# Multi-terminal Source Coding With Unreliable Sensors

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***Abstract*— The problem of multi-terminal source coding with unreliable sensors is studied. In this scenario, two correlated discrete memoryless sources are separately encoded. There are three decoders at the receiver; a central one for lossless decoding and two side decoders for reconstructing the sources with prescribed distortion levels when one of the sensors fails. Single-letter achievable and converse rate-distortion regions are derived. Although these regions do not coincide in general, in the special setting where only one sensor is unreliable, a complete single-letter characterization is successfully derived by strengthening the converse. According to this characterization, there is no rate loss incurred on the system if the unreliable sensor expends a sufficiently low rate.**

## I. Introduction

Consider a sensor network with two sensors and one central receiver, where the sensors (or equivalently, the links between the sensors and the receiver) are unreliable. It is desirable to obtain a lossless reconstruction of both sensor data at the receiver whenever the sensors operate properly. This can be achieved through Slepian-Wolf coding [4], i.e., random assignment of typical vectors to bins and decoding using the joint typicality principle. However, when one of the sensors fails, the corresponding bin index is permanently lost and even the data of the other sensor cannot be recovered.

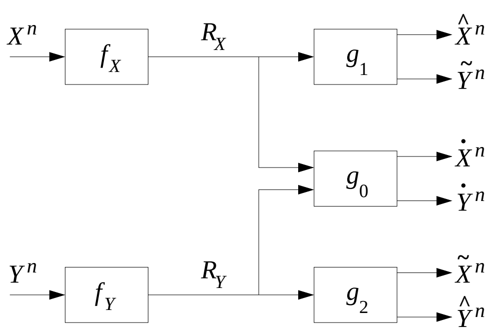


Fig. 1. Multi-terminal source coding with unreliable sensors.

In this work, we analyze the rate-distortion tradeoff of a modified Slepian-Wolf scheme, where both sensor data can still be recovered within prescribed distortion levels when one of the sensors fails. This source coding phenomenon is illustrated in Figure 1. The sensor data $\{(X_t, Y_t)\}_{t=1}^{\infty}$ are independent copies of a pair of dependent random variables $(X, Y)$ that take values in the finite sets $\mathcal{X}, \mathcal{Y}$, respectively. The rates at the encoder outputs are $R_X$ and $R_Y$. All the encodings and decodings are performed in blocks of length $n$. The outputs of the "side" decoder $g_1$ are $\hat{X}^n$, a direct lossy reconstruction of $X^n$, and $\tilde{Y}^n$, an indirect lossy reconstruction of $Y^n$. Similarly, the side decoder $g_2$ outputs $\tilde{X}^n$, an indirect lossy reconstruction of $X^n$, and $\hat{Y}^n$, a direct lossy reconstruction of $Y^n$. The outputs of $g_1$ and $g_2$ are obtained using only the bits received from the $X$-sensor and the $Y$-sensor, respectively. The "central" decoder, $g_0$, is a regular Slepian-Wolf decoder, i.e., it outputs $(\dot{X}^n, \dot{Y}^n)$ which is a noiseless reconstruction of $(X^n, Y^n)$ with probability approaching 1 as $n \to \infty$.

While Chen et al. [2] studied the rate-distortion performance of a system with unreliable sensors, their framework is different in that sensors are observing noisy versions of an underlying source which is to be reconstructed at the decoder. This is sometimes referred to as "indirect coding" in the literature. In contrast, we analyze direct coding, where it is $\{(X_t, Y_t)\}_{t=1}^{\infty}$ to be reconstructed, either losslessly or in a lossy manner.

We derive a single-letter achievable rate-distortion region, as well as a single-letter converse. Although the achievable and the converse regions do not coincide generally, the only difference between the two is a long Markov chain in the achievable region versus two shorter Markov chains in the converse. We then consider a special case where one of the sensors, say the $X$-sensor, is very reliable. Thus, $\tilde{X}^n$ and $\hat{Y}^n$ need not be reconstructed. In this setting, we are able to strengthen the converse and obtain a complete single-letter characterization of the achievable rates and distortions. Furthermore, it turns out that this rate-distortion region coincides with the Slepian-Wolf region when $R_Y$ is sufficiently low (but still higher than $H(Y|X)$.) Thus, we do not necessarily have to pay extra rate because of the proposed modification on the Slepian-Wolf scheme when only one sensor is unreliable.

