Communication Using Helping Repeaters

Vladimir B. Balakirsky and A. J. Han Vinck
Institute for Experimental Mathematics
University of Duisburg-Essen
45326 Essen, Germany
Email : v.b.balakirsky@tue.nl, vinck@iem.uni-due.de

Abstract— We analyze communication schemes over bus systems that are of interest for creating effective data transmission algorithms when the destination can be hardly reached by the sender in one round, and some number of intermediate repeaters should be included. Possible strategies are investigated for Reed–Solomon codes under assumption that a certain maximum number of errors always has to be corrected.

I. Introduction

There are different data transmission situations where the quality of the received signal essentially depends on the distance between the receiver and the sender. The list of examples includes power–line communication ([1], [2], [3] and other papers). To organize a reliable communication between the sender and receiver one includes repeaters [4], [5]. Each repeater is usually understood as a device, which makes a decision about the transmitted signal and retransmits this signal again. The known approaches are based on the optimization of the number of repeaters and the duration of the signal under conditions that certain reliability at the destination is provided and a total transmission delay is fixed. We pay attention to the point that a repeater can also be "a Shannon repeater" in the sense that, instead of signal-by-signal decisions, it decodes the message of the sender and transmits a sequence of signals corresponding to the codeword for the decoded message. We demonstrate the advantage of this approach for Reed–Solomon codes.

II. Communication aspects of data transmission over a bus system

Let us consider the following general situation.

- ⊙ There is a data transmission system consisting of L users that are located at distance d meters from each other and connected to the same cable.
- ⊙ At each time instant, only one of the users is allowed to send a signal to the cable according to a fixed time sharing protocol.
- ⊙ At each time instant, all the users receive corrupted versions of the transmitted signal with the quality that depends on their distances from the sender.

The problem can be formulated as organizing the data transmission in such a way that User 0 can reliably deliver his message to User L with a possible help of $L-1$ intermediate users.

One can easily construct examples, which show that protocols with active intermediate participants who are capable of encoding and decoding the message can give much better performance than the uncoded transmission with non-Shannon repeaters. Although this conclusion is promising, "the capacity point of view" is non-constructive in a sense that it does not bring specific codes of finite lengths providing certain results. Our main concern in the present correspondence is translation of this point to constructive schemes where we make use of helping data to decode the message at the destination. At the same time, we come to some conclusions that are of information theoretical interest. This is the case because the data transmission scheme under consideration offers more than the organization of pair–wise transmissions. In particular, when User 0 transmits a codeword over a cable, all the users in the system receive corrupted versions of that codeword. While only few of them can reliable decode the message, all other users receive some information, and they can reduce their uncertainty about the message. Thus, we are dealing with data transmission scheme over a degrading broadcast channel [7] where all, but the last user, serve as helpers at the future stages of the protocol.

We will concentrate on a combinatorial model for data transmission when a certain number of errors has to be corrected. This model can be translated into a probabilistic model. The derivation of corresponding communication protocols for that model allows us to inspect information theory approaches based on random codes. We remark on results obtained in this direction in the conclusion.

III. Combinatorial model for data transmission over a bus system

Let us consider a bus system with three users (see Figure 1). Suppose that User 0 sends a q-ary codeword $\mathbf{x} \in \mathcal{Q}^n$ over a channel to inform User 2 about one of

$$M \triangleq q^{nR} \tag{1}$$

messages $\mathbf{u} \in \mathcal{Q}^{nR}$, where $\mathcal{Q} \triangleq \{0, \ldots, q-1\}$. Users 1 and 2 receive the vectors $\mathbf{y}$ and $\mathbf{y}'$, respectively, where

$$\mathbf{y} = \mathbf{x} \oplus \mathbf{e} \oplus \tilde{\mathbf{e}}, \;\; \mathbf{y}' = \mathbf{x} \oplus \mathbf{e} \oplus \Delta\mathbf{e},$$

where $\oplus$ denotes the component-wise sum modulo q. We assume that the Hamming weights of the error vectors

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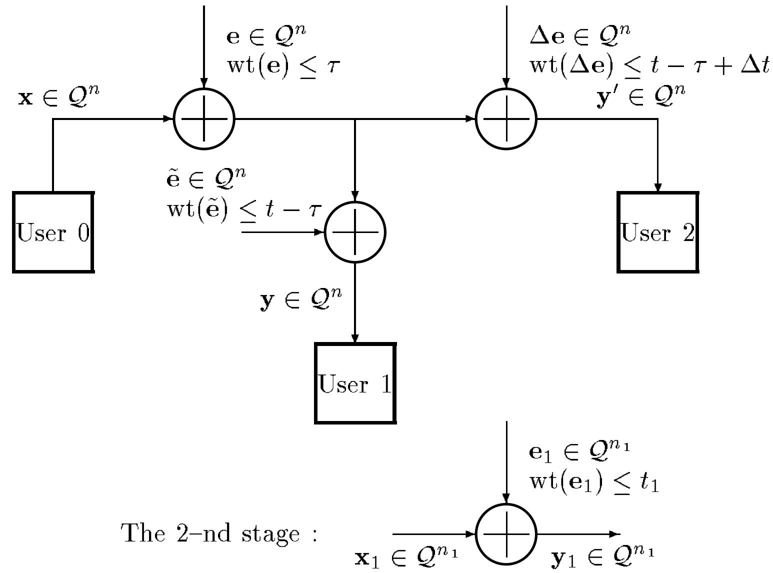

Fig. 1. Combinatorial model for data transmission over a bus system with three users; the codeword $\mathbf{x}_1$ can be formed depending on the message of User 0 and a partial knowledge about the error vector $\mathbf{e}$.

$\mathbf{e}, