A Graph-based Framework for Transmission of Correlated Sources over Broadcast Channels

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***Abstract*— In this paper we consider the communication problem that involves transmission of correlated sources over broadcast channels. We consider a graph-based framework for this information transmission problem. The system involves a source coding module and a channel coding module. In the source coding module, the sources are efficiently represented using a nearly semi-regular bipartite graph, and in the channel coding module, the edges of this graph are reliably transmitted over a broadcast channel. We consider nearly semi-regular bipartite graphs as discrete interface between source coding and channel coding in this multiterminal setting. We provide an information-theoretic characterization of (1) the rate of growth of the exponent (as a function of the number of channel uses) of the size of the bipartite graphs whose edges can be reliably transmitted over a broadcast channel and (2) the rate of growth of the exponent (as a function of the number of source samples) of the size of the bipartite graphs which can reliably represent a pair of correlated sources to be transmitted over a broadcast channel.**

I. Introduction

The broadcast channel, where a transmitter wishes to accomplish reliable simultaneous communication with a set of receivers was first introduced by Cover [1] in 1972. Since then, the broadcast channel has been studied extensively, and the capacity region for several special classes of broadcast channels have been found. For the discrete memoryless broadcast channel, Marton [2] established an inner bound to the capacity region which contains all known achievable rate regions. Later, Han and Costa [3] presented a new coding theorem for transmission of correlated sources over broadcast channels. They established a new coding scheme that specifies, instead of achievable rates, a class of source-channel matching conditions between the source and the channel. They also gave an interesting example, which reveals that separate source and channel coding is not optimal for the transmission of correlated sources over broadcast channels.

The essence of Shannon's separation approach in the point-to-point case is to have an efficient architecture for transmission problems by having a discrete interface (a finite set) to represent information sources. The fundamental concept which facilitates this is the notion of typicality, which states that when grouped into large blocks (with size n), only those sequences, called typical sequences, coming from a set of size nearly equal to $2^{nH(X)}$ are observed most of the time, and each such sequence has nearly the same probability, where $H(X)$ denotes Shannon entropy [4].

This notion of typicality can be extended to multiple sources (say S and T) called joint typicality. It says that even though there are roughly $2^{n(H(S)+H(T))}$ individually typical sequence pairs, most of the time only jointly typical sequence pairs with size nearly equal to $2^{nH(S,T)}$ are observed with high probability. Further, using these ideas, it can be shown that for every typical sequence of X (respectively Y), there exist roughly $2^{nH(Y|X)}$ (respectively $2^{nH(X|Y)}$) typical Y (respectively X) sequences that are jointly typical, where $H(Y|X)$ is the conditional entropy [4] of Y given X. This leads us naturally to consider a bipartite undirected graph on the sets of individually typical sequences induced from the property of joint typicality. That is, the vertexes of this graph denote the individually typical sequences, and the jointly typical sequences are connected through an edge. We refer to this graph as the typicality-graph of two correlated sources.

Based on this observation, a graph-based framework for transmission of correlated sources over multiple-access channels was considered in our earlier work [5]. In essence, the transmission system would involve two modules: a source coding module and a channel coding module. In the source coding module, the sources are represented using a bipartite graph, and in the channel coding module, the edges coming from this graph are reliably transmitted over a multiple-access channel.

In this work we consider this approach for the broadcast channels. In particular, we consider the problem of simultaneous transmission of correlated sources over broadcast channels. For this problem, we consider a modular framework based on bipartite graphs. In this transmission system, we have a source coding module and a channel coding module. In the source coding module, the sources are efficiently represented using a bipartite graph, and in the channel coding module, the edges coming from this graph are reliably transmitted over the broadcast channels.

We provide an information-theoretic characterization of (1) the rate of growth of the exponent (as a function of the number of channel uses) of the size of the bipartite graphs whose edges can be reliably transmitted over a broadcast channel and (2) the rate of growth of the exponent (as a function of the number of source samples) of the size of the bipartite graphs which can reliably represent a pair of correlated sources to be transmitted over a broadcast channel.