## II. Results

*Definition 1:* The region $\mathcal{A}$ consists of 6-tuples $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{X}}, D_{\hat{Y}}, D_{\tilde{Y}})$ such that there exists encoders

$$
\begin{aligned}
f_X &: \mathcal{X}^n \longrightarrow \{1, 2, \ldots, M_X\} \\
f_Y &: \mathcal{Y}^n \longrightarrow \{1, 2, \ldots, M_Y\}
\end{aligned}
$$

and decoders

$$
\begin{aligned}
g_0 &: \{1, 2, \ldots, M_X\} \times \{1, 2, \ldots, M_Y\} \longrightarrow \mathcal{X}^n \times \mathcal{Y}^n \\
g_1 &: \{1, 2, \ldots, M_X\} \longrightarrow \hat{\mathcal{X}}^n \times \tilde{\mathcal{Y}}^n \\
g_2 &: \{1, 2, \ldots, M_Y\} \longrightarrow \tilde{\mathcal{X}}^n \times \hat{\mathcal{Y}}^n
\end{aligned}
$$

satisfying

$$\begin{aligned}
\frac{1}{n}\log M_X &\leq R_X+\epsilon && (1)\\
\frac{1}{n}\log M_Y &\leq R_Y+\epsilon && (2)\\
E\{d_{\hat{X}}(X^n,\hat{X}^n)\} &\leq D_{\hat{X}}+\epsilon && (3)\\
E\{d_{\tilde{X}}(X^n,\tilde{X}^n)\} &\leq D_{\tilde{X}}+\epsilon && (4)\\
E\{d_{\hat{Y}}(Y^n,\hat{Y}^n)\} &\leq D_{\hat{Y}}+\epsilon && (5)\\
E\{d_{\tilde{Y}}(Y^n,\tilde{Y}^n)\} &\leq D_{\tilde{Y}}+\epsilon && (6)\\
\Pr[(X^n,Y^n)\neq(\dot{X}^n,\dot{Y}^n)] &\leq \epsilon && (7)
\end{aligned}$$

for any $\epsilon \geq 0$ and arbitrarily large $n$, where

$$\begin{aligned}
(\dot{X}^n,\dot{Y}^n) &= g_0(f_X(X^n),f_Y(Y^n))\\
(\hat{X}^n,\tilde{Y}^n) &= g_1(f_X(X^n))\\
(\tilde{X}^n,\hat{Y}^n) &= g_2(f_Y(Y^n))
\end{aligned}$$

and $d_{\hat{X}}$, $d_{\tilde{X}}$, $d_{\hat{Y}}$, and $d_{\tilde{Y}}$ are all single-letter distortion measures with respective maximum values $d_{\hat{X},\max}$, $d_{\tilde{X},\max}$, $d_{\hat{Y},\max}$, and $d_{\tilde{Y},\max}$.

We next define two single-letter regions, $\mathcal{A}_{\text{out}}$ and $\mathcal{A}_{\text{in}}$, and show that

$$\mathcal{A}_{\text{in}} \subset \mathcal{A} \subset \mathcal{A}_{\text{out}} \ .$$

*Definition 2:* The region $\mathcal{A}_{\text{in}}$ consists of the convex closure of 6-tuples $(R_X,R_Y,D_{\hat{X}},D_{\tilde{X}},D_{\hat{Y}},D_{\tilde{Y}})$ such that there exist auxiliary random variables $U$ and $V$ forming a Markov chain $V-Y-X-U$ and deterministic functions $\hat{X}(U)$, $\tilde{X}(V)$, $\hat{Y}(V)$, and $\tilde{Y}(U)$, altogether satisfying

$$\begin{aligned}
E\{d_{\hat{X}}(X,\hat{X}(U))\} &\leq D_{\hat{X}} && (8)\\
E\{d_{\tilde{X}}(X,\tilde{X}(V))\} &\leq D_{\tilde{X}} && (9)\\
E\{d_{\hat{Y}}(Y,\hat{Y}(V))\} &\leq D_{\hat{Y}} && (10)\\
E\{d_{\tilde{Y}}(Y,\tilde{Y}(U))\} &\leq D_{\tilde{Y}} && (11)\\
H(X|Y)+I(Y;U) &\leq R_X && (12)\\
H(Y|X)+I(X;V) &\leq R_Y && (13)\\
H(X,Y)+I(U;V) &\leq R_X+R_Y \ . && (14)
\end{aligned}$$

*Remark 1:* Due to $V-Y-X-U$, (12)-(14) can alternatively be written as

$$\begin{aligned}
H(X|Y,U,V)+I(X;U) &\leq R_X && (15)\\
H(Y|X,U,V)+I(Y;V) &\leq R_Y && (16)\\
I(X;U)+I(Y;V)+H(X,Y|U,V) &\leq R_X+R_Y \ . && (17)
\end{aligned}$$

In proving that $\mathcal{A}_{\text{in}} \subset \mathcal{A}$, (15)-(17) is easier to employ compared to (12)-(14). However, (12)-(14) sheds more light on the extra rate expended on top of traditional Slepian-Wolf coding.

