Connectivity, Devolution, and Lacunae in Geometric Random Digraphs

Santosh S. Venkatesh
Department of Electrical and Systems Engineering
University of Pennsylvania
Philadelphia, PA 19104, USA
Email: venkatesh@ee.upenn.edu

***Abstract*— A mosaic process is formed by placing a small random set at each point in a collection of a large number of random points in the unit disc. The mosaic induces a geometric random digraph on the collection of random points by adding a directed edge outward from each point in the collection to all the other points in the collection that lie within the random set centred at that point. If the random sets are sufficiently regular then a threshold function for graph connectivity is manifested at a critical rate of coverage by the small sets.**

I. Mosaics

A *mosaic process* formalizes the idea of throwing down sets at random. Suppose n is a parameter considered large and $\mathbb{S}_n$ is a random set of the Euclidean plane governed by a law $\mathcal{L}_n$ parametrized by n. For our purposes, we think of $\mathbb{S}_n$ as a small, plump random set centred at the origin and whose size decreases (in a suitable probabilistic sense) as n increases. While not essential, it will be convenient to suppose that the distribution $\mathcal{L}_n$ of $\mathbb{S}_n$ is unchanged with respect to rotations of the axes.

The simplest illustration is provided by discs and, to fix ideas, we may suppose that the random set $\mathbb{S}_n$ is a disc of random radius R, centred at the origin, and governed by law $F_n(r)$. We formalize the idea that the random sets get increasingly small as n increases by imposing an asymptotic condition on the coverage; for instance, the condition $\int r^2\,dF_n(r) \to 0$ as $n \to \infty$ says that the expected radius and the expected area of coverage of the random discs $\mathbb{S}_n$ vanish asymptotically.

Now suppose $X_1, \ldots, X_n$ is a collection of random points in the unit disc. We think of n as a large parameter. For each X_k we pick a random set $\mathbb{S}_{n,k}$ distributed as $\mathbb{S}_n$, independent for different X_k. Then the mosaic process $\bigcup_{k=1}^n (X_k + \mathbb{S}_{n,k})$ represents a collection of many small random sets thrown down at random.

II. Geometric Random Digraphs

The mosaic process induces a *geometric random digraph* $\mathcal{G} = \mathcal{G}_{n,\mathcal{L}_n}$. The vertices of $\mathcal{G}$ are the points $X_1, \ldots, X_n$. For each k, we associate the directed edge (X_k, X_l) originating from X_k and terminating in X_l for all points $X_l \in X_k + \mathbb{S}_{n,k}$ that are not X_k itself.

The simplest setting for the mosaic and the induced graph occurs when $\mathbb{S}_n$ is a disc of fixed radius r_n. In this case the mosaic is formed by n identical copies of discs of radius r_n distributed randomly in the unit disc. The graph that obtains is then an ordinary undirected graph with points connected by edges if, and only if, they are within Euclidean distance r_n of each other. Such graphs were first studied by Gilbert [1] and are called geometric random graphs. In recent years these models have seen renewed interest spurred by applications in computational geometry, randomly deployed sensor networks, and cluster analysis; see the monograph by Penrose for a slew of references [6].

The digraph induced by a general mosaic process generalises this idea of a geometric random graph by replacing the "disc of influence of radius r_n" at each point X_k by a random "region of influence" defined by the random set $\mathbb{S}_{n,k}$ placed at that location. The edges of the digraph are directed as associations are not, in general, commutative: if $X_l \in X_k + \mathbb{S}_{n,k}$ it is not necessary that $X_k \in X_l + \mathbb{S}_{n,l}$.

The general mosaic process may be used to model connectivity in a randomly deployed sensor network to account for observed asymmetries in connectivity between elements. Differing transmission powers at the sensors, for instance, may be modelled by placing random discs at each sensor, the radius determined by the available power at the sensor. More generally, a typical foliated antenna pattern with a fat main lobe may be scaled by random amounts to account for differing transmission powers, and provided with a random rotation to account for a random orientation on deployment at each sensor.

III. Connectivity

A vertex X_k of the digraph $\mathcal{G}_{n,\mathcal{L}_n}$ is *isolated* if there are no other points X_l in the set $X_k + \mathbb{S}_{n,k}$. The key implication to connectivity of the underlying graph is the discovery of the feature that as the coverage of the random sets $\mathbb{S}_n$ decreases there is an abrupt appearance of isolated vertices in the graph.

Roughly speaking, suppose the random sets $\mathbb{S}_n$ are Lebesgue measurable and have area $A_n = A(\mathbb{S}_n)$. Write $\psi_n = \mathbf{E}\big(e^{-nA_n/\pi}\big)$ for the Laplace transform of the distribution of A_n. Interactions at the boundary of the unit disc make for fierce complications and are ultimately negligible in the range of interest. We finesse these details here by calling the random sets $\mathbb{S}_n$ "regular" and reserve the gory details for elsewhere. The random disc model may serve as a useful guide to focus thought. The main result: *for regular mosaic processes, the number of isolated vertices in the induced geometric random digraph has an asymptotic Poisson distribution with mean* e^{-c} *if* $\log \psi_n + \log n \to c$. Here c is an arbitrary real constant.

