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# On Integer Codes

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***Abstract*— Integer codes are important in single–error correction and the study of perfect codes. Several further applications in coding theory, computer science, graph theory as well as related concepts in algebra and geometry are discussed.**

***Index Terms*— Single-error correction, perfect codes, codes on graphs, tiling, splitting of groups.**

## I. INTRODUCTION

An integer code as defined by Vinck and Morita in [1] consists of all words $(c_1, \ldots, c_n) \in Z_m^n$, where $Z_m$ is the ring of integers modulo $m$, fulfilling

$$\sum_{i=1}^{n} w_i \cdot c_i = d \text{ mod } m. \quad (1)$$

Here $(w_1, \ldots, w_n) \in Z_m^n$ is a sequence of weights and $d$ is an element of $Z_m$. So, $n$ is the length of the code and $m$ is the size of the code alphabet.

Integer codes have a variety of applications in information theory, computer science, graph theory, and algebra. In coding theory they are a useful tool in single–error–correction [3] – [16]. The advantage is that the proper choice of the weights allows great flexibility and enables to correct single errors in many error models (substitution, insertion/deletion, synchronization, etc.) and types (symmetric, asymmetric, unidirectional, etc.). The analysis of the syndromes of integer codes often yields conditions for perfectness, such that they also contribute to the theory of perfect codes in various metrics [17] – [21]. Further applications in coding theory have been presented in [1].

A different motivation comes from computer science. The goal here is not error correction but the efficient placement of resources in distributed and parallel computations. This leads to the concept of codes in a graph, since it has to be avoided that processors are placed in vertices within a certain distance in this graph [22] – [28]. An example is presented in section II.

The proper choice of the weight sequence depends on the type of error to be corrected. The effect of a single error is reflected in the behaviour of the syndrome, which should be changed to a value different from $d$ by a linear combination of the weights corresponding to the codeword's coordinates involved in this error. It turned out that the analysis of the syndrome is deeply related to splitting of groups and tiling of the Euclidean space by certain star bodies. This had been intensively discussed in algebra [29] – [36].

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We shall discuss the choice of the parameters of an integer code in the next section. After that, in Section III several results important for coding theory are presented followed by remarks on packings and coverings in Section IV.

## II. CHOICE OF THE PARAMETERS

*The weight sequence:* There are two possibilities to arrange the weight sequence $w_1, \ldots, w_n$. One might work with a fixed sequence of weights chosen in advance and then immediately obtain the code or one might leave the sequence variable for further analysis.

The most well - known example for codes with a fixed weight sequence are the Varshamov – Tenengolts codes defined by

$$\sum_{i=1}^{n} i \cdot c_i = d \text{ mod } n+1.$$

This construction with a special $n$ had been used already by Ulrich [2] in 1957 for nonbinary codes. Varshamov – Tenengotls codes have many applications, for instance, they are a useful tool to correct single asymmetric errors [3] and single deletions [4]. Often, the codewords $(c_1, \ldots, c_n)$ are assumed to be binary [4], [5], such that the resulting codes are not integer codes in the sense of (1).

Another fixed–weight sequence was recently studied by Dorbec and Mollard [24]. They discussed the following formula

$$\sum_{i=0}^{k}\sum_{j=1}^{p}((2p+1)i+j)c_{ip+j} = l(2p+1) \text{ mod } (k+1)(2p+1)$$

for some $l \in \{0, \ldots, k\}$. The codewords fulfilling this equation followed by some further components form a perfect code in the graph obtained as cartesian product of the grid $Z^{p(k+1)}$ and the hypercube of dimension $k$.

As mentioned above, often the weights are left variable and then the code behaviour is studied, for instance, by analyzing the possible syndromes as explained in detail later on. Since in many cases only existence results are of interest, an explicit code construction may not be necessary. Otherwise, the weights then later can be optimized according to the special requirements.

*The parameter $d$:* Usually, the parameter $d$ in (1) will be chosen as 0. This is quite natural and the resulting code has a group structure, which might be of advantage. However, there might be reasons to choose another $d$, for instance, when the resulting code has better properties or is concatenated with another code as in the previous example. A combinatorial analysis of the code size for different choices of $d$ has been carried out by Martirosyan [6]. Sloane [7] discussed surprising relations of the code sizes for Varshamov–Tenengolts codes to further enumeration problems.

*The syndrome*: In order to be able to correct one single error, the syndromes of an integer code have to be pairwisely different. As pointed out before, integer codes allow to correct single errors in several code concepts, as substitution, deletion, transposition, etc. The code concept is reflected by the syndrome as the following two examples illustrate.