*Definition 3:* The region $\mathcal{A}_{\text{out}}$ consists of the convex closure of 6-tuples $(R_X,R_Y,D_{\hat{X}},D_{\tilde{X}},D_{\hat{Y}},D_{\tilde{Y}})$ such that there exist auxiliary random variables $U$ and $V$ forming Markov chains $Y-X-U$ and $V-Y-X$ and deterministic functions $\hat{X}(U)$, $\tilde{X}(V)$, $\hat{Y}(V)$, and $\tilde{Y}(U)$, altogether satisfying (8)-(13) and (17).

*Remark 2:* If (17) is adopted instead of (14) in the definition of $\mathcal{A}_{\text{in}}$, the only difference between $\mathcal{A}_{\text{in}}$ and $\mathcal{A}_{\text{out}}$ becomes the long Markov chain $V-Y-X-U$ versus two short Markov chains $V-Y-X$ and $Y-X-U$.

*Theorem 1:* $\mathcal{A}_{\text{in}} \subset \mathcal{A}$.

*Proof:* We use standard random coding arguments. We also employ strong typicality and use the notation of Csiszár and Körner [3].

Assume that $(R_X,R_Y,D_{\hat{X}},D_{\tilde{X}},D_{\hat{Y}},D_{\tilde{Y}}) \in \mathcal{A}_{\text{in}}$. Then there exist auxiliary random variables $U$ and $V$, and deterministic functions $\hat{X}(U)$, $\tilde{X}(V)$, $\hat{Y}(V)$, and $\tilde{Y}(U)$, satisfying $V-Y-X-U$, (8)-(11), and (15)-(17). Randomly create $\{U^n(i_1)\}_{i_1=1}^{M'_1}$ and $\{V^n(i_2)\}_{i_2=1}^{M'_2}$ by picking vectors from $T^n_{[U]}$ and $T^n_{[V]}$, respectively, independently and according to uniform distribution. Next, for $1\leq j\leq M'_X, 1\leq k\leq N'_X, 1\leq l\leq M'_Y$, and $1\leq m\leq N'_Y$, create vectors $\{\dot{X}^n(k|j)\}$ and $\{\dot{Y}^n(m|l)\}$, respectively from $T^n_{[X]}$ and $T^n_{[Y]}$, independently and according to uniform distribution. Essentially, the collection of vectors $B_X(j)=\{\dot{X}^n(k|j)\}_{k=1}^{N'_X}$ and $B_Y(l)=\{\dot{Y}^n(m|l)\}_{m=1}^{N'_Y}$ can be considered as randomly created *bins*. Inform the encoder and decoder about the vectors $\{U^n(i_1)\}_{i_1=1}^{M'_1}$ and $\{V^n(i_2)\}_{i_2=1}^{M'_2}$ and the resultant binning structure.

*X-encoding:* Find any $i_1$ such that

$$(X^n,U^n(i_1)) \in T^n_{[X,U]} \ . \tag{18}$$

If no such $i_1$ is found, set $i_1=1$. Also find any $(j,k)$ such that $\dot{X}^n(k|j)=X^n$. Set $j=1$ if no such j exists. Transmit $(i_1,j)$, expending a rate of

$$\frac{1}{n}\log M'_1+\frac{1}{n}\log M'_X \ .$$

*Y-encoding:* Similarly, find any $i_2$ such that

$$(Y^n,V^n(i_2)) \in T^n_{[Y,V]} \ . \tag{19}$$

If no such $i_2$ is found, set $i_2=1$. Also find any $(l,m)$ such that $\dot{Y}^n(m|l)=Y^n$. Set $l=1$ if no such $l$ exists. Transmit $(i_2,l)$, expending a rate of

$$\frac{1}{n}\log M'_2+\frac{1}{n}\log M'_Y \ .$$

*Side decoding:* At decoder $g_1$, upon receiving $i_1$, output $\hat{X}_t(U_t(i_1))$ and $\tilde{Y}_t(U_t(i_1))$ at time instant $t$, for $1\leq t\leq n$. Similarly, at decoder $g_2$, output $\tilde{X}_t(V_t(i_2))$ and $\hat{Y}_t(V_t(i_2))$ for $1\leq t\leq n$.

