On Superposition Coding and Beamforming for the Multi-Antenna Gaussian Broadcast Channel

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***Abstract*— The capacity region of the Gaussian multiple-input, single-output (per user) Broadcast Channel (BC) using superposition coding is analyzed. Perfect channel state information is assumed both at the transmitter and the receivers. The achievable Signal-to-Interference-plus-Noise Ratio (SINR) region of BC and MAC using beamforming is considered, and it is shown that existing SINR-balancing results extend to the case of any cross-talk matrix between the different users. Optimal beamforming vectors are found, simplifying thus the analysis of the achievable rate-region. Finally, a modified form of beamforming is applied as an inner bound for the Van der Meulen - Hajek - Pursley (CMHP) rate-region in this model, and this is compared to the optimal capacity region and also to other sub-optimal strategies such as zero-forcing, for the BC.**

I. Introduction

In this work, we consider the problem of efficient coding for the memoryless gaussian vector BC with perfect CSI at both the transmitter and the receiver. Contrary to the scalar gaussian BC, it is a non-degraded BC. Only recently the optimal transmission strategy and capacity region of MIMO gaussian BC were found, by the use of Dirty-Paper Coding (DPC), capitalizing on the fact that interference from other users may be viewed as channel side information known non-causally to the transmitter [1]. It was also shown that capacity region in this case equals Marton's lower bound [2], defined as

$$\begin{aligned} \mathbf{R}^{DPC} = & \bigcup_{P_{U_0,U_1,U_2} P_{X|U} P_{Y|X}} \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : \right. \\ & R_1 \le I\left(U_0, U_1; Y_1\right), \\ & R_2 \le I\left(U_0, U_2; Y_2\right). \\ & R_1 + R_2 \le \min\left\{ I\left(U_0; Y_1\right), I\left(U_0; Y_2\right)\right\} + \\ & \quad I\left(U_1; Y_1|U_0\right) + I\left(U_2; Y_2|U_0\right) - I\left(U_1; U_2|U_0\right), \end{aligned} \tag{1}$$

where P_{U_0,U_1,U_2} is a joint distribution. Some practical techniques for implementation of DPC have been suggested in the literature (e.g. [3], [4], [5]), however the precoding implementation still remains a difficult and costly task. The complexity of DPC and sensitivity on availability of high quality channel state information on the transmitter side has motivated the search of other efficient and robust sub-optimal coding techniques. One major group of those methods is beamforming.

Optimal beamforming and power control for the vector BC with linear transmitter and receivers was found in [6]. Characterization of the capacity region was achieved using SINR-balancing technique. While the result just mentioned deals with optimal beamforming design, there are many other beamforming strategies which assume a given transmitter and receiver structure. Perhaps the most popular and simple one is the Zero-Forcing (ZF) precoder [7]. A nice comparison between the performance of some coding techniques just mentioned and some others could be found at [8].

Recently, the employment of CMHP inner bound for MIMO BC was considered as a sub-optimal coding strategy [9]. The information theoretic definition of the two-user CMHP rate region due to [10], [11], [12], is given by (1), where U_0, U_1, U_2 are chosen to be independent. The main difference compared to regular beamforming is that the receiver is not restricted to be linear any longer. More specifically, the receivers may performs joint or successive decoding. In this work the CMHP rate-region of the vector Gaussian BC is analyzed. A lower bound of the achievable rate-region, assuming Gaussian signaling and successive decoders with common information being decoded first on both receivers, is being analyzed. Such a decoding order is known to be sum-capacity optimal. For simplicity, we demonstrate the arguments for a two user case.

First step of the analysis requires the derivation of a dual MAC. Beamforming duality was already found by [13], [14]. The general up-link - down-link duality presented in [14] was employed for SINR duality and DPC duality. In this work, this duality result is extended to take into consideration any Cross-Talk Matrix (CTM) between the different users. Although the resulting dual MAC is in general non-convex, deriving the dual channel is beneficial as it reveals the optimal beamforming vectors, given any power allocation. Optimal beamformers turn out to be the Minimum Mean Square Error (MMSE) receivers on the MAC domain. As a result, the sumrate optimization problem boils down to a power control problem and the SINR-balancing problem [6] extends to BC - MAC with any CTM.

Finally, analysis of CMHP rate-region is being performed. The main obstacle introduced during the analysis is treating the "min" operator. Overcoming this obstacle requires the development of an extended BC system where not all the optimal beamforming vectors are identical to the MMSE receiver. An iterative algorithm which serves as a lower bound to the two-user CMHP rate-region, is presented. A more detailed analysis of the problem may be found in [15].