If the error is a substitution of the letter $c_i$ by $c_i'$ then the resulting syndrome is

$$w_1c_1 + \ldots + w_{i-1}c_{i-1} + w_ic_i' + w_{i+1}c_{i+1} + \ldots + w_nc_n$$
$$= d + w_i(c_i' - c_i), \text{ for } i = 1, \ldots, n$$

A single error may involve more than one component. For instance, if the two adjacent letters $c_i$ and $c_{i+1}$ are permuted, then the syndrome will be

$$w_1c_1 + \ldots + w_{i-1}c_{i-1} + w_ic_{i+1} + w_{i+1}c_i + w_{i+2}c_{i+2} +$$
$$\ldots + w_nc_n = d + (w_i - w_{i+1})(c_{i+1} - c_i), \text{ for } i = 1, \ldots, n-1.$$

A very similar syndrome occurs in the correction of peak shifts discussed by Levenshtein and Vinck [8]. Here $n$ is the number of runs in a run–length limited sequence of 0s and 1s and the $c_i$s are the lengths of the successive runs, i. e., the numbers of 0s between two consecutive 1s. A peak shift results in a shift by at most $k$ positions of the $i$th 1 to the left or to the right. This has the effect that the length of the $i$–th run $c_i$ increases by a number $0 \leq j \leq k$ and the length of the $(i+1)$th run decreases by the same number $j$, or vice versa. The resulting syndrome is

$$w_1c_1 + \ldots + w_{i-1}c_{i-1} + w_i(c_i \pm j) + w_{i+1}(c_{i+1} \mp j)$$
$$+ w_{i+2}c_{i+2} + \ldots + w_nc_n = d \pm (w_i - w_{i+1})j.$$

*Possible errors*:Without loss of generality, we assume $d = 0$. So if the possible distortions, which can be corrected by the integer code, are from an error set $\mathcal{E} = \{a_0, a_1, \ldots, a_{k-1}\} \subset Z_m$ and the linear combinations of the weights are from a set $\mathcal{H} \subset Z_m$, then we have to assure that all possible products by the appropriate weight combinations and codeword combinations are different, i. e.,

$$e \cdot h \neq e' \cdot h' \text{ for all } e, e' \in \mathcal{E} \text{ and } h, h' \in \mathcal{H}. \quad (2)$$

For the special error set $\mathcal{E} = \{1, 2, \ldots, k\}$ the set $\mathcal{H}$ is denoted as shift design [8] or shift code [9]. In this sense, we shall denote $\mathcal{H}$ corresponding to an arbitrary error set $\mathcal{E}$ as syndrome code. If, all elements from the set $Z_m \setminus \{0\}$ occur as a product in (2), then the syndrome code is said to be perfect. In this case also the corresponding integer code is perfect.

A perfect syndrome code in (2) jointly with the set $\mathcal{E}$ is also known in algebra as splitting $(\mathcal{E}, \mathcal{H})$ of the additive group $Z_m$ [29], [30], [31]. Further, if $m = p$ is a prime number, then $\mathcal{E} \cdot \mathcal{H}$ yields a factorization [32] of the multiplicative group $Z_p^*$ (here multiplication of two sets means the set of all possible products of one element in one set with an element of the other set).

## III. Special Perfect Codes and Splittings

*A general construction*: We shall concentrate on groups $Z_p$ where $p$ is a prime number – results for composite moduli can easily be derived from this ([9]). In this case the multiplicative group $Z_p^* = (Z_p \setminus \{0\}, \cdot)$ consists of all numbers $\{1, \ldots, p-1\}$ and a splitting $(\mathcal{E}, \mathcal{H})$ corresponds to a factorization $\mathcal{E} \cdot \mathcal{H}$ of the group $Z_p^*$.

In many cases it can be shown that the set

$$\mathcal{H} = \{g^{jk} : j = 0, \ldots, \frac{p-1}{k}\}. \quad (3)$$

is a subgroup in $Z_p^*$, where $g$ is a generator of $Z_p^*$.

In [10] the elements of $\mathcal{E}$ are expressed as powers of the generator $g$, namely $a_i = g^{\mu_i}$ for $i = 0, \ldots, k-1$. In order to assure that all products $e \cdot h$ ($e \in \mathcal{E}, h \in \mathcal{H}$) are different, one has to guarantee that all products $g^{\mu_i} \cdot g^{jk} = g^{jk+\mu_i}$ are different for all possible choices $j = 0, \ldots, \frac{p-1}{k}$, $i = 0, \ldots, k-1$. This obviously holds if and only if the $\mu_i$ fall into the different congruence classes modulo $k$, i. e., if

$$\{\mu_0 \bmod k, \ldots, \mu_{k-1} \bmod k\} = \{0, \ldots, k-1\}. \quad (4)$$

If, additionally, $|\mathcal{E}| \cdot |\mathcal{H}| = p - 1$, then obviously $\mathcal{H}$ is a perfect syndrome code.