*Expected distortion at side decoders:* The distortion at decoder $g_1$ may occur either because no $i_1$ satisfies (18), or due to the distortion between $X_t$ and $\hat{X}_t(U_t(i_1))$ and between $Y_t$ and $\tilde{Y}_t(U_t(i_1))$ for an $i_1$ satisfying (18). Define $\hat{X}^n(i_1)=\{\hat{X}_t(U_t(i_1))\}_{t=1}^n$, and $\tilde{Y}^n(i_1)$, $\tilde{X}^n(i_2)$, and $\hat{Y}^n(i_2)$ similarly. If (18) is satisfied, we readily have

$$\begin{aligned}
d_{\hat{X}}(X^n,\hat{X}^n(i_1)) &= \frac{1}{n}\sum_{t=1}^n d_{\hat{X}}(X_t,\hat{X}_t(U_t(i_1)))\\
&\leq E\{d_{\hat{X}}(X,\hat{X}(U))\}+\epsilon/2\\
&\leq D_{\hat{X}}+\epsilon/2
\end{aligned}$$

for $\epsilon > 0$ and large enough $n$. Otherwise, we have $d_{\hat{X}}(X^n,\hat{X}^n(i_1)) \leq d_{\hat{X},\max}$. Now, choosing $M'_1$ such that

$$\frac{1}{n}\log M'_1 \geq I(X;U)+\epsilon/4 \tag{20}$$

the probability of finding an $i_1$ satisfying (18) approaches 1 as $n \to \infty$. With this choice of $M_1'$, we then achieve

$$E\{d_{\hat{X}}(X^n, \hat{X}^n)\} \leq D_{\hat{X}} + \epsilon \, .$$

Similarly, at decoder $g_2$, with the choice of

$$\frac{1}{n} \log M_2' \geq I(Y;V) + \epsilon/4 \tag{21}$$

the probability of finding an $i_2$ satisfying (19) approaches 1 as $n \to \infty$, yielding

$$E\{d_{\hat{Y}}(Y^n, \hat{Y}^n)\} \leq D_{\hat{Y}} + \epsilon \, .$$

Since $Y - X - U - V$, and $\Pr[X^n = x^n, Y^n = y^n] = \prod_{t=1}^n p(x_t, y_t)$, it follows from the extension of the Markov lemma in [1] that when (18) and (19) are satisfied,

$$\Pr[(X^n, Y^n, U^n(i_1), V^n(i_2)) \in T^n_{[X,Y,U,V]}] > 1 - \frac{\epsilon}{2} \tag{22}$$

for $n$ sufficiently large. This implies with high probability that $(Y^n, U^n(i_1)) \in T^n_{[Y,U]}$ and $(X^n, V^n(i_2)) \in T^n_{[X,V]}$, and thus

$$E\{d_{\tilde{Y}}(Y^n, \tilde{Y}^n)\} \leq D_{\tilde{Y}} + \epsilon$$

and

$$E\{d_{\tilde{X}}(X^n, \tilde{X}^n)\} \leq D_{\tilde{X}} + \epsilon \, .$$

*Central decoding:* Decode $(U^n(i_1), V^n(i_2))$ first. Then output any $(\dot{X}^n(k|j), \dot{Y}^n(m|l)) \in B_X(j) \times B_Y(l)$ satisfying

$$(\dot{X}^n(k|j), \dot{Y}^n(m|l), U^n(i_1), V^n(i_2)) \in T^n_{[X,Y,U,V]} \, . \tag{23}$$

*Probability of error at the central decoder:* As long as there exists $k^*$ and $m^*$ such that $X^n = \dot{X}^n(k^*|j)$ and $Y^n = \dot{Y}^n(m^*|l)$, which will be guaranteed with probability approaching 1 if

$$H(X) + \frac{\epsilon}{12} \geq \frac{1}{n} \log M_X' N_X' \geq H(X) + \frac{\epsilon}{13}$$

and

$$H(Y) + \frac{\epsilon}{12} \geq \frac{1}{n} \log M_Y' N_Y' \geq H(Y) + \frac{\epsilon}{13} \, ,$$

(22) implies that with probability more than $1 - \frac{\epsilon}{2}$, $(X^n, Y^n)$ itself constitutes a pair in $B_X(j) \times B_Y(l)$ satisfying (23). Hence, it suffices to show that the probability of finding *another* $(\dot{X}^n(k|j), \dot{Y}^n(m|l))$ pair, i.e., with $(k, m) \neq (k^*, m^*)$, satisfying (23) is less than $\frac{\epsilon}{2}$. Below, we upper-bound this probability by separately analyzing three cases:

i) Let $p$ denote the probability that a randomly chosen pair from $T^n_{[X]} \times T^n_{[Y]}$ satisfies (23). Also let $P_1$ denote the probability that there exists at least one $(k, m)$ with $k \neq k^*$ and $m \neq m^*$ such that $(\dot{X}^n(k|j), \dot{Y}^n(m|l)) \in B_X(j) \times B_Y(l)$ satisfies (23). It easily follows from standard arguments that