In the conventional separate source and channel coding scheme the correlated sources are first represented into one common message W_0 and two private messages W_1 and W_2 in the source coding module, and then (W_0, W_1) and (W_0, W_2) are reliably transmitted to the receiver 1 and the receiver 2, respectively in the channel coding module. Note that here the messages W_0, W_1 and W_2 are regarded as independent. The outline of the remaining part of this paper is as follows. In Section II, we formulate the problem and provide definitions of graphs of our interest. Then, we discuss the channel coding part in Section III. After that, the source coding part is presented in Section IV. In Section V some interpretations of correlated sources in terms of graphs are presented. Finally, Section VI provides concluding remarks.

II. Problem Formulation

The problem we are addressing is the simultaneous transmission of two discrete memoryless stationary correlated sources S and T with finite alphabets $\mathcal{S}$ and $\mathcal{T}$ respectively, over a discrete memoryless stationary broadcast channel with one sender and two receivers as shown in Figure 1. The encoder is given by a mapping $f : \mathcal{S}^n \times \mathcal{T}^n \rightarrow \mathcal{X}^n$. The decoders are given by mappings $g_1 : \mathcal{Y}_1^n \rightarrow \mathcal{S}^n$ and $g_2 : \mathcal{Y}_2^n \rightarrow \mathcal{T}^n$. The performance measure associated with this transmission system is the probability of decoding error:

$$Pr[(S^n, T^n) \neq (g_1(Y_1^n), g_2(Y_2^n))]. \quad (1)$$

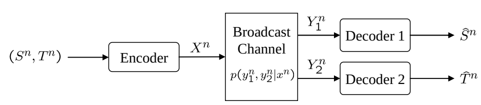

Fig. 1. Transmission of correlated sources over a broadcast channel

A. Basic Concepts

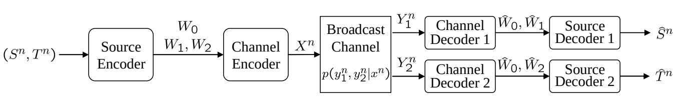

Fig. 2. Separate source coding and channel coding in the broadcast channel

We consider an approach to this problem, which is shown in Figure 2. The system has two modules: the source coding module and the channel coding module. The correlated sources (S, T) are first represented efficiently using nearly semi-regular bipartite graphs in the source coding module. Then, the edges coming from these graphs are reliably transmitted over the broadcast channel in the channel coding module.

We first discuss the channel coding part on the assumption that the source coding module is going to produce three messages W_0, W_1 and W_2 where $W_0 \in \mathcal{W}_0$ is a common message to both receivers and $W_1 \in \mathcal{W}_1$ and $W_2 \in \mathcal{W}_2$ are private messages where $\mathcal{W}_0$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are finite integer message sets. We also assume that there is some kind of "*correlation*" between two message sets $\mathcal{W}_1$ and $\mathcal{W}_2$, i.e., private messages for each receiver can not be chosen independently. In more detail, we can think of these messages as follows.

- If the messages of senders are independent, the message pairs (W_1, W_2) are equally likely with probability $\frac{1}{|\mathcal{W}_1 \times \mathcal{W}_2|}$.
- If the messages of senders are correlated, the message pairs $(W_1, W_2) \in A$ are equally likely with probability $\frac{1}{|A|}$, and the message pairs $(W_1, W_2) \notin A$ have probability zero.

We then consider the source coding part, showing that the assumptions are indeed reasonable, i.e., the correlated sources (S, T) can be represented with a common message set and two correlated private message sets with arbitrarily small probability of error.

Before we discuss the main problem, let us first define a bipartite graph and related mathematical terms which will be used in our discussion.

Definition 1:

- A *bipartite graph* G is defined as $G = (A_1, A_2, B)$ where A_1 and A_2 are two non-empty sets of vertices, and B is a set of edges where every edge of B joins a vertex in A_1 to a vertex in A_2. In general, $B \subseteq A_1 \times A_2$.
- If G is a bipartite graph, $V_1(G)$ and $V_2(G)$ are the first (left) and the second (right) vertex sets of G and $E(G)$ is the edge set of G.
- G is said to be a *complete* if $E(G) = V_1(G) \times V_2(G)$.
- The *degree*, or *valency*, of a vertex $v \in V_i(G)$ in a graph G, denoted by $\deg_{G,i}(v)$ is the number of edges incident to v for $i = 1, 2$.
- A *subgraph* of a graph G is a graph whose vertex and edge sets are subsets of those of G

Since we consider particular type of bipartite graphs in our discussion, let us define those bipartite graphs.