In the following we are only interested in symmetric errors, such that the error sets $\mathcal{E}$ under consideration are of the form $\{\pm a_0, \pm a_1, \ldots, \pm a_{k-1}\}$. It is easily seen that one can identify the elements $x$ and $-x$ in $Z_p^*$ and hence simply carry out all previous calculations with the numbers $a_0, a_1, \ldots, a_{k-1}$ now in $Z_p^*/\{1, -1\}$.

*The error set* $\mathcal{E} = \{\pm 1, \pm a\}$: In [11] the error set $\mathcal{E} = \{\pm 1, \pm a\}$ is discussed. This corresponds to the error model, in which a letter $c_i$ is changed to one of its nearest neighbours on the $a \times a$ – grid, where a component $(x, y)$ is represented by the number $x + y \cdot a$ This can be described in such a way that the received letter is contained in the set $\{c_i \pm 1, c_i \pm a\}$. given. The condition on the existence of a splitting in $Z_p$ is that the element $a^2$ has an even order modulo $p$. The set $\mathcal{H}$ then consists of the group of even powers of $a$ in $Z_p^*/\{1, -1\}$ and its translates in the respective cosets, cf. also [8] for $a = 2$.

*The error set* $\mathcal{E} = \{\pm 1, \pm a, \ldots, \pm a^r\}$: This error set was studied among others in [12]. The condition on perfectness here is that the element $a$ has order divisible by $r + 1$ in $Z_p^*/\{1, -1\}$.

*The error set* $\mathcal{E} = \{\pm 1, \pm a, \ldots, \pm a^r, \pm b, \ldots, \pm b^s\}$:

For this set the conditions on the existence of a splitting in $Z_p$ derived in [13] are.

1 The orders of $a$ and $b$ are both divisible by $r + s + 1$.
2 Whenever $b^{l_1} = a^{l_2}$ for some integers $l_1, l_2$, then $l_1 + l_2 \equiv 0 \bmod (r + s + 1)$.

These conditions guarantee that the the syndrome code $\mathcal{H} = \{a^i \cdot b^j, i - j \equiv 0 \bmod (r+s+1)\}$ is really the proper subgroup

(3), which is then generated by the elements $a^{r+s}, b^{r+s}$ and $a \cdot b$.

*The error set* $\mathcal{E} = \{\pm 1, \pm a, \ldots, \pm a^r, \pm b, \ldots, \pm b^s, \pm c, \ldots, \pm c^t\}$: This is much more difficult to analyze. If $a = g^{\mu_1}$, $b = g^{\mu_2}$, $c = g^{\mu_3}$, then it has to be arranged that

$\{\mu_1, \ldots, r\mu_1\} \cup \{\mu_2, \ldots, s\mu_2\} \cup \{\mu_3, \ldots, t\mu_3\} = \{1, \ldots, r + s + t\}$,

where the numbers are reduced modulo $r + s + 1$.

If one would have to control only the two parameters $a$ and $b$ as previously discussed, one could simply choose $\mu_1 \equiv 1 \bmod (r + s)$ and $\mu_2 \equiv -1 \bmod (r + s)$ and the results above follow easily.

By the same approach it is still possible to analyze the case $s = 1$, i.e., $\mathcal{E} = \{\pm 1, \pm a, \ldots, \pm a^r, \pm b, \pm c, \ldots, \pm c^t\}$. Fortunately, the two sets most important for applications are of this form, as we shall see now – in general, for three parameters $a$,$b$,$c$, a general formula seems to be hard to find.

The error set $\mathcal{E} = \{\pm 1, \pm a, \pm b, \pm c\}$ has recently been studied in [10]. The motivation came from another concept for codes on the $a \times a$ grid, which was also discussed is [11]. In this error model a letter cannot only be changed to the nearest neighbours on the grid, but also to closest elements on the diagonals passing through the vertex representing this letter, such that the special error set $\mathcal{E} = \{\pm 1, \pm(a-1), \pm a, \pm(a+1)\}$ arises.

The syndrome code $\mathcal{H}$ here is the group generated by the elements $a^4, b^2, c^4, a \cdot c, a^2 \cdot b$ (plus possibly its cosets) in $Z_p^*/\{1, -1\}$.

Again it has to be guaranteed that this group is really of the form (3) The conditions here are:

1 In $Z_p * /\{1, -1\}$ the orders of $a$ and $c$ are divisible by $4$ and the order of $b$ is divisible by 2,
2 whenever $a^i \cdot c^j \in \mathcal{H}$ then $i - j \equiv 0 \bmod 4$,
3 whenever $a^i \cdot b^j \in \mathcal{H}$ then $2i + j \equiv 0 \bmod 4$,
4 whenever $c^i \cdot b^j \in \mathcal{H}$ then $2i + j \equiv 0 \bmod 4$.

