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The Multiterminal Source Coding Problem for Spatial Waves

Sergio D. Servetto Joseph M. Rosenblatt

***Abstract*— We study a problem of data compression for spatial waves. This is a new multiterminal source coding problem, in which the objects to be represented by a network of encoders are solutions of a partial differential equation. In this paper we formulate the problem, and discuss some intial results towards the determination of the rate-distortion region for waves.**

I. Introduction

A. Spatial Waves

Spatial waves arise in a number of natural and man-made phenomena. In nature, sound perceived by humans consists of small pressure waves in the range of 20Hz-20KHz, propagating in the air. Earthquakes/tsunamis occur as large pressure waves propagating in the earth's crust/oceans. Waves are also generated by humans: electromagnetic wave fields are created for purposes such as communication and object tracking, and acoustic waves are used for example to build images of internal human organs, or to locate oil. Thus, it is not hard to argue that there is great interest in understanding how to perform signal processing tasks on spatial waves. Of particular interest in this work is the task of *compressing* these waves.

Spatial waves form a fundamentally different class of objects from those traditionally studied in signal processing theory. Classical DSP is built on the assumption that signals are *bandlimited* functions of their arguments, meaning that their Fourier transform is compactly supported. Spatial waves are not bandlimited functions of their arguments – the proper abstraction for waves is to regard them as solutions of a partial differential equation. And this shift in point of view (from signals being bandlimited functions to signals being solutions of the wave equation), when it comes to the data compression problem for waves, brings out some new challenges, which we begin to address in this work, whose long term goal is to determine the rate-distortion characteristics of waves.

B. A Simple Abstraction

What is a wave field? Perhaps the simplest explanation can be given in terms of the spring-mass model for an oscillator [11, Ch. 1], as illustrated in Fig. 1.

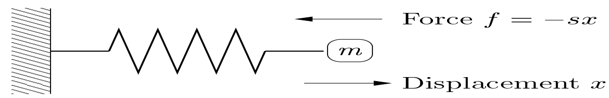

Fig. 1. Spring-mass model for an elementary (small) oscillation. A small mass is tied to one end of a spring, which at the other end is fixed. Motion is one-dimensional. The mass is displaced from its rest position by a distance x, and the spring exerts a force on that mass proportional to this distance.

In the simple physical system of Fig. 1, when energy is supplied in the form of displacing the elemental mass from its rest position, the spring produces another force which attempts to bring the mass back to rest. This force sets the mass in motion, and results in a certain pressure on the mass. Clearly, the motion will be an oscillation in the horizontal direction: the spring pulls the mass back, then pushes it away, then pulls it back again, and so on, until all the energy supplied by the initial displacement is dissipated (or forever, if there is no dissipation at all).

The next step is to extend this model to a one-dimensional vibration. And for this purpose, consider a chain of masses all interconnected by springs, as illustrated in Fig. 2. A numerical solution to the 1D wave equation, for a simple excitation applied at the center of the membrane, is illustrated in Fig. 3.

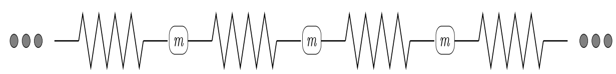

Fig. 2. Multiple springs and masses. In this setup, displacing one of the elemental masses from its rest position, besides setting that mass in motion as in the simpler model of Fig. 1, also has the effect of inducing motion on neighboring masses. Therefore, the oscillations of individual masses are clearly not independent, but coupled through the springs that bind them.

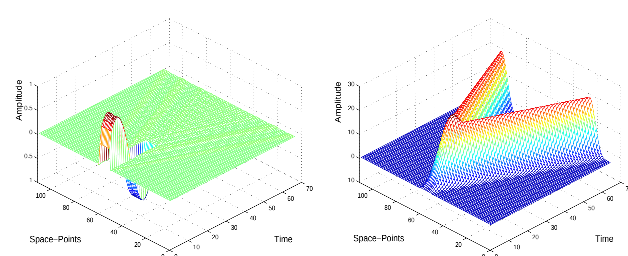

Fig. 3. An example of a 1D vibration, illustrated in space and time. Left: excitation applied at the center of the membrane. Right: resulting wave.

With this example in one dimension clear, it is not difficult to extend this concept to multiple dimensions, and allowing for inhomogeneities (springs with different characteristics in terms of stiffness), and boundary conditions. In this work, we will limit our attention to the simple homogeneous 1D case.