Definition 2:

- A bipartite graph G is called *semi-regular* [6] with parameters $(\theta_1, \theta_2, \theta_1', \theta_2')$, denoted by $G(\theta_1, \theta_2, \theta_1', \theta_2')$, if it satisfies:
  - $|V_i(G)| = \theta_i$ for i=1, 2,
  - $\forall u \in V_1(G)$, $\deg_{G,1}(u) = \theta_2'$,
  - $\forall v \in V_2(G)$, $\deg_{G,2}(v) = \theta_1'$.

Definition 3: A nearly semi-regular bipartite graph G is said to have parameters $(\Delta_0, \Delta_1, \Delta_2, \Delta_1', \Delta_2', \mu)$ for $\mu \geq 1$ if it satisfies the following:

- G is composed of Δ_0 disjoint subgraphs $\tilde{G}_m$ for $m = 1, 2, \ldots, \Delta_0$,
- $\forall m$, $V_i(\tilde{G}_m) = \{1, 2, \ldots, \Delta_i\}$ for i=1, 2.
- $V_i(G) = \bigcup_{m=1}^{\Delta_0} V_i(\tilde{G}_m)$ for i=1, 2,
- $E(G) = \bigcup_{m=1}^{\Delta_0} E(\tilde{G}_m)$,
- $\forall u \in V_1(G)$, $\Delta_2' \mu^{-1} \leq \deg_{G,1}(u) \leq \Delta_2' \mu$,
- $\forall v \in V_2(G)$, $\Delta_1' \mu^{-1} \leq \deg_{G,2}(v) \leq \Delta_1' \mu$.

Figure 3 illustrates an example of bipartite message-graph $G(2, 4, 4, 2, 2, 1)$ composed of two disjoint subgraphs.

Definition 4: With a bipartite graph G with parameters

$(\Delta_0, \Delta_1, \Delta_2, \Delta_1', \Delta_2', \mu)$, one can associate a triple of messages W_0, W_1 and W_2 with message sets given, respectively, by $\mathcal{W}_i = \{1, 2, \ldots, \Delta_i\}$ for $i = 0, 1, 2$, referred to as a message-graph. W_0 is the common message and W_1 and W_2 are private messages. Every edge in $E(G)$ denotes a message pair which occurs with nonzero equal probability, given by $1/|E(G)|$.

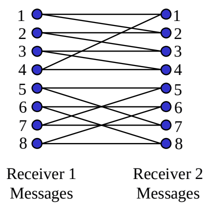

Fig. 3. Examples of bipartite message graph $G(2, 4, 4, 2, 2, 1)$ composed of two disjoint subgraphs $\tilde{G}$ where $\mathcal{W}_0 = \{1, 2\}$, $\mathcal{W}_1 = \mathcal{W}_2 = \{1, 2, 3, 4\}$

In Fig. 3, the message sets are given by $\mathcal{W}_0 = \{1, 2\}$, and $\mathcal{W}_1 = \mathcal{W}_2 = \{1, 2, 3, 4\}$.

Remark 1: The goal of the channel encoder is to transmit a common message and a pair of private messages (or edges in the graph associated with the messages) reliably over the channel. As an analogy, the conventional Shannon's channel coding theorem can be interpreted as finding the maximum number of codewords (colors, if each codeword has a different color) that are distinguishable at the noisy receiver. In conventional broadcast channels, the goal is to distinguish colors at the noisy receivers, where the common color comes from one set and the private colors come from two other sets, and all possible combination of pairs in the two other sets are allowed. A natural question to ask is: if only a fraction of all possible combination of pairs of colors in the two other sets is permitted, what is the maximum size of the sets of these colors which can be reliably distinguished at the receivers.

