$$\varepsilon > \frac{2(1-\frac{1}{L})}{\sqrt{(1-Db)^2 + \frac{4Db}{L}} + 1 - Db + \frac{2Db}{L}} \cdot (Db)^2\,. \quad (18)$$

When L is large, the above condition becomes

$$\varepsilon > \frac{1}{1-Db}\cdot(Db)^2\,.$$

Set

$$\nu \stackrel{\triangle}{=} \frac{\kappa_2 - (L-1)^2\left\{\kappa_1 - (D(a+b+c))^{-1}\right\}}{(L-1)\{\kappa_1 - (D(a+b+c))^{-1}\}}\,. \quad (19)$$

Under the above choice of a, b, c, D, we have the following. For each $i \in \Lambda$ and each fixed $L-1$ values $u_j, j \neq i$ with $u_j \in (\nu, D(a+c))$, we have

$$\begin{aligned} &\kappa_1 - \frac{1}{D(a+b+c)} \\ &> \kappa_1 - \frac{1}{u_i+Db} \\ &\geq \frac{\kappa_2 - \sum_{\substack{k\neq i \\ l\neq i}}\left(\kappa_1 - \frac{1}{u_k+Db}\right)\left(\kappa_1 - \frac{1}{u_l+Db}\right)}{\sum_{j\neq i}\left(\kappa_1 - \frac{1}{u_i+Db}\right)}\,. \end{aligned} \quad (20)$$

The equality of the second inequality in (20) corresponds to the boundary $\partial\mathcal{B}_L(D)$. Our result in the case of L correlated remote sources is the following.

Theorem 4: Suppose that ε, a, b, c, D are chosen so as to satisfy (16)(or equivalent to (18)) and (17). If ε, b, D further satisfy

$$\varepsilon + (L-1)(Db) \leq \frac{1}{L-1}\,, \quad (21)$$

then, we have $\mathcal{R}_L^{(in)}(D) = \mathcal{R}_L(D) = \mathcal{R}_L^{(out)}(D)$.

Next, we examine a particular part of the rate distortion region $\mathcal{R}_L(D)$. Define

$$R_L(D) \stackrel{\triangle}{=} \min_{R^L\in\mathcal{R}_L(D)}\left\{\sum_{i=1}^{L}R_i\right\}\,. \quad (22)$$

The determination problem of $R_L(D)$ was first investigated by Pandya et. al [15]. They derived upper and lower bound of $R_L(D)$. Pandya et. al [15] also numerically compared those two bounds to show that for some numerical examples the gap between them is relatively small. In this paper we determine $R_L(D)$ for some nontrivial case of Gaussian sources. Our result is the following.

Theorem 5: We assume (16)(or equivalent to (18)) and (17). If a, b, c, D satisfy the condition (21) of Theorem 4, we have

$$R_L(D) = \frac{L}{2}\log\left(\frac{L_1(Dc)}{D(a+b+c)-L_1}\right) + \frac{1}{2}\log\left(1 - \frac{L(Db)}{L_1}\right) ,$$

where

$$\begin{aligned} L_1 &= L_0 + Db \\ &= \frac{L}{2}\left[1 + Db + \sqrt{(1-Db)^2 + \frac{4Db}{L}}\right] . \end{aligned}$$

V. Proofs of the Theorems

In this section we state outlines of the proofs of Theorems 2- 5. The following lemma is a key result for the proofs of Theorems 2 and 4.

Lemma 1: Let $\mathcal{B}_{\{i\}^c}(D)$ be a set of all nonnegative vectors $r_{\{i\}^c}$ such that $r_{\{i\}^c \cup \{i\}} \in \mathcal{B}_L(D)$ for some nonnegative r_i. If we have the following two conditions

C1: For any $i \in \Lambda$ and any $r_{\{i\}^c} \in \mathcal{B}_{\{i\}^c}(D)$ there exists nonnegative r_i such that $r_{\{i\}^c \cup \{i\}} \in \partial\mathcal{B}_L(D)$.

C2: For $r^L \in \mathcal{B}(D)$ and for $i \in \Lambda$,

$$\frac{\partial}{\partial r_i} J_{\{i\}}(r^L, \theta(D, r^L)) \geq 0 .$$

then $\mathcal{R}_L^{(out)}(D) = \mathcal{R}_L^{(in)}(D)$.

We examine a sufficient condition for C2 to hold. Let $\eta_i = \eta_i(r^L), i \in \Lambda$ be L eigen values of $D(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1})$. By the transformation (13), we can regard $\eta_i, i \in \Lambda$ as functions of u^L. We consider the following regularity condition with respect to $\eta_i, i \in \Lambda$.

C3: For $i \in \Lambda$, the eigen values $\eta_i = \eta_i(u^L)$ are smooth functions of u^L and

$$\frac{\partial \eta_i}{\partial u_j} \geq 0 \text{ for } j \in \Lambda .$$

The following lemma presents an important property of $\eta_i = \eta_i(u^L), i \in \Lambda$.

Lemma 2: We assume C3. Then, for $i \in \Lambda$, we have

$$\sum_{j=1}^{L} \frac{\partial \eta_j}{\partial u_i} = 1 .$$

Using (15), we can see that when

$$D(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1}) = \begin{bmatrix} u_1 & -Db & \ldots & -Db \\ -Db & u_2 & \ldots & -Db \\ \vdots & \vdots & \ddots & \vdots \\ -Db & -Db & \ldots & u_L \end{bmatrix} ,$$

$\eta_i, i = 1, 2, \cdots, L$ are L solutions to the following eigen value equation:

$$\left\{\prod_{i=1}^{L}(u_i + Db - \eta)\right\}\left(1 - (Db)\sum_{i=1}^{L}\frac{1}{u_i + Db - \eta}\right) = 0 . \quad (23)$$

Then, we have the following proposition.

Proposition 1: The L solutions $\eta_i, i \in \Lambda$ to the eigen value equation (23) satisfy C3.

Using Lemma 2 we have the following proposition.

Proposition 2: We assume C3. Let $\eta = \eta(r^L)$ be the minimum eigen value of $D(\Sigma_{X^L}^{-1} + \Sigma_{N^L(r^L)}^{-1})$. Let $\mathcal{D}$ be some compact subset of $[0, +\infty)^L$. If we have

$$\eta(r^L) \geq L \cdot \frac{D(a_i + c)}{D(a_i + c) + 1} \text{ on } \mathcal{D} ,$$

then,

$$\frac{\partial}{\partial r_i} J_{\{i\}}(r^L, \theta(D, r^L)) \geq 0 \text{ on } \mathcal{D} .$$

From Lemma 1, Propositions 1, 2 and some analytical arguments we obtain Theorems 2-5.

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