$$\begin{aligned} p &= \frac{\left| T^n_{[X,Y|U,V]}(U^n(i_1), V^n(i_2)) \right|}{\left| T^n_{[X]} \right| \left| T^n_{[Y]} \right|} \\ &\leq \frac{2^{n[H(X,Y|U,V)+\epsilon/12]}}{2^{n[H(X)-\epsilon/12]} 2^{n[H(Y)-\epsilon/12]}} \end{aligned}$$

for large $n$. Using the union bound, $P_1$ can be upper-bounded as

$$\begin{aligned} P_1 &\leq (N_X' - 1)(N_Y' - 1)p \\ &\leq N_X' N_Y' \frac{2^{n[H(X,Y|U,V)+\epsilon/12]}}{2^{n[H(X)-\epsilon/12]} 2^{n[H(Y)-\epsilon/12]}} \\ &\leq \frac{2^{nH(X)}}{M_X'} \frac{2^{nH(Y)}}{M_Y'} \frac{2^{n[H(XY|U,V)+5\epsilon/12]}}{2^{nH(X)} 2^{nH(Y)}} \, . \end{aligned}$$

Thus if $M_X'$ and $M_Y'$ are chosen such that

$$H(X,Y|U,V) + \frac{5\epsilon}{12} < \frac{1}{n} \log M_X' + \frac{1}{n} \log M_Y' \, , \tag{24}$$

$P_1 \to 0$ as $n \to \infty$.

ii) Denote by $P_2$ the probability that there exists at least one $(k, m)$ with $k = k^*$ and $m \neq m^*$ such that $(\dot{X}^n(k|j), \dot{Y}^n(m|l))$ satisfies (23). Using similar arguments as above, one can show

$$\begin{aligned} P_2 &\leq N_Y' \frac{\left| T^n_{[Y|X,U,V]}(X^n, U^n(i_1), V^n(i_2)) \right|}{\left| T^n_{[Y]} \right|} \\ &\leq N_Y' \frac{2^{n[H(Y|X,U,V)+\epsilon/12]}}{2^{n[H(Y)-\epsilon/12]}} \\ &\leq \frac{2^{nH(Y)}}{M_Y'} \frac{2^{n[H(Y|X,U,V)+\epsilon/4]}}{2^{nH(Y)}} \, . \end{aligned}$$

Thus, one needs to choose

$$H(Y|X,U,V) + \frac{\epsilon}{4} < \frac{1}{n} \log M_Y' \tag{25}$$

in order for $P_2$ to vanish for large $n$.

iii) Finally, denote by $P_3$ the probability that there exists at least one $(k, m)$ with $k \neq k^*$ and $m = m^*$ such that $(\dot{X}^n(k|j), \dot{Y}^n(m|l))$ satisfies (23). We similarly have

$$\begin{aligned} P_3 &\leq N_X' \frac{\left| T^n_{[X|Y,U,V]}(Y^n, U^n(i_1), V^n(i_2)) \right|}{\left| T^n_{[X]} \right|} \\ &\leq N_X' \frac{2^{n[H(X|Y,U,V)+\epsilon/12]}}{2^{n[H(X)-\epsilon/12]}} \\ &\leq \frac{2^{nH(X)}}{M_X'} \frac{2^{n[H(X|Y,U,V)+\epsilon/4]}}{2^{nH(X)}} \end{aligned}$$

and it suffices to choose

$$H(X|Y,U,V) + \frac{\epsilon}{4} < \frac{1}{n} \log M_X' \tag{26}$$

to let $P_3 \to 0$ as $n \to \infty$.

Comparing (20), (21), (24), (25), and (26) with (15)-(17), we conclude that $M_1'$, $M_2'$, $M_X'$, and $M_Y'$ can be chosen such that

$$\begin{aligned} \frac{1}{n} \log M_1' + \frac{1}{n} \log M_X' &\leq R_X + \epsilon \\ \frac{1}{n} \log M_2' + \frac{1}{n} \log M_Y' &\leq R_Y + \epsilon \, . \end{aligned}$$

Thus $\mathcal{A}_{\text{in}} \subset \mathcal{A}$. ■

*Theorem 2:* $\mathcal{A} \subset \mathcal{A}_{\text{out}}$.

*Proof:* Let $S_X = f_X(X^n)$ and $S_Y = f_Y(Y^n)$. Also define $U_t = (S_X, Y_1^{t-1})$ and $V_t = (S_Y, X_1^{t-1})$ and observe that $U_t - X_t - Y_t$ and $V_t - Y_t - X_t$. Further, $\hat{X}_t$ is deterministically related to $U_t$, i.e., $\hat{X}_t = \hat{X}_t(U_t)$, and similarly $\tilde{X}_t = \tilde{X}_t(V_t)$, $\hat{Y}_t = \hat{Y}_t(V_t)$, and $\tilde{Y}_t = \tilde{Y}_t(U_t)$.