II. System Model

We consider the Gaussian, complex and memoryless, discrete-time vector BC with K receivers, each equipped with a single antenna, and a base-station transmitter with M antennas, defined by

$$\begin{bmatrix} y_1 \\ \vdots \\ y_K \end{bmatrix} = \mathbf{H}\mathbf{x} + \begin{bmatrix} n_1 \\ \vdots \\ n_K \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_K \end{bmatrix} \tag{2}$$

where $\mathbf{x} \in \mathbb{C}^{\mathbf{M}}$, $y_k \in \mathbb{C}$ and $n_k \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_{z,k})$ are the input, the outputs and the circularly symmetric Gaussian noise. $\mathbf{h}_k \in \mathbb{C}^{1\times M}$ are the frequency flat channel vectors from the transmitter to the k-th receiver. Without loss of generality, we assume from now on that $E(n_k n_k^*) = 1$. The covariance matrix of the input signal is subject to a total power constraint $tr\left(E[\mathbf{x}\mathbf{x}^\dagger]\right) \leq P_{max}$. The channel matrix is assumed to stay constant and perfectly known at the receivers and the transmitter. We assume the transmitter chooses codewords meant for different users independently, and that codewords are Gaussian distributed. Making this assumption, the vector BC may be alternatively defined by

$$y_k = \mathbf{h}_k \sum_{i=1}^{K} \mathbf{x}_i + n_k, \quad 1 \leq k \leq K \tag{3}$$

where $\mathbf{x} = \sum_{i=1}^{K} \mathbf{x}_i$. The covariance matrix of each independent input $\mathbf{x}_i$ is defined as $\mathbf{Q}_i \triangleq E[\mathbf{x}_i\mathbf{x}_i^\dagger]$.

We assume that the base-station sends a single beam to each user, so that the covariance matrices are rank 1. Each covariance matrix is therefore given by $\mathbf{Q}_k \triangleq \mathbf{u}_k q_k \mathbf{u}_k^\dagger$, where q_k is the power allocation vector for the k-th beam and $\mathbf{u}_k \in \mathbb{C}^{M\times 1}$ is the beamforming vector, assumed normalized so that $\sqrt{u_1^2 + \ldots + u_K^2} = 1$, $\forall k$. For simplicity of notation, all beamforming vectors are gathered together into a single matrix $\mathbf{U}^{M\times K} \triangleq [\mathbf{u}_1, \ldots, \mathbf{u}_K]$, and we define $\mathbf{q} \triangleq [q_1, \ldots, q_K]$ so that the total power constraint translates into $\sum_{j=1}^{K} q_j \leq P_{max}$. The achievable rate vector is denoted by $\left[R_1^{bc}, \ldots, R_K^{bc}\right]$. The BC system is described by the left half of Fig. 1.

Consider now the dual vector MAC which is derived by flipping the roles of transmitter and receivers of the vector BC. The dual MAC may be mathematically defined by

$$\tilde{\mathbf{y}} = \sum_{k=1}^{K} \mathbf{h}_k^\dagger \tilde{x}_k + \tilde{n} \tag{4}$$

where $\tilde{x}_k \in \mathbb{C}$, $\tilde{\mathbf{y}} \in \mathbb{C}^{\mathrm{M}}$ and $\tilde{\mathbf{n}} \sim N_{\mathbb{C}}(0, \Sigma_z)$ are the inputs, the output and the circularly symmetric Gaussian noise vector at the base-station. We assume $E\left(\mathbf{n}\mathbf{n}^\dagger\right) = \mathbf{I}$. The vector MAC transmitters power are defined as $p_k \triangleq E[\tilde{x}_k\tilde{x}_k^\dagger]$ and each user is subject to individual power constraint $p_k \leq P_k^{max}$. The receiver is assumed to implement linear reception strategy by utilizing a different beamforming vector $\tilde{\mathbf{u}}_k$ for demodulating the data stream sent for each user k. The receiver estimation of data stream $\tilde{x}_k$ is denoted by $\hat{x}_k$.

For simplicity of notation, all beamforming vectors are gathered together into a single matrix $\tilde{\mathbf{U}}^{M\times K} \triangleq [\tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_K]$, and we define $\mathbf{p} \triangleq [p_1, \ldots, p_K]$. The achievable rate vector is denoted by $[R_1^{mac}, \ldots, R_K^{mac}]$. The MAC system is described by the right half of Fig. 1.