We give a partial answer to this question in this paper. It was shown in [5], [7] that the achievable rates for multiple access channel (MAC) with correlated messages can be increased by adopting special channel codes which exploit the existing correlation structure in the messages. Further, it was suggested that, possibly, graphs could be used as digital interface between multiuser source coding and channel coding problems. In this work we similarly consider special channel codes and graphs for broadcast channels with correlated messages. The conventional channel codes for broadcast channels do not consider the existing correlation structure in the given messages. In other words, even though messages are correlated, they are treated as independent messages. However, if we can design special channel codes which translate the existing correlation between the messages into the channel input, we might achieve higher transmission rates than are bounded by the conventional codes.

B. Equivalence classes of graphs

Let us consider a set $\mathcal{G}$ of all the bipartite message-graphs G with fixed parameters $(\theta_1, \theta_2, \theta_1', \theta_2')$. Our previous study [5], [7] showed that this set can be divided into equivalence classes where equivalence relation is permutation and relabeling of the vertices in the graphs. In more detail, one element (or graph) in a class can be obtained from the other in the same class by permutation and relabeling of the vertices. However, if two elements(or graphs) belong to different classes, one can not be obtained from the other by permutation and relabeling since they have different correlation structures.

This property is closely related to the minimum number of channel codebooks required to reliably transmit all the message-graphs in the set $\mathcal{G}$. If we have a channel code which can reliably transmit message pairs coming from a message-graph (say G_1), then this channel code can be also used to transmit reliably message pairs coming from any message-graph that belongs to the equivalence class of G_1. Therefore, we need much less number of channel codebooks than the cardinality of the set $\mathcal{G}$ for reliable transmission by reusing the existing codebooks.

III. Achievable rate region for broadcast channels

In this section we characterize transmissibility of certain message-graphs over broadcast channels. We consider a stationary discrete memoryless broadcast channel characterized by a conditional probability distribution $p(y_1, y_2|x)$, with input alphabet given by a finite set $\mathcal{X}$, and output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$. In other words, a broadcast channel is a tuple given by $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1, \mathcal{Y}_2)$.

Definition 5: An (n, τ)-transmission system for a bipartite graph G with parameters $(\Delta_0, \Delta_1, \Delta_2, \Delta_1', \Delta_2', \mu)$ and a broadcast channel $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1, \mathcal{Y}_2)$ with "correlated" messages would involve:

1) an encoding mapping where:

$$f : E(G) \to \mathcal{X}^n, \tag{2}$$
$$\text{i.e.} \forall (i, j) \in E(\tilde{G}_m), \text{ assign } x^n = f(m, i, j), \tag{3}$$

2) decoding mappings where:

$$g_i : \mathcal{Y}_i^n \to V_i(G) \quad \text{for} \quad i = 1, 2, \quad \text{i.e.,} \tag{4}$$
$$g_1 : \mathcal{Y}_1^n \to \{1, 2, \ldots, \Delta_0\} \times \{1, 2, \ldots, \Delta_1\}, \tag{5}$$
$$g_2 : \mathcal{Y}_2^n \to \{1, 2, \ldots, \Delta_0\} \times \{1, 2, \ldots, \Delta_2\}, \tag{6}$$

3) a performance measure given by the following average probability of error criterion:

$$\tau = \frac{1}{|E(G)|} \sum_{m=1}^{\Delta_0} \sum_{(i,j) \in E(\tilde{G}_m)} Pr\Big[(g_1(Y_1^n), g_2(Y_2^n)) \neq ((m, i), (m, j)) | X^n = f(m, i, j)\Big]. \tag{7}$$

Definition 6: A tuple of rates $(R_0, R_1, R_2, R_1', R_2')$ is said to be *achievable* for a given broadcast channel with correlated message sets, if for any $\epsilon > 0$, and for sufficiently large n, there exists a bipartite graph G with parameters $(\Delta_0, \Delta_1, \Delta_2, \Delta_1', \Delta_2', \mu)$ and an associated (n, τ)-transmission system as defined above satisfying: $R_0 - \epsilon < \frac{1}{n} \log \Delta_0$,

$R_i - \epsilon < \frac{1}{n}\log\Delta_i$, $R'_i - \epsilon < \frac{1}{n}\log\Delta'_i$ for $i = 1, 2$, $\frac{1}{n}\log\mu < \epsilon$ and the corresponding average probability of error $\tau < \epsilon$.