Now, assume that $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{X}}, D_{\hat{Y}}, D_{\tilde{Y}}) \in \mathcal{A}$. We then have for any $\epsilon > 0$ and arbitrarily large $n$ that

$$\begin{aligned} n(R_X+\epsilon) &\geq H(S_X) \\ &\geq I(S_X; X^n, Y^n) \\ &= I(S_X; Y^n) + I(S_X; X^n|Y^n) \\ &= I(S_X; Y^n) + H(X^n|Y^n) \\ &\quad -H(X^n|S_X, Y^n) \\ &\overset{(a)}{\geq} I(S_X; Y^n) + H(X^n|Y^n) \\ &\quad -H(X^n|S_X, S_Y) \\ &\overset{(b)}{\geq} I(S_X; Y^n) + H(X^n|Y^n) \\ &\quad -H(X^n|g_0(S_X, S_Y)) \\ &\geq I(S_X; Y^n) + H(X^n|Y^n) \\ &\quad -H(X^n, Y^n|g_0(S_X, S_Y)) \end{aligned}$$

where $(a)$ and $(b)$ follow since $H(A|B) \leq H(A|f(B))$ for an arbitrary function $f$ and random variables $A$ and $B$. Defining $P_e = \Pr[(X^n, Y^n) \neq g_0(S_X, S_Y)]$ and using Fano's inequality

$$1 + nP_e \log|\mathcal{X}||\mathcal{Y}| \geq H(X^n, Y^n|g_0(S_X, S_Y)) \qquad (27)$$

together with $P_e \leq \epsilon$, we obtain for large $n$ and a constant $k_1$ that

$$\begin{aligned} R_X + k_1\epsilon &\geq \frac{1}{n}\left[I(S_X; Y^n) + H(X^n|Y^n)\right] \\ &= \frac{1}{n}\left[H(Y^n) - H(Y^n|S_X) + H(X^n|Y^n)\right] \\ &= \frac{1}{n}\sum_{t=1}^{n}\Big[H(X_t|Y_t) \\ &\quad +H(Y_t) - H(Y_t|S_X, Y_1^{t-1})\Big] \\ &= \frac{1}{n}\sum_{t=1}^{n}\left[H(X_t|Y_t) + H(Y_t) - H(Y_t|U_t)\right] \\ &= \frac{1}{n}\sum_{t=1}^{n}\left[H(X_t|Y_t) + I(Y_t; U_t)\right] . \qquad (28) \end{aligned}$$

From symmetry, this also implies

$$R_Y + k_1\epsilon \geq \frac{1}{n}\sum_{t=1}^{n}\left[H(Y_t|X_t) + I(X_t; V_t)\right] . \qquad (29)$$

We also have

$$\begin{aligned} n(R_X + R_Y + 2\epsilon) &\geq H(S_X) + H(S_Y) \\ &\geq I(S_X; X^n, Y^n) + I(S_Y; X^n, Y^n) . \qquad (30) \end{aligned}$$

Let us further lower-bound the first mutual information in (30).

$$\begin{aligned} &I(S_X; X^n, Y^n) \\ &\quad = H(X^n, Y^n) - H(X^n, Y^n|S_X) \\ &\quad = \sum_{t=1}^{n}\left[H(X_t, Y_t) - H(X_t, Y_t|S_X, X_1^{t-1}, Y_1^{t-1})\right] \\ &\quad = \sum_{t=1}^{n}\left[H(X_t, Y_t) - H(X_t, Y_t|U_t, X_1^{t-1})\right] \\ &\quad \geq \sum_{t=1}^{n}\left[H(X_t, Y_t) - H(X_t, Y_t|U_t)\right] \\ &\quad = \sum_{t=1}^{n} I(X_t, Y_t; U_t) \\ &\quad = \sum_{t=1}^{n} I(X_t; U_t) \qquad (31) \end{aligned}$$

where (31) follows from $U_t - X_t - Y_t$. Similarly,

$$I(S_Y; X^n, Y^n) \geq \sum_{t=1}^{n} I(Y_t; V_t) . \qquad (32)$$

Combining (30), (31), and (32), we obtain

$$R_X + R_Y + 2\epsilon \geq \frac{1}{n}\sum_{t=1}^{n}\left[I(X_t; U_t) + I(Y_t; V_t)\right] . \qquad (33)$$