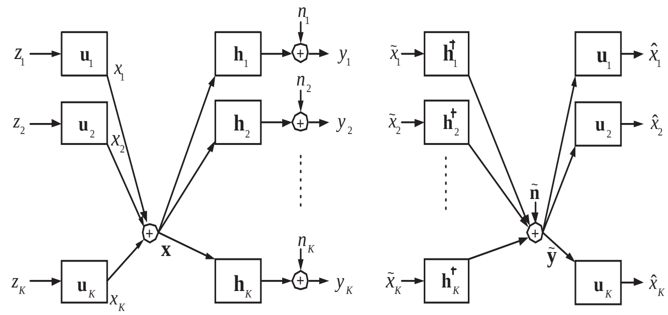

Fig. 1. System model for vector BC (left) and vector MAC (right) channels with beamforming

Assume now that the beamforming processing scheme, depicted by Fig. 1, is combined with interference cancellation which corresponds to any general CTM. Dirty-paper coding, successive cancellation, off-line cooperation or any other method that comes to mind may be used for the interference cancellation. Consider first the BC case, characterized its binary cross-talk matrix $\mathbf{\Psi}^{K\times K}$, $\mathbf{\Psi}_{m,n} \in \{0,1\}$. If the transmission to user n interferes with the reception of user m, we set $\mathbf{\Psi}_{m,n} = 1$. Otherwise $\mathbf{\Psi}_{m,n} = 0$. Note that the diagonal elements of $\mathbf{\Psi}$ equal zero by definition. Similarly, the dual MAC with general CTM is characterized by the same binary cross-talk matrix $\mathbf{\Psi}$ defined before so that if the transmission by user n interferes with the reception of user m $\mathbf{\Psi}_{n,m} = 1$ and otherwise $\mathbf{\Psi}_{n,m} = 0$.

The SINRs achieved by such pair of uplink-downlink systems, for each of the K users are given by the following equations

$$SINR_k^{bc}(\mathbf{q}, \mathbf{H}, \mathbf{U}, \mathbf{\Psi}) = \frac{\mathbf{h}_k \mathbf{u}_k q_k \mathbf{u}_k^\dagger \mathbf{h}_k^\dagger}{1 + \sum_i \mathbf{\Psi}_{k,i} \mathbf{h}_k \mathbf{u}_i q_i \mathbf{u}_i^\dagger \mathbf{h}_k^\dagger}, \tag{5}$$

$$SINR_k^{mac}(\mathbf{p}, \mathbf{H}, \mathbf{U}, \mathbf{\Psi}) = \frac{\mathbf{u}_k^\dagger \mathbf{h}_k^\dagger p_k \mathbf{h}_k \mathbf{u}_k}{1 + \sum_i \mathbf{\Psi}_{k,i}^T \mathbf{u}_k^\dagger \mathbf{h}_i^\dagger p_i \mathbf{h}_i \mathbf{u}_k}, \tag{6}$$

so the set over which the denominator is summed depends on $\mathbf{\Psi}$. Two specific examples are two-user BC with dirty paper coding (or MAC with successive decoding) [16], described by the following CTM

$$\mathbf{\Psi}^{dp1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{\Psi}^{dp2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{7}$$

for each encoding order respectively, or the case of no interference cancellation (linear transmitter and SINR receiver) [6]

$$\mathbf{\Psi}^{sinr} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{8}$$

III. Beamforming Duality

The purpose of this section is to prove that for any beamforming matrix $\mathbf{U}$, there exists a BC - MAC power transformation that keeps the total power equal. In addition, using the same beamformers on the downlink and uplink domains results in the same SINRs and achievable rates. The BC achievable rate-region, under total power constraint $P_{\max}$, equals

$$\mathbf{R}^{bc}\left(P_{\max},\mathbf{H},\mathbf{\Psi}\right) \triangleq \bigcup_{\mathbf{U},\sum_{\mathbf{k}}\mathbf{q_k}\leq \mathbf{P}_{\max}} \left\{\mathbf{R}: R_k = log\left(1+SINR_k^{bc}\right)\right\}, \quad (9)$$

and the MAC achievable rate-region, under total power constraint $P_{\max}$, equals

$$\mathbf{R}^{mac}\left(P_{\max},\mathbf{H},\mathbf{\Psi}\right) \triangleq \bigcup_{\mathbf{U},\sum_{\mathbf{k}}\mathbf{p_k}\leq \mathbf{P}_{\max}} \left\{\mathbf{R}: R_k = log\left(1+SINR_k^{mac}\right)\right\}. \quad (10)$$