The goal is to find the achievable rate region $\mathcal{R}$ which is the set of all achievable tuple of rates $(R_0, R_1, R_2, R'_1, R'_2)$. In the following we provide an information-theoretic characterization of an achievable rate region.

Theorem 1: For a discrete memoryless broadcast channel $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)$, a tuple of rates $(R_0, R_1, R_2, R'_1, R'_2)$ belongs to the achievable rate region if

$$R_0 < \min\{I(Z;Y_1), I(Z;Y_2)\}, \tag{8}$$

$$R_0 + R_1 < I(Z,U;Y_1), \tag{9}$$

$$R_0 + R_2 < I(Z,V;Y_2), \tag{10}$$

$$R_0 + R_1 + R_2 < \min\{I(Z;Y_1), I(Z;Y_2)\} + I(U;Y_1|Z) + I(V;Y_2|Z) - I(U;V|Z) + \alpha, \tag{11}$$

$$R'_i < R_i - \alpha, \quad \text{for } i = 1, 2, \tag{12}$$

$$R_1 + R'_2 = R'_1 + R_2 < I(U;Y_1|Z) + I(V;Y_2|Z) - I(U;V|Z) \tag{13}$$

for some distribution $p(z,u,v,x)$=$p(z)p(u,v|z)p(x|z,u,v)$ where α is a nonnegative constant which characterizes the amount of correlation between the messages, $0 \leq \alpha \leq \min\{R_1, R_2\}$ such that $(Z,U,V) \to X \to (Y_1, Y_2)$.

Note that in this characterization, there is a constraint on the distribution $p(z,u,v,x)$ that one can choose for determining the rate region. For any $p(z,u,v,x)$, and for a fixed sum rate of $\min\{I(Z;Y_1), I(Z;Y_2)\} + I(U;Y_1|Z) + I(V;Y_2|Z) - I(U;V|Z) + \alpha$, the minimum values that R_1 and R_2 can take are $I(U;Y_1|Z) - I(U;V|Z) + \alpha$ and $I(V;Y_2|Z) - I(U;V|Z) + \alpha$, respectively. Also, the rates R'_1 and R'_2 must be nonnegative. Hence the constraint on $p(z,u,v,x)$ is given by $\min\{I(U;Y_1|Z), I(V;Y_2|Z)\} > I(U;V|Z)$.

Remark 2: Note that the achievable rate region given in Theorem 1 enhances Marton's region [2] with the addition of a nonnegative constant α which is determined by the correlation between messages. When the private messages are independent, i.e., when all the elements in the set $\mathcal{W}_1 \times \mathcal{W}_2$ can occur, the rate region become exactly same as Marton's since $\alpha = 0$. However, when the private messages are correlated, i.e., only some elements in the set $\mathcal{W}_1 \times \mathcal{W}_2$ can occur, the rate region can be larger since $\alpha > 0$ in this case. As the correlation between the messages increases, in other words, as α increases, the achievable rate region also become larger.

Remark 3: The limitations of this theorem are as follows. Note that this theorem gives only a partial characterization of the set of all nearly semi-regular graphs whose edges can be reliably transmitted over a broadcast channel. In the formulation of the achievable rate region, we have the freedom of choosing a particular message-graph for every block-length n. The theorem characterizes exponential rate of growth (as a function of the number of channel uses) of size of certain nearly semi-regular graphs, such that edges coming from any such graph can be reliably transmitted over the broadcast channel. This obviously also means that it is possible to transmit edges of a graph that belongs to the equivalence class of any of these graphs. This theorem, however, does not claim that the edges of any graph with those parameters can be reliably transmitted.