S. D. Servetto is with the School of Electrical and Computer Engineering, Cornell University. URL: http://cn.ece.cornell.edu/. J. M. Rosenblatt is with the Department of Mathematics, University of Illinois, Urbana-Champaign. URL: http://www.math.uiuc.edu/~jrsnbltt/. Work supported by the National Science Foundation, under awards CCR-0238271 (CAREER), CCR-0330059, and ANR-0325556.

C. Paper Summary

In this paper we study the problem of efficiently encoding waves. First, in Section II we develop a simple mathematical abstraction for waves, as well as the tools for their analysis. Then, in Section III we formulate the problem of coding waves. In developing a solution to the coding problem, we identify a basic sampling problem, for which a number of results from our work in progress are reviewed in Section IV. A survey of some related work is presented in Section V. The paper concludes with Section VI.

II. A Simple Abstraction for Spatial Waves

A. Finding Good Bases to Represent Waves

Sensing and actuation problems on waves do not fit into the classical signal processing paradigm of modeling signals as bandlimited processes.

The sinc basis functions used to represent a signal in terms of its samples, and the basis of complex exponentials used in the Fourier transform, provide a powerful set of tools for processing bandlimited signals using linear and time-invariant operators:

- Using the sinc basis, a bandlimited signal can be represented in terms of a set of its samples. Thus, any operation we may wish to apply to the continuous-time signal can be expressed in terms of operations on these samples.
- Using the basis of complex exponentials, the design and analysis of linear and time-invariant operators is greatly simplified, since the former are eigenfunctions of the latter.

These two observations provide the foundation for digital signal processing: a linear and time-invariant operator acting on a bandlimited signal can be implemented as an A/D coverter, a digital filter, and a D/A converter [16].

Whereas this is a richly developed theory with countless applications in real life, waves are simply not well modeled as bandlimited signals, and the type of operations we may want to apply on them are just not well modeled by linear and time-invariant operators. In this section we will see however that a different basis, chosen to diagonalize a different operator, plays an equally fundamental role for waves.

B. The Wave Equation

Consider a 1D membrane of length π, consisting of a large number of elemental masses connected by springs, as depicted in Fig. 2. Acting on this membrane is a source that at time t, at location x, applies a force $s(x,t)$. The source s generates waves in the membrane, inducing a pressure field p. And from elementary newtonian mechanics relating mass, pressure, force, position, velocity and acceleration, we can obtain a differential equation that constrains the values that p and s can take simultaneously: for all $x \in [0,\pi]$, for all $t \in \mathbb{R}$, we must have that

$$\frac{\partial^2 p(x,t)}{\partial^2 x} + \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial^2 t} + s(x,t) = 0.$$

See, for example, the textbook of Berkhout [5] for a thorough treatment on this subject.

For a given source s and fixed boundary conditions, it is not difficult to prove that the wave equation admits a unique solution for p. Thus, our first goal is to express that unique solution p in terms of s. To do that, we proceed as follows [9, Ch. 2]:

1) First we consider the problem in the Fourier domain: regarding the spatial coordinates as constants, we take a Fourier transform of both $p(x,t)$ and $s(x,t)$ with respect to their time variable t. The resulting spectra are denoted by $\hat{p}(x,\omega)$ and $\hat{s}(x,\omega)$.
2) We expand both the source $\hat{s}(x,\omega)$, and the sought solution $\hat{p}(x,\omega)$, into sine and cosine spatial eigenfunctions of the differential equation [6].
3) We plug both the source function and the solution that we seek into the forced differential equation, and we collect the coefficients associated with each eigenfunction [7].
4) We pick a set of boundary conditions, in our case indicating that the boundaries are perfectly reflecting. This is captured by requiring that for all t, $\frac{\partial p}{\partial x}|_{x=0} = \frac{\partial p}{\partial x}|_{x=\pi} = 0$.

Following these steps, it is not difficult to prove that the solution can be expressed as

$$\hat{p}(x,a) = \sum_{n=0}^{\infty} p_n(a)\cos(nx),$$

with coefficients

$$p_n(a) = \frac{\frac{2}{\pi}\int_0^{\pi}\hat{s}(x,a)\cos(nx)\mathrm{d}x}{n^2 - a^2},$$

and where c is the speed at which waves propagate in the membrane, $a \triangleq \frac{\omega}{c}$ is a new frequency variable (normalized by the speed of propagation c), and $\hat{s}$ is the spectrum of the source [21].

C. A Linear System Interpretation for Wave Fields in Membranes with Point Sources.

1) Point Sources in Membranes of Finite Length: The solution to the wave equation shown above makes no assumptions whatsoever about the shape of the source: this source was assumed to be a general function acting anywhere on the membrane. We do have a special interest though in wave fields that arise from considering *point sources*, i.e., sources that occupy a contiguous region of space that is very small compared to the size of the membrane. An arbitrary point source at location x_0, in the frequency domain, has the form

$$\hat{s}(x,a) = \hat{s}(a)\delta(x - x_0),$$

where $\hat{s}$ is the temporal spectrum of a temporal source. Our interest in point sources is motivated by the fact that most fields arising in the applications that motivate us to study this problem can be modeled quite accurately as being generated by a finite number of point sources.

2) The Frequency Response of a Finite Membrane: To obtain a simpler expression for p, we need to evaluate the coefficients $p_n(a)$, now taking into account the structure of this source. By a simple substitution in the solution for p, we get that

$$\hat{p}(x,a) = \hat{s}(a)\hat{h}(x,a),$$

where $\hat{h}$ is defined by $\hat{h}(x,a) = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(nx_0)\cos(nx)}{n^2-a^2}$. And this expression provides much insight into the structure of wave fields generated by point sources:

- The pressure signal observed at an arbitrary location x can be expressed as the convolution of the source signal $s(t)$ with a linear filter $h(x,t)$. For obvious reasons, we refer to this function as the *impulse response of the membrane*, and to its Fourier transform as the *frequency response*.
- The filter $h(x,t)$ depends on a number of parameters:
  - The location x of the field measurement.
  - The location x_0 of the source signal.
  - The dimension of the membrane (π here, chosen for convenience).
  - The speed c of wave propagation in the membrane.
- $\hat{h}(x,a)$ has an asymptote at all integer frequencies $a \in \mathbb{Z}$. These special frequencies are referred to as the *modes* of the membrane [11].

Equipped with this toolbox, we formulate next the problem of encoding solutions to the wave equation driven by a spatial source of information $s(x,t)$.

III. A Multiterminal Source Coding Problem

In this section, we formulate the problem of coding waves. To do so, we define a class of codes intended to capture what we deem to be the essence of a fundamental problem of data compression in sensor networks. We ask whether it is possible to find N locations such that, if we observe a spatial data field (waves, in this case) at each one of them, and deliver an encoding of these observations to a central decoder, then that central encoder can produce an estimate of the entire field within a given distortion D. Formal definitions follow.

A. Formal Definitions

Let $(X_t)_{t\in\mathbb{R}}$ denote a continuous-time, stationary random process, and let $\mathcal{X}$ denote the set of all such processes. Let $(P_{x,t})_{0\le x\le\pi, t\in\mathbb{R}}$ denote the unique solution of the wave equation, when forced using X at some fixed and known location $x_0 \in [0,\pi]$, and let $\mathcal{P}$ denote the set of all such solutions. For a fixed location $\xi \in [0,\pi]$, $P(\xi,t)_{t\in\mathbb{R}}$ denotes the wave measured at ξ, and the set of all such measurements is denoted by $\mathcal{P}(\xi)$.

For two solutions $P, Q \in \mathcal{P}$, at time t and location x, define $d_{x,t}(P,Q) = E\big((P_{x,t} - Q_{x,t})^2\big)$. Based on this, define a distortion measure between P and Q as

$$d(P,Q) \;=\; \lim_{T\to\infty}\frac{1}{\pi T}\int_{x=0}^{\pi}\int_{t=-\frac{T}{2}}^{\frac{T}{2}} d_{x,t}(P,Q)\;\mathrm{d}x\mathrm{d}t.$$

For a fixed block length n, and fixed locations $0 \le \xi_1 < ... < \xi_N \le \pi$, the *encoders* are N functions

$$f_k : \mathcal{P}(\xi_k) \to \{1...2^{nR_k}\},$$

for $k = 1...N$. The *decoder* is a function

$$g : \Pi_{k=1}^{N}\{1...2^{nR_k}\} \to \mathcal{P}.$$

A $\left(2^{nR_1}...2^{nR_N}, \xi_1...\xi_N, n, N, \bar{D}\right)$ code is defined by:

- a block length n;
- a set of N measurement locations $\xi_1...\xi_N$, for some finite value of N;
- encoding functions f_k, $k = 1...N$;
- a decoding function g;
- and a distortion $\bar{D} = d\Big(P, g\big(f_1(P_{\xi_1,t})...f_1(P_{\xi_N,t})\big)\Big)$.

We say that the rate-distortion tuple $(R_1...R_N, D)$ is *achievable with N encoders* if there exists a fixed finite value of N such that, for any $\epsilon > 0$ and sufficiently large n, there is a $\left(2^{nR_1}...2^{nR_N}, \xi_1...\xi_N, n, N, \bar{D}\right)$ code for which $\bar{D} < D + \epsilon$. The *rate region with N encoders* $\mathcal{R}_N(D)$ is the closure of set of rates $(R_1...R_N)$ such that, for fixed D, we have that $(R_1...R_N, D)$ is an achievable rate-distortion tuple with N encoders. Finally, we define the *distributed rate-distortion function* $\mathcal{R}(D)$, by

$$\mathcal{R}(D) \quad = \quad \inf_{N\ge 1}\sum_{k=1}^{N} R_k, \tag{1}$$

where $(R_1...R_N) \in \mathcal{R}_N(D)$.

B. Problem Statement

In this problem, we have two goals:

- First, we need to make sure that the object $\mathcal{R}_N(D)$ is well defined. For that purpose, we need to verify that there exists a value of N that is independent of D, for all possible distortion levels.[1]
- Second, but more fundamentally, we need to give a complete and computable characterization of the object $\mathcal{R}(D)$, in terms of information theoretic quantities.

Issues related to the first of these problems are addressed next.

IV. Sampling Waves

A. Source Coding of Continuous-Parameter Processes

Perhaps the simplest way to highlight the challenges involved in coding waves is to draw a parallel with other similar compression tasks that are better understood: source coding of continuous-time bandlimited processes. So consider a stationary Gaussian process X_t, with mean zero and a bandlimited autocorrelation function $R_X(\tau)$, of bandwidth Ω. It is well known that the discrete-time process X_{nT}, with $T < \frac{1}{2\Omega}$, can be interpolated to recover the original process X_t in the mean-squared sense [17], [22]. Then, the discrete-time process can be compressed using standard techniques [4].

[1] Without this condition we would have to specify, for each possible distortion value, how many encoders are required to achieve it. This would be very cumbersome, and unsatisfactory too.

We see in the example above that a key step in solving the source coding problem for continuous-time bandlimited processes is the *sampling* step. This form of discretization has its roots in Shannon's sampling theorem, which states that any signal with finite squared integral and Fourier transform supported over $[-\pi, \pi]$ can be expressed as

$$f(t) \;=\; \sum_{n\in\mathbb{Z}} f(n)\frac{\sin\big(\pi(t-n)\big)}{\pi(t-n)}. \qquad (2)$$

For waves however, there are two reasons that make finding one such discrete representation a somewhat more difficult matter:

- Waves, defined over a finite membrane, cannot be bandlimited functions of space. So in the classical sense, there is a severe aliasing effect to deal with.
- Practical considerations, stemming from difficulties in physically realizing devices capable of observing the entire signal field in space (and already embedded in our codes defined in Section III), preclude the use of bandlimiting operators such as an analog anti-aliasing prefilter.

Our first result towards the computation of $\mathcal{R}(D)$ is that a representation such as that of eqn. (2) for bandlimited signals is indeed possible for waves.

B. The Sampling Problem

1) Temporal Sampling: Assume the source $\hat{s}(a)$ itself is a bandlimited function in time, of normalized bandwidth a_0. Thus, for any fixed location $0 \le x \le \pi$, the support of $\hat{p}(x,a)$ is also a confined to a range of size at most a_0. Therefore, it follows from classical arguments that the set of samples $\{p(x,mT)\}_{m\in\mathbb{Z}}$ uniquely specifies the continuous time function $p(x,t)$, provided $T < \frac{1}{2a_0}$. If the source itself is not bandlimited, then the temporal sampling problem can be dealt with using standard techniques for dealing with such cases [23], [24], [25].

2) Spatial Sampling: From the definition of the impulse response of the membrane earlier, it is easy to see that in general, the pressure signal at arbitrary locations x and x' are related by

$$\hat{p}(x',a) = \frac{\hat{p}(x,a)\hat{h}(x',a)}{\hat{h}(x,a)}. \qquad (3)$$

This is immediate, since $\frac{\hat{p}(x,a)}{\hat{h}(x,a)} = \hat{s}(a)$. This relationship needs some qualifiers though, since two conditions may occur to render it invalid:

(1) If for some mode $m \in \mathbb{Z}$, $\hat{s}(m) \neq 0$, then $\hat{p}(x,m)$ is not defined.

(2) If at some location x there is a frequency a for which $\hat{h}(x,a) = 0$, then a "cannot be heard" at x, and so any information in a will be missing from the measurements collected at x.

The relationship in eqn. (3) allows us to highlight what exactly are the challenges involved in the spatial sampling problem. To sample in space, essentially what we have to do is "deal" with zeros in the impulse response $\hat{h}(x,a)$. And our way of dealing with those zeros is as follows: we write, for all $0 \le x \le \pi$ and $a \in \mathbb{R}$,

$$\hat{p}(x,a) = \sum_{k=1}^{n} \hat{f}_k(x,a)\hat{p}(\xi_k,a). \qquad (4)$$

Such an identity essentially says that we can express $\hat{p}$ as a linear combination of some interpolating functions $\hat{f}_k$, and where the "coefficients" of this linear combination are given by samples of $\hat{p}$ taken at a finite set of locations ξ_k ($k = 1...n$). Note that this relationship is given in the frequency domain: in time, what we are saying is that we want to be able to find a linear filter to interpolate the time functions in space.

C. Existence of a Sampled Representation

To ensure the existence of a representation such as that of eqn. (4), it is sufficient to show that for a fixed and finite set of locations $\{\xi_1...\xi_N\}$, and for all a in the support of $\hat{s}$, $\hat{p}(\xi_k,a) \neq 0$, for at least one ξ_k.

Assume $\widehat{p}(x,a) = \widehat{s}(a)\widehat{h}(x,a)$ is defined on a compact set I such that I does not contain any integers n ($n \ge 0$).

Theorem 1: There exists $N \ge 1$ and locations $\{\xi_1...\xi_N\}$, such that for all $a \in I$, there is an ξ_k for which $\widehat{h}(\xi_k,a) \neq 0$.

Proof: See [21]. ■

V. Related Work

There is an extensive body of related work.

Wave propagation problems have been studied for a long time in physics and engineering, and they are reasonably well understood. Some textbooks we found particularly useful are [5], [10], [11]. The standard mathematical tool for dealing with these problems are differential equations with boundary value constraints, and Hilbert space methods for their solution. In this area, textbooks we relied on heavily are [6], [7], [9].

Our approach to the sampling problem relies heavily on properties of analytic functions, and on the theory of uniform distribution of sequences. For this, we have found of particular help the textbooks of Ahlfors on complex analysis [1], and of Kuipers and Niederreiter on uniform distribution [13]. For elementary concepts on real variables we relied on [18].

Data modeling issues for problems originating in sensor networking applications (one of the motivations for this work), have been considered in [20], in [19], in [15], and in [3].

In the signal processing literature, questions involving spatial signal fields have been studied in the context of sound and light fields. A number of references on data compression for light fields are available from `http://www.stanford.edu/~chuoling/lightfield.html`. For sound fields, the impulse response $h(x,t)$ has been referred to as the *plenacoustic* function [12]. Spatial sampling and reconstruction of this function has been considered in [2], and spectral properties of this function were studied in [8], under a far-field assumption. A closely related signal processing problem is that of wave field *synthesis*: in this case, the goal is to generate from a finite number of fixed point sources a pre-specified wave field [14]. Various groups, primarily in Germany and in the Netherlands, have studied this problem. A number of references on various aspects of this problem are available from http://www.lnt.de/LMS/.

In summary, most of the related work we are aware of has focused on applications and signal processing aspects of the problem under consideration, but little attention seems to have been paid to pure information theoretic questions.

VI. Conclusions

In this paper, we have formulated a new multiterminal source coding problem. In our problem, the goal is to determine the least number of bits required to represent a wave field in space, subject to a fidelity criterion, and subject to the constraints encountered in sensor networking problems which prevent any one encoder from observing the entire signal field. This is only the beginning of our work on this problem, and much remains to be done.

For Theorem 1 to be a complete solution to the sampling problem in space, we still need to provide an *estimate* of how large N needs to be made, in terms of known parameters in the problem – this would be a condition equivalent to the determination of the Nyquist rate for bandlimited signals. We do not have complete answers yet to this issue, but this is a topic currently under investigation, on which we hope to report more in the final version of [21]. We have obtained one preliminary estimate, which we do *not* believe to be tight in any sense, but which we mention to illustrate the kind of characterization we are looking for. We have found that the solution of the forced equation $\widehat{p}(x,a)$ can be reconstructed everywhere in a closed interval $I \subset (n_o, n_o+1)$ where a varies and $n_o \in \mathbb{Z}$, from at most N linear filters, provided $N \geq C_o \frac{n_o \log(n_o+1)}{\Delta^2}$, and where Δ is the minimum distance from I to one of the resonant integer frequencies n_o or n_o+1.

Our goal for this paper was to introduce the problem, and to illustrate our first steps towards its solution. The end goal though, as stated before, is to effectively compute the distributed rate-distortion function $\mathcal{R}(D)$ of eqn. (1).

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