Theorem 1 (BC - MAC beamforming duality): The rate-region of vector BC with power constraint $P_{\max}$ and cross-talk matrix $\mathbf{\Psi}$ is equal to the rate-region of the dual vector MAC with the same total sum-power constraint

$$\mathbf{R}^{bc}\left(P_{\max},\mathbf{H},\mathbf{\Psi}\right) = \mathbf{R}^{mac}\left(P_{\max},\mathbf{H},\mathbf{\Psi}\right)$$

In addition, any rate-tuple achieved on one channel given any power allocation and a given beamforming matrix $\mathbf{U}$, is also achieved on its dual channel using the same beamforming matrix $\mathbf{U}$ and a different power allocation with the same total power. The BC power allocation is defined by:

$$b_i \triangleq \frac{SINR_i^{mac}}{\mathbf{u}_i^\dagger\mathbf{h}_i^\dagger\mathbf{h}_i\mathbf{u}_i\left(1+SINR_i^{mac}\right)} \quad (11)$$

$$\mathbf{q} = \left(\mathbf{I} - diag\left(b_i\mathbf{u}_i^\dagger\mathbf{h}_i^\dagger\mathbf{h}_i\mathbf{u}_i\right) - diag\left(b_i\right)\mathbf{B}\right)^{-1}\mathbf{b}, \quad (12)$$

where $\mathbf{B}\left(i,j\right) \triangleq \mathbf{\Psi}_{i,j}\cdot\mathbf{u}_j^\dagger\mathbf{h}_i^\dagger\mathbf{h}_i\mathbf{u}_j$. The MAC power allocation is defined by:

$$a_i \triangleq \frac{SINR_i^{BC}}{\mathbf{u}_i^\dagger\mathbf{h}_i^\dagger\mathbf{h}_i\mathbf{u}_i\left(1+SINR_i^{BC}\right)}, \quad (13)$$

and

$$\mathbf{p} = \left(\mathbf{I} - diag\left(b_i\mathbf{u}_i^\dagger\mathbf{h}_i^\dagger\mathbf{h}_i\mathbf{u}_i\right) - diag\left(b_i\right)\mathbf{B}^T\right)^{-1}\mathbf{a}. \quad (14)$$

The proof of the theorem is a direct extension of [17] to the case of general CTM.

IV. Optimal Beamforming - No Common Information

In this section we find the optimal beamforming for achieving the rate-region boundaries of BC with general CTM. When no common information is introduced, beamforming turns out to be the optimal transmit strategy [15]. Although this section stands for itself, it also provides the basis for the analysis of the less trivial CMHP BC problem, which will be discussed next. Two complementary techniques for characterizing the boundaries of the achievable rate-region are examined, SINR-balancing and rate-weighting. Those two techniques have different quality of service interpretations. The quasi-convex SINR-balancing problem [18] is defined by the following maxmin optimization

$$\max\left(\min_{1\leq i\leq K}\left(SINR_i^{bc}/\gamma_i\right)\right)$$

while the rate-weighting optimization problem is defined as

$$\max\left(\sum_{k=1}^{K}\mu_k R_k^{bc}\right).$$

Both optimizations are performed given a total power constraint $P_{\max}$. Note that the rate weighting optimizations problem is non-convex in general, except for the DPC case.

We start by addressing the SINR balancing problem. Optimal beamforming analysis of the SINR balancing problem for linear transmitter and linear SINR receivers was given in [6]. The specific problem of DPC for the BC was also solved in [16], however using a direct analysis. We show that the a similar solution to [6] holds for BC - MAC with any CTM. Instead of analyzing the original problem on the BC domain, the analysis is performed on the dual MAC domain. The main motivation behind this strategy is that optimal beamforming vectors are easily found on the MAC domain and equal the well-known MMSE receiver

$$\mathbf{u}_k^{MMSE} = \left(\mathbf{I} + \sum_i \mathbf{\Psi}_{k,i}^T\mathbf{h}_i^\dagger p_i\mathbf{h}_i\right)^{-1}\mathbf{h}_k^\dagger, \quad 1\leq i\leq K \quad (15)$$