Using (27), we also observe

$$\begin{aligned} &1 + nP_e \log|\mathcal{X}||\mathcal{Y}| \\ &\quad \geq H(X^n, Y^n|g_0(S_X, S_Y)) \\ &\quad \geq H(X^n, Y^n|S_X, S_Y) \\ &\quad = \sum_{t=1}^{n} H(X_t, Y_t|S_X, S_Y, X_1^{t-1}, Y_1^{t-1}) \\ &\quad = \sum_{t=1}^{n} H(X_t, Y_t|U_t, V_t) . \qquad (34) \end{aligned}$$

Thus, (33) and (34) yields

$$\begin{aligned} &R_X + R_Y + k_2\epsilon \\ &\quad \geq \frac{1}{n}\sum_{t=1}^{n}\left[I(X_t; U_t) + I(Y_t; V_t) + H(X_t, Y_t|U_t, V_t)\right] \\ &\qquad (35) \end{aligned}$$

for large $n$ and some constant $k_2$. By (3), (4), (5), (6), we also have

$$D_{\hat{X}} + \epsilon \geq \frac{1}{n}\sum_{t=1}^{n} E\{d_{\hat{X}}(X_t, \hat{X}_t(U_t))\} \qquad (36)$$

$$D_{\tilde{X}} + \epsilon \geq \frac{1}{n}\sum_{t=1}^{n} E\{d_{\tilde{X}}(X_t, \tilde{X}_t(V_t))\} \qquad (37)$$

$$D_{\hat{Y}} + \epsilon \geq \frac{1}{n}\sum_{t=1}^{n} E\{d_{\hat{Y}}(Y_t, \hat{Y}_t(V_t))\} \qquad (38)$$

$$D_{\tilde{Y}} + \epsilon \geq \frac{1}{n}\sum_{t=1}^{n} E\{d_{\tilde{Y}}(Y_t, \tilde{Y}_t(U_t))\} . \qquad (39)$$

Thus, (28), (29), (35), and (36)-(39) imply that $\mathcal{A}$ is outer-bounded by the arithmetic average of points in the region we define as $\mathcal{A}_{\text{out}}$. Since $\mathcal{A}_{\text{out}}$ is by definition convex, we conclude that $\mathcal{A} \subset \mathcal{A}_{\text{out}}$. ■

We next analyze the special case where the $X$-sensor is very reliable. More specifically, assume that $f_X(X^n)$ will always be received, whereas $f_Y(Y^n)$ can be unavailable sometimes. In that case, there is no need for a side decoder $g_2$. Equivalently, we can set $D_{\tilde{X}} \to \infty$ and $D_{\hat{Y}} \to \infty$, and analyze the cross-section of $\mathcal{A}$ corresponding to achievable $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{Y}})$. We denote by $\bar{\mathcal{A}}$ that cross-section, and present in the next theorem a complete single-letter characterization for $\bar{\mathcal{A}}$.

*Theorem 3:* $\bar{\mathcal{A}} = \mathcal{A}^*$, where $\mathcal{A}^*$ consists of the quadruplets $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{Y}})$ such that there exists auxiliary random variables $\hat{X}$ and $\tilde{Y}$ forming a Markov chain $Y - X - (\hat{X}, \tilde{Y})$ satisfying

$$\begin{aligned} E\{d_{\hat{X}}(X,\hat{X})\} &\leq D_{\hat{X}} && (40)\\ E\{d_{\tilde{Y}}(Y,\tilde{Y})\} &\leq D_{\tilde{Y}} && (41)\\ H(X|Y) + I(Y;\hat{X},\tilde{Y}) &\leq R_X && (42)\\ H(Y|X) &\leq R_Y && (43)\\ H(X,Y) &\leq R_X + R_Y\ . && (44)\end{aligned}$$

*Proof:* The direct part, i.e., that $\mathcal{A}^* \subset \bar{\mathcal{A}}$, follows from Theorem 1. More specifically, for any $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{Y}}) \in \mathcal{A}^*$ and corresponding $\hat{X}$ and $\tilde{Y}$ satisfying $Y - X - (\hat{X}, \tilde{Y})$ and (40)-(44), set $U = (\hat{X}, \tilde{Y})$, $\hat{X}(U) = \hat{X}$, $\tilde{Y}(U) = \tilde{Y}$, and $V$ to be a constant. With this choice, (8)-(14), as well as $V - Y - X - U$, are automatically satisfied. Thus, from Theorem 1, $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{Y}}) \in \bar{\mathcal{A}}$.