The Poisson approximation ideas behind the proof are classical and follow the approach outlined in Kunniyur and Venkatesh [3], [4]. The key ideas may be sketched quite simply.

If we ignore complications at the boundary of the unit disc, the probability that a given point X_k is isolated in the graph $\mathcal{G}_{n,\mathcal{L}_n}$ is given by $\mathbf{E}\bigl\{\bigl(1 - A_{n,k}/\pi\bigr)^{n-1}\bigr\} \sim \psi_n$ with suitable asymptotic regularity conditions on the random sets. For any given k, the sets $X_{i_1} + \mathbb{S}_{n,i_1}, \dots, X_{i_k} + \mathbb{S}_{n,i_k}$ will with high probability be non-overlapping so that the probability that each of $X_{i_1}, \dots, X_{i_k}$ is isolated is asymptotic to $\sigma_{n,k} := \mathbf{E}\bigl\{\bigl(1 - \frac{1}{\pi}\sum_{j=1}^{k} A_{n,i_j}\bigr)^{n-k}\bigr\} \sim \psi_n^k$ so that vertex isolations are asymptotically independent. Then $S_{n,k} := \binom{n}{k}\sigma_{n,k} \to e^{-kc}/k!$ and the inclusion-exclusion formula shows that the probability that there are exactly m isolated vertices is given by

$$\sum_{k=0}^{n-m} (-1)^k \binom{m+k}{m} S_{n,m+k} \to e^{-e^{-c}} \frac{(e^{-c})^m}{m!}.$$

In words, the number of isolated vertices is asymptotically Poisson with mean e^{-c}.

As in the classical Erdös-Rényi random graph models, isolated vertices are the main agents for the loss of connectivity and a Peierls argument shows that *the probability that the digraph* $\mathcal{G}_{n,\mathcal{L}_n}$ *is strongly connected tends to* $e^{-e^{-c}}$. We provide details elsewhere.

When the mosaic is built up of fixed discs of radius r_n, the condition on ψ_n translates into an explicit threshold for the radius: if $r_n^2 = \frac{\log n}{n} + \frac{c}{n} + o\bigl(\frac{1}{n}\bigr)$ then the probability that the corresponding geometric random graph is connected tends to $e^{-e^{-c}}$. Results of this stripe were obtained by Penrose [5] for random points in the unit square, and more generally, in the unit cube in d-dimensions, using the Stein-Chen method of Poisson approximation. An application in the sensory network context was outlined by Gupta and Kumar [2] for random points deployed in a disc.

IV. Emergent Lacunae

Vertex extinctions were considered by Kunniyur and Venkatesh [3], [4] as a means of modelling sensor death in a sensor network. Suppose point X_k and the associated random set $X_k + \mathbb{S}_{n,k}$ are deleted after a random time T_k. We suppose the random variables $T_1, \dots, T_n$ are independent and subscribe to a common law $G(t) = G_{\mathcal{L}_n}(t)$ which may be parametrised by the parameters of the mosaic process, for instance, the mean diameter of the random sets. If we start with a strongly connected digraph, as vertices are extinguished the graph *devolves*: first isolated vertices and then regions of isolation, or *lacunae*, centred at the vertices begin to appear. Write $\phi_n(t) := \mathbf{E}\bigl(e^{-nA_n G_{\mathcal{L}_n}(t)/\pi}\bigr)$. Then, under the same conditions of regularity of the random sets, if the sequence of time epochs $\{t_n\}$ satisfies $\log \phi(t_n) + \log n \to c$ then the number of isolated vertices at time t_n is asymptotically Poisson with mean e^{-c}. Arguments similar to those of Kunniyur and Venkatesh [3], [4] may be adduced to obtain a similar Poisson law for lacunae of a given size. Thus, lacunae emerge abruptly at a critical time determined by the parametrised extinction process.

Acknowledgment

The author gratefully acknowledges the NSF Cybertrust program for support of this work.

References

[1] E. N. Gilbert, "Random plane networks," *J. SIAM*, vol. 9, pp. 533–553, 1961.

[2] P. Gupta and P. R. Kumar, "Critical power for asymptotic connectivity in wireless networks." In *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming* (eds. W. M. McEneany, G. Yin, and Q. Zhang). Birkhäuser, Boston, pp. 547–566, 1998.

[3] S. S. Kunniyur and S. S. Venkatesh, "Network devolution and the growth of sensory lacunae in sensor networks," in *Proc. WiOpt'04: Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks*, University of Cambridge, UK, March 2004.

[4] S. S. Kunniyur and S. S. Venkatesh, "Threshold functions, node isolation, and emergent lacunae in sensory networks," to appear, 2006. Prepress copies may be obtained at http://www.seas.upenn.edu/∼venkates under Publications.

[5] M. D. Penrose, "On k-connectivity for a geometric random graph," *Random Structures and Algorithms*, vol. 15, pp. 145–164, 1999.

[6] M. Penrose, *Random Geometric Graphs*. New York: Oxford University Press, 2003.