For the sake of brevity we omit the detailed analysis and give only the final iterative algorithm, without any proof of convergence. For more details, the reader is referred to [6], [15]. Defining

$$\mathbf{d} \triangleq \left[\gamma_1\Big/\mathbf{u}_1^\dagger\mathbf{h}_1^\dagger\mathbf{h}_1\mathbf{u}_1,\ldots,\gamma_K\Big/\mathbf{u}_K^\dagger\mathbf{h}_K^\dagger\mathbf{h}_K\mathbf{u}_K\right]^T,$$

$$\mathbf{D} \triangleq diag\left(\mathbf{d}\right),$$

$$\mathbf{B}^{K\times K}:\ \mathbf{B_{i,j}} \triangleq \mathbf{\Psi_{i,j}}\mathbf{u_j}^\dagger\mathbf{h_i}^\dagger\mathbf{h_i}\mathbf{u_j},\ \forall \mathrm{i,j},$$

$$\mathbf{\Upsilon} = \begin{bmatrix} \mathbf{DB} & \mathbf{d} \\ \frac{1}{P_{max}}\boldsymbol{I}^T\mathbf{DB} & \frac{1}{P_{max}}\boldsymbol{I}^T\mathbf{d}\end{bmatrix},$$

and

$$\mathbf{\Lambda} \triangleq \begin{bmatrix} \mathbf{DB}^T & \mathbf{d} \\ \frac{1}{P_{max}}\boldsymbol{I}^T\mathbf{DB}^T & \frac{1}{P_{max}}\boldsymbol{I}^T\mathbf{d}\end{bmatrix},$$

the following iterative algorithm will converge to the optimal solution of the SINR-balancing problem

Algorithm 4.1 (Iterative SINR-balancing):

$initialize\ \mathbf{p}^{(0)} = \mathbf{0}$

$i = 0$

$repeat$

$i = i + 1$

$calculate:\ \mathbf{U}^{(i)} = \mathbf{U}_{MMSE}\left(\mathbf{H},\mathbf{p}^{(i-1)},\mathbf{\Psi}\right)$

$$solve:\ \mathbf{\Lambda}\left(\mathbf{U^{(i)}}\right)\begin{bmatrix}\mathbf{p}^{(i)}\\ 1\end{bmatrix}\triangleq\lambda^{(\mathbf{n})}_{\max\_\mathbf{real}}\begin{bmatrix}\mathbf{p}^{(i)}\\ 1\end{bmatrix},$$

$$\mathrm{C}^{(\mathrm{n})}\triangleq 1\Big/\lambda^{(n)}_{\max\_real}$$

until the desired accuracy is reached

end

if downlink power allocation $\mathbf{q}$ *is required*

$$solve:\ \mathbf{\Upsilon}\left(\mathbf{U^{(i)}}\right)\begin{bmatrix}\mathbf{q}\\ 1\end{bmatrix}\triangleq\lambda_{\max\_\mathbf{real}}\begin{bmatrix}\mathbf{q}\\ 1\end{bmatrix},$$

endif

where $\lambda^{(n)}_{\max\_real}$ stands for the maximal eigenvalues among the real eigenvalues of $\mathbf{\Upsilon}^{(\mathbf{n})}$. To conclude this section, we refer to the rate-weighting problem. Like SINR-balancing, rate-weighting is efficiently optimized on the dual MAC domain. Optimal power allocation must fulfill $\sum_i p_i^{opt} = P_{\max}$, and the optimal beamforming for a given power allocation are the MMSE receiver vectors (15). A closed form solution, as in the SINR-balancing case, was not found. However since optimal beamforming is known for a given power allocation, the problem boils down to finding the optimal power control, under the restriction $\sum_i p_i = P_{\max}$. This is a power control problem that has a closed form solution based on λ_{max}, for which efficient algorithms that lead to a global maximization exist.

V. Lower Bound of the CMHP BC Rate-Region

In this section we analyze the CMHP BC rate-region, which is the most general rate-region achieved by superposition coding. For simplicity we consider the two-user channel only. By definition, the two-user CMHP rate-region consists of a common information term, even if transmission of true common information is not of interest. The common information term could be interpreted as representing a data stream which may be successfully decoded by both users. In practice, it may contain pure common information, private information or any mixture of both. We limit the scope to beamforming, also for the common information term, although this may turn out sub-optimal [19]. In order to account for the common information, we modify the notation used so far, and define: $\mathbf{q}^{cmhp}\triangleq[q_0,q_1,q_2]$, $\mathbf{U}^{cmhp}\triangleq[\mathbf{u}_0,\mathbf{u}_1,\mathbf{u}_2]$, and $\mathbf{R}^{cmhp}\triangleq[R_0^{cmhp},R_1^{cmhp},R_2^{cmhp}]$, where zero indexed term relate to the common-information. In this work we consider the special type of successive decoding receivers. The two-user CMHP BC rate-region, *with successive decoding receivers and common information decoded first* on both receivers and under total power constraint $P_{\max}$, is defined as follows