Conversely, for any $(R_X, R_Y, D_{\hat{X}}, D_{\tilde{Y}}) \in \bar{\mathcal{A}}$, following the same steps in the proof of Theorem 2, we obtain

$$R_X + k_1\epsilon \geq \frac{1}{n}\left[I(S_X;Y^n) + H(X^n|Y^n)\right]$$

for some constant $k_1$ and large enough $n$. Continuing,

$$\begin{aligned} R_X + k_1\epsilon &\geq \frac{1}{n}\left[I(\hat{X}^n,\tilde{Y}^n;Y^n) + H(X^n|Y^n)\right]\\ &= \frac{1}{n}\left[H(X^n|Y^n) + H(Y^n) - H(Y^n|\hat{X}^n,\tilde{Y}^n)\right]\\ &= \frac{1}{n}\sum_{t=1}^{n}\Big[H(X_t|Y_t) + H(Y_t)\\ &\qquad -H(Y_t|\hat{X}^n,\tilde{Y}^n,Y_1^{t-1})\Big]\\ &\geq \frac{1}{n}\sum_{t=1}^{n}\left[H(X_t|Y_t) + H(Y_t) - H(Y_t|\hat{X}_t,\tilde{Y}_t)\right]\\ &= \frac{1}{n}\sum_{t=1}^{n}\left[H(X_t|Y_t) + I(Y_t;\hat{X}_t,\tilde{Y}_t)\right]. \end{aligned} \quad (45)$$

Now, it is straightforward to show using noiseless coding arguments that

$$R_Y + \epsilon \geq H(Y|X) = \frac{1}{n}\sum_{t=1}^{n} H(Y_t|X_t) \quad (46)$$

and

$$R_X + R_Y + \epsilon \geq H(X,Y) = \frac{1}{n}\sum_{t=1}^{n} H(X_t,Y_t)\ . \quad (47)$$

By (3) and (6), we also have

$$\begin{aligned} D_{\hat{X}} + \epsilon &\geq \frac{1}{n}\sum_{t=1}^{n} E\{d_{\hat{X}}(X_t,\hat{X}_t(U_t))\} && (48)\\ D_{\tilde{Y}} + \epsilon &\geq \frac{1}{n}\sum_{t=1}^{n} E\{d_{\tilde{Y}}(Y_t,\tilde{Y}_t(U_t))\}\ . && (49)\end{aligned}$$

Finally, it is obvious that $Y_t - X_t - (\hat{X}_t, \tilde{Y}_t)$ forms a Markov chain. This, together with (45)-(49) implies that $\bar{\mathcal{A}}$ is outer-bounded by the arithmetic average of points in the region we define as $\mathcal{A}^*$. Since $\mathcal{A}^*$ can easily be shown to be convex, we conclude that $\bar{\mathcal{A}} \subset \mathcal{A}^*$. ■

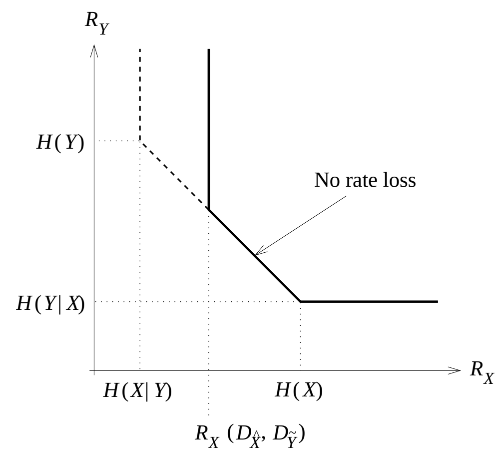


Fig. 2. Comparison of the rate cross-section of $\mathcal{A}^*$ (indicated in bold) and the Slepian-Wolf region (dashed lines).

Now, we compare the rate cross-section of $\mathcal{A}^*$, obtained by fixing $D_{\hat{X}}, D_{\tilde{Y}}$, to the Slepian-Wolf region. Define

$$R_X(D_{\hat{X}}, D_{\tilde{Y}}) = \min\{R_X : (R_X, R_Y, D_{\hat{X}}, D_{\tilde{Y}}) \in \mathcal{A}^*\}\ .$$

According to Theorem 3, the single-letter characterization of $R_X(D_{\hat{X}}, D_{\tilde{Y}})$ is given by

$$R_X(D_{\hat{X}}, D_{\tilde{Y}}) = \min\left\{H(X|Y) + I(Y;\hat{X},\tilde{Y})\right\} \quad (50)$$

where the minimization is over all $(\hat{X}, \tilde{Y})$ satisfying $Y - X - (\hat{X}, \tilde{Y})$, (40), and (41). Thus, the region of achievable $(R_X, R_Y)$ pairs is as indicated in Figure 2. Note that when $R_Y$ is low enough, there is no rate loss incurred on the system compared to the Slepian-Wolf scenario.

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