IV. Representation of Correlated Sources into Graphs

In broadcast channels, the source encoder can have access to both sources to be transmitted, but the source decoders can not collaborate with each other. We consider two correlated sources S and T with a joint probability distribution $p(s,t)$ and the alphabets are given by finite sets $\mathcal{S}$ and $\mathcal{T}$, respectively. For ease of exposition, we assume that the sources do not have a common part [8], [9], [10]. The goal is to represent these sources into message sets which can be associated with message-graphs with parameters $(\Delta_0, \Delta_1, \Delta_2, \Delta'_1, \Delta'_2, \mu)$.

Definition 7: An (n,τ)-transmission system for a bipartite graph $G(\Delta_0, \Delta_1, \Delta_2, \Delta'_1, \Delta'_2, \mu)$ and a pair of correlated sources (S,T) would involve:

1) An encoding mapping:

$$f : \mathcal{S}^n \times \mathcal{T}^n \to E(G), \tag{14}$$

2) Decoding mappings g_1 and g_2:

$$g_1 : V_1(G) \to \mathcal{S}^n, \tag{15}$$

$$g_2 : V_2(G) \to \mathcal{T}^n, \tag{16}$$

3) A performance measure given by the probability of error:

$$\tau = Pr\big[f(S^n,T^n) \notin E(G)\big] + Pr\big[\{f(S^n,T^n) \in E(G)\} \cap \{(g_1(f(S^n,T^n)), g_2(f(S^n,T^n))) \neq (S^n,T^n)\}\big]. \tag{17}$$

Note that the rationale for choosing the above performance measure is the following. In the previous section it was shown that a channel coder for a broadcast channel with correlated messages provides guarantees on the probability of error only if the message pair belongs to a graph of certain parameters and no guarantees will be given otherwise. So, the source coder has to take this event into account while calculating the probability that the reconstruction source vectors are not equal to the vectors observed by the encoders.

Definition 8: A tuple of rates $(R_0, R_1, R_2, R'_1, R'_2)$ is said to be *achievable* for a source coding problem with correlated sources (S,T) (for transmission over broadcast channels), if for any $\epsilon > 0$, and for all sufficiently large n, there exists a bipartite graph G with parameters $(\Delta_0, \Delta_1, \Delta_2, \Delta'_1, \Delta'_2, \mu)$ and an associated (n,τ)-transmission system as defined above satisfying: $R_i + \epsilon > \frac{1}{n}\log\Delta_i$ for $i = 0, 1, 2$, $R'_i + \epsilon > \frac{1}{n}\log\Delta'_i$ for $i = 1, 2$, $\frac{1}{n}\log\mu < \epsilon$ and the corresponding average probability of error $\tau < \epsilon$.

The goal is to find the achievable rate region which is the set of all achievable tuple of rates $(R_0, R_1, R_2, R'_1, R'_2)$. An achievable rate region is given by the following theorem.

Theorem 2: An achievable rate region for a source coding problem with correlated sources (S,T) is given by the set of all $(R_0, R_1, R_2, R'_1, R'_2)$ such that

$$R_0 \geq I(S,T;W), \tag{18}$$

$$R_1 \geq H(S|W), \tag{19}$$
$$R_2 \geq H(T|W), \tag{20}$$
$$R_1' \geq H(S|T, W), \tag{21}$$
$$R_2' \geq H(T|S, W), \tag{22}$$
$$R_1 + R_2' = R_1' + R_2 \geq H(S, T|W), \tag{23}$$
$$R_0 + R_1 + R_2' = R_0 + R_1' + R_2 \geq H(S, T) \tag{24}$$

where W is an auxiliary random variable such that $p(w, s, t) = p(s, t)p(w|s, t)$.

V. Different Message-Graphs for a Pair of Correlated Sources in the Broadcast channel

We have shown that in the broadcast channel, for sufficiently large block length n, a pair correlated sources (S, T) can be represented as nearly semi-regular message-graphs G with parameters $(2^{nR_0}, 2^{nR_1}, 2^{nR_2}, 2^{nR_1'}, 2^{nR_2'}, 2^{n\epsilon'})$ as shown in Figure 4.