$$\mathbf{R}^{cmhp}\left(P_{\max},\mathbf{H}\right)\triangleq\bigcup_{\mathbf{U},\sum_{\mathbf{k=0}}^{\mathbf{2}}\mathbf{q_k}\le\mathbf{P}_{\max}}\left\{\mathbf{R}: R_k = log\left(1+SINR_k^{cmhp}\right)\right\},\tag{16}$$

where

$$SINR_k^{cmhp}\left(\mathbf{q},\mathbf{H},\mathbf{U}\right)\triangleq\begin{cases}\dfrac{\mathbf{h}_1\mathbf{u}_1q_1\mathbf{u}_1^\dagger\mathbf{h}_1^\dagger}{1+\mathbf{h}_2\mathbf{u}_2q_2\mathbf{u}_2^\dagger\mathbf{h}_2^\dagger}, & k=1\\ \dfrac{\mathbf{h}_2\mathbf{u}_2q_2\mathbf{u}_2^\dagger\mathbf{h}_2^\dagger}{1+\mathbf{h}_1\mathbf{u}_1q_1\mathbf{u}_1^\dagger\mathbf{h}_1^\dagger} & k=2\\ \min\limits_{i=1,2}\left\{\dfrac{\mathbf{h}_i\mathbf{u}_0q_0\mathbf{u}_0^\dagger\mathbf{h}_i^\dagger}{1+\mathbf{h}_i(\mathbf{u}_1q_1\mathbf{u}_1^\dagger+\mathbf{u}_2q_2\mathbf{u}_2^\dagger)\mathbf{h}_i^\dagger}\right\}, & k=0.\end{cases}\tag{17}$$

In order to ease the analysis and over come the difficulty in treating the "min" operator introduced by equation (17), we define an extended CMHP system, with both terms within the "min" operator treated as separate users. We also use the general CTM representation for defining the extended system as follows

$$R_k^{bc\_ext}\left(\mathbf{q}^{ext},\mathbf{U}^{ext},\mathbf{H}^{ext}\right)\triangleq R_k^{bc}\left(\mathbf{q}^{ext},\mathbf{U}^{ext},\mathbf{H}^{ext},\mathbf{\Psi}^{cmhp}\right),\tag{18}$$

where $\mathbf{H}^{ext}\triangleq[\mathbf{h}_1^T,\mathbf{h}_2^T,\mathbf{h}_1^T,\mathbf{h}_2^T]$, $\mathbf{q}^{ext}\triangleq[q_1,q_2,q_0,q_0]$, $\mathbf{U}^{ext}\triangleq[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_0,\mathbf{u}_0]$, $\mathbf{R}^{bc\_ext}\triangleq[R_1,R_2,R_3,R_4]$, and

$$\mathbf{\Psi}^{cmhp}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix},$$

The original CMHP system rates are related to the extended system by

$$R_k^{cmhp}\triangleq\begin{cases}R_k^{bc\_ext}, & 1\le k\le 2\\ \min\left\{R_3^{bc\_ext},R_4^{bc\_ext}\right\}, & k=0\end{cases}\tag{19}$$

The following lemma states the limitations involved in processing the CMHP on the dual MAC domain

Lemma 1: The extended BC and the extended MAC, defined by

$$R_k^{mac\_ext}\triangleq R_k^{mac}\left(\mathbf{p},\mathbf{U^{ext}},\mathbf{H}^{ext},\mathbf{\Psi}^{cmhp}\right),$$

are partly dual. By partly dual we mean that any extended BC has a dual extended MAC, but total power conservation is non-trivial and takes the following form: $p_1+p_2+p_3+p_4 = q_1+q_2+2q_0$, so the resulting total power on the extended MAC may be larger then the power on the original BC. In addition, the CMHP BC rate-region with total power constraint P_{max} is fully contained within extended MAC rate-region with total power constraint $2P_{max}$. On the other hand, a given power control and beamforming allocation on the extended MAC domain represented an extended CMHP BC iff the following conditions are satisfied:

1) $\mathbf{u_3}=\mathbf{u_4}$.
2) The MAC to BC power transformation results in $q_3 = q_4\triangleq q_0$.
3) $q_1+q_2+q_0\leqslant P_{max}$

Proof of the lemma is a direct consequence of the BC - MAC duality and of the limitations posed by the problem structure. An important observation is that the first limitation is easily handled on the MAC domain while the last two limitations are more naturally treated on the BC domain.