Another important example for this kind of error set, is the set $\{1, 2, 3, 4, 5\} = \{1, a, a^2, b, c\}$ for $(a, b, c) = (2, 3, 5)$. Here the syndrome code $\mathcal{H}$ is basically the group generated by the elements $a^5$, $b^5$, $c^5$, $a \cdot c$, and $a^2 \cdot b$.

*The error set* $\mathcal{E} = \{\pm 1, \pm 2, \ldots, \pm k\}$: By far the most important error set is $\mathcal{E} = \{\pm 1, \pm 2, \ldots, \pm k\}$. As mentioned before, this set arose in the study of run–length limited codes and the syndrome code was denoted by Levenshtein and Vinck [8] as $k$ – shift design and later by Munemasa [9] as $k$ – shift code.

As a second application in coding theory, $\{\pm 1, \pm 2, \ldots, \pm k\}$ is also the error set for codes correcting single errors in the so–called Stein sphere [14] – [16], where a single component is distorted in such a way that the received letter $c_i'$ is of the form $c_i' = c_i + j, j \in \{\pm 1, \pm 2, \ldots, \pm k\}$. A special case are single–error–correcting codes in the Lee metric. The results for the error sets previously discussed yield conditions for the existence of of perfect $k$–shift codes for the small parameters $k = 2, 3, 4, 5$.

This is deeply related to algebra and geometry, where the set $\{\pm 1, \pm 2, \ldots, \pm k\}$ jointly with the corresponding perfect shift code had been studied before under the name group splitting. The concept arose in the analysis of tilings of the $n$-dimensional space by the $(k, n)$–cross [29], [30], [31].

A $(k, n)$–cross is the cluster consisting of the $2kn + 1$ unit $n$ – dimensional cubes with centers (for $j = 1, \ldots, k$)

$$(0, \ldots, 0), (\pm j, 0, \ldots, 0), (0, \pm j, \ldots, 0), \ldots, (0, \ldots, 0, \pm j).$$

These star bodies correspond to the error spheres denoted by Golomb [14] as Stein sphere and Stein corner, respectively.

It turned out in [33] that a lattice tiling of the Euclidean space $R^n$ by the $(k, n)$–cross exists exactly if $\{\pm 1, \ldots \pm k\}$ splits some Abelian group of order $2kn + 1$. The splitting of Abelian groups can often be reduced to those for cyclic groups $Z_{2kn+1}$, such that the syndromes of integer codes fulfilling (2) come into play.

The Stein sphere is also a special kind of polyomino [41] and has a further application in the study of memory with defects [42].

## IV. Packings and Coverings

It can be shown by combinatorial arguments that perfect integer codes are quite sparsely distributed [10]. One might relax the condition and also look for constructions of almost perfect codes, i. e., condition (2) is fulfilled and most elements in $Z_m \setminus \{0\}$ can be represented as a product in (2).

Packings and coverings have been considered e.g in [37], [38], [39]. Some applications to Information Theory are discussed in [15] and [16]. The following asymptotical result is known.

$$\lim_{k \to \infty} \frac{f(k, n)}{k^2} = 1$$

Here $f(k, n)$ denotes the smallest $m$ such that a packing by $\{\pm 1, \pm 2, \ldots, \pm k\}$ of size $n$ exists in $Z_m$

Contrasting to the group splittings, the results on packings and coverings of groups cannot be directly applied to coding theory. Motivated by the geometric application, here the parameter $n$ is fixed and $k$ tends to infinity. The result shows that good packings in this case cannot be expected, since $f(k, n)$ is about $k^2$, which is much bigger than $2kn + 1$ for $n$ small compared to $k$.

For applications in coding theory, however, one would rather fix $k$ and look for code constructions suitable for any $n$. Very close packings of cyclic groups $Z_p$ of prime order can also be obtained if condition (4) is fulfilled. However, if several $\mu_i$s in (4) fall into the same congruence class, it is also obvious by (4) that a close packing cannot exist in this case. So here it might be advantageous to study packings of $Z_m$ for composite numbers $m$. For the special sets $\mathcal{E} = \{\pm 1, \pm 2, \pm 3\}$ and $\mathcal{E} = \{\pm 1, \pm 2, \pm 3, \pm 4\}$ a greedy construction yielding very close packed shift codes has been provided in [40]. For instance, in $Z_{40}$ the 3–shift code $\{1, 4, 5, 7, 9, 17\}$ of size 6 improves the value in Table V-4 on p. 316 in [30] and the 4–shift code $\{1, 5, 8, 9, 11, 13, 14, 17, 23, 35, 37, 40\}$ of size 12 in $Z_{99}$ is almost perfect.

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