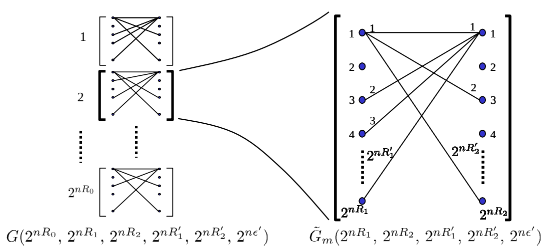

Fig. 4. Message-graph G with parameters $(2^{nR_0}, 2^{nR_1}, 2^{nR_2}, 2^{nR_1'}, 2^{nR_2'}, 2^{n\epsilon'})$ composed of 2^{nR_0} subgraphs $\tilde{G}_m(2^{nR_1}, 2^{nR_2}, 2^{nR_1'}, 2^{nR_2'}, 2^{n\epsilon'})$

Note that many different message-graphs can represent the same correlated sources without increasing the redundancy. Let us consider a case where the sum rate is minimum, i.e., $R_0 + R_1 + R_2' = H(S, T)$. In this case, the total number of edges in the graph, $|E(G)| = 2^{n(I(S,T;W)+H(S,T|W))} = 2^{nH(S,T)}$. Since we can vary the mutual information $I(S, T; W)$ such that $0 \leq I(S, T; W) \leq H(S, T)$, we can have many message-graphs without increasing redundancy in the graphs.

As examples, let us consider some special cases as follows.

- Case A: If $I(S, T; W) = 0$, then $R_0 = 0$, $R_1 = H(S)$, $R_2 = H(T)$, $R_1' = H(S|T)$, $R_2' = H(T|S)$. Roughly speaking, this corresponds to the typicality-graph of (S, T).
- Case B: If $I(S, T; W) = H(S, T)$, then $R_0 = H(S, T)$, $R_1 = 0$, $R_2 = 0$, $R_1' = 0$, $R_2' = 0$.
- Case C: If $0 < I(S, T; W) < H(S, T)$, then $R_0 = I(S, T; W)$, $R_1 = H(S|W)$, $R_2 = H(T|W)$, $R_1' = H(S|T, W)$, $R_2' = H(T|S, W)$.

In particular, consider the case when $R_0 = C(S; T)$, where $C(S; T)$ denotes the "common information" defined by Wyner [10] as follows. $C(S; T) = \inf I(S, T; W)$ where the infimum is taken over all triples of random variables S, T, W, where auxiliary random variable W take values on a finite set, and such that (1) the marginal distribution for S, T is $p(s, t)$, (2) S and T are conditionally independent given W, i.e., $S \rightarrow W \rightarrow T$ and $p(s, t, w) = p(w)p(s|w)p(t|w)$.

In this case, $R_0 = C(S; T)$, $R_1 = H(S|W)$, $R_2 = H(T|W)$, $R_1' = H(S|W)$, $R_2' = H(T|W)$ since S and T are conditionally independent given W. So, each subgraph $\tilde{G}_m$ become complete. This also means that the private messages for receiver 1 and 2 become independent. Note that a subgraph $\tilde{G}_m$ can not be complete if $\tilde{R}_0 < C(S; T)$ because, by the definition, $C(S; T)$ is the infimum of $I(S, T; W)$ such that $S \rightarrow W \rightarrow T$.

Hence Case A can be thought of as situated at one end of the spectrum, and Case B as situated on the other end of the spectrum. For every value of $I(S, T; W)$, we get an equally efficient representation of the sources into a nearly semi-regular graph.

VI. Conclusion

We have discussed the transmission of correlated sources over broadcast channels. It is shown that if one knows that only certain pairs of messages occur, the transmission rate can be increased by using the special channel encoding scheme which exploits the known characteristics of the messages. We have also considered representation of correlated sources into a common message set and a pair of correlated private message sets, which can be associated with an undirected nearly semi-regular bipartite graph (called message-graph). We have shown that a pair of correlated sources can be represented into many different message-graphs without increasing the redundancy. The goal of this work is to show that message-graphs can be used as discrete interface in this Shannon-style modular approach to multiterminal communication problems.

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