Next, we seek for a lower bound to the SINR-balancing and rate-weighting problems, this time using the CMHP coding strategy. In order to handle the "min" operator we analyze the extended system instead of the original one. We also note that any achievable rate-tuple in the CMHP rate-region must result with one of the following:

1) $R_3 < R_4$, or
2) $R_3 > R_4$, or
3) $R_3 = R_4$,

so the original optimization problem may split into three sub-problems, which we nickname $min1$, $min2$ and eql, respectively. The total solution is defined as the best between the three. Assuming $min1$ holds true, the optimization problem is equivalent to a three dimensional problem, with user 4 of the extended system completely ignored. The cross-talk matrix representing this case is given by

$$\mathbf{\Psi}^{min} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{20}$$

Since this problem may be treated as a BC with general CTM, it could be easily solved by the tools described in section IV. One must keep in mind to verify the assumption, by placing the resulting power and beamforming vectors back to the extended system and choosing $R_0 \triangleq \min\{R_3, R_4\}$. The $min2$ problem is treated in a similar manner. Next, we deal with the eql problem. We start by finding beamforming vectors for a given power allocation, by examining the dual extended MAC. We recall that in the general CTM case optimal beamforming for users 3 and 4 were chosen as their MMSE receivers. Such a choice would not do well in this case, first of all because the beamforming vectors of users 3 and 4 need to equal $\mathbf{u}_3 = \mathbf{u}_4 \triangleq \mathbf{u}_0$. Secondly, because they must result with $SINR_3^{ext} = SINR_4^{ext}$. More explicitly, the common beamforming vector must fulfill

$$\mathbf{u}_0^\dagger \mathbf{h}_3^\dagger p_3 \mathbf{h}_3 \mathbf{u}_0 = \mathbf{u}_0^\dagger \mathbf{h}_4^\dagger p_4 \mathbf{h}_4 \mathbf{u}_0 \tag{21}$$

In general, this is an vector projection problem which may have many solutions, where for performance purposes the solution that maximize both sides of the equation is the preferable one. Let us define $\mathbf{u}_0$ that solve this equation with maximal rates as $\mathbf{u}_0^{R_3=R_4}$, and the following beamforming matrix

$$\mathbf{U}_{EQL}(\mathbf{p}) = \left[\mathbf{u}_0^{R_3=R_4}, \mathbf{u}_1^{MMSE}, \mathbf{u}_2^{MMSE}\right]. \tag{22}$$

We suggest the following iterative algorithm to lower bound the eql problem:

Algorithm 5.1 (Iterative lower bound):

$initialize\ \mathbf{p}^{(0)} = \mathbf{0}$
$i = 0$
$repeat$
  $i = i + 1$
  $calculate:\ \mathbf{U}^{(i)} = \mathbf{U}_{EQL}\left(\mathbf{H}^{\mathbf{ext}}, \mathbf{p}^{(i-1)}, \mathbf{\Psi}^{cmhp}\right)$
  $solve:\ \mathbf{\Upsilon}\left(\mathbf{U}^{(\mathbf{i})}, \mathbf{\Psi}^{\mathbf{min}}\right)\begin{bmatrix} \mathbf{q}^{(i)} \\ 1 \end{bmatrix} \triangleq \lambda_{\max\_\mathbf{real}}^{(\mathbf{n})}\begin{bmatrix} \mathbf{q}^{(i)} \\ 1 \end{bmatrix},$
  $calculate:\ \mathbf{p}^{(i)}\ using\ (13)$
  $\mathrm{C}^{(\mathrm{n})} \triangleq 1\big/\lambda_{\max\_real}^{(n)}$
  $until\ the\ desired\ accuracy\ is\ reached$
end

Simulations show that the algorithm converges well within 5 iterations. Note that the algorithm iterated between the BC and MAC domains, and that power allocation is performed on the BC domain, while beamforming are found on the MAC domain. Algorithm 5.1 may be slightly changed for finding beamforming vector for a given $\mathbf{q}$ by skipping the BC power allocation stage. This way, rate-weighting of the eql problem may be treated as a power allocation optimization problem, although finding the beamforming vector per given power allocation involves an iterative algorithm, and may be quite computational expensive.

It should be emphasized that the achievable rate-region presented in this sub-section serves as a lower bound only for the following reasons:

1) Gaussian signaling was assumed optimal without a proof.
2) Successive decoding receivers were assumed optimal.
3) Common massage being decoded first on both receivers was assumed optimal.
4) Rank 1 $\mathbf{Q}_0$ was assumed optimal.
5) Beamforming matrix (22) was not claimed of being optimal for a given power allocation $\mathbf{q}$.

It is worth mentioning that the sinr balancing problem with common information may be alternatively tackled directly on the BC domain. It is a problem that recently attracted considerable attention and was shown to be NP-hard [20]. However, this problem may be alternatively tackled without imposing beamforming, which turns the problem into convex. In such case, successive cancellation receivers with any decoding order and combined with rate splitting would result with the optimal CMHP solution, up to the Gaussian signaling assumption. The computation burden of such a technique turns out polynomial [15].

To conclude this section, we consider the more practical ZF - CMHP suggested in [9]. Like with CMHP coding, this setting assumes a common information term that is being decoded first on both receivers, but private information streams are being precoded as with ZF coding. This scheme is handled in a similar manner to the more general CMHP rate-region [15].

VI. Numerical Results

In this section we provide a two-user numerical example to better illustrate the capacity region achieved by different coding methods. The example is the same 2×1 symmetric vector BC channel channel analyzed by [21], defined by $\mathbf{h_1} = [\mathbf{1}\ .\mathbf{4}]$; $\mathbf{h_2} = [.\mathbf{4}\ \mathbf{1}]$. We compare the dirty paper capacity region to the rate-region achieved by TDMA, SPC with SINR receiver, ZF, ZF-CMHP and CMHP lower bound. The rate-region achieved by DPC, SPC and ZF (two active users only) are characterized using the SINR-balance technique. The CMHP and ZF-CMHP rate-regions are characterized by rate-weighting. Since no true common information is considered in this example, the CMHP common information term is allocated to the higher priority user. The results are presented in Fig. 2. Note that in the presented example the sum-capacity

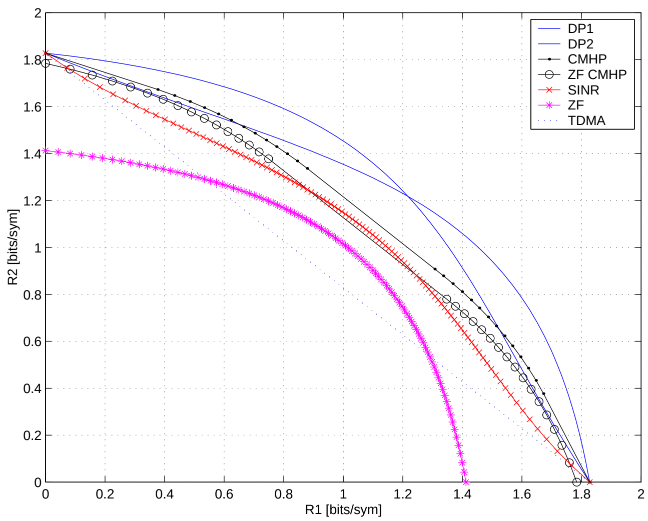

Fig. 2. DPC, CMHP, TDMA, ZF and ZF-CMHP broadcast regions: $K = 2$, $M = 2$, $N_1 = N_2 = 1$, $\mathbf{h_1} = [\mathbf{1}\ .\mathbf{4}]$; $\mathbf{h_2} = [.\mathbf{4}\ \mathbf{1}]$, $P = 10$.

of the SINR curve exhibits good performance (also in [8]), and the sum-rate advantage of the lower bound of CHMP is relatively moderate here.

VII. Conclusion

In this paper, we have considered vector BC with general cross-talk between the different users. It was shown that any rate achieved on the BC, is also achieved on the dual MAC with the same total power and the the same CTM. Using duality, SINR balancing of vector BC - MAC with general CTM was analyzed. It was shown that with no common information the optimal beamformers are simply the MMSE filters on the MAC domain. Next, the two-user Cover - Van-der-Meulen - Hajek - Pursley rate-region was analyzed. Successive decoding receivers, with common information decoded first were assumed. Such receivers are sumrate optimal, but otherwise serves only as a lower bound. A lower bound to the CMHP rate-region is achieved by finding a sub-optimal solution to the described setting. A numerical examples of a symmetric channel was given, with comparison to other coding strategies. It be interesting to compute the optimal CMHP rate region, and compare to the SINR approach, identifying the channel characteristics where this advantage is maximum. Another interesting question is how does the CMHP rate-region compares with other transmission techniques when the transmitter is exposed to limited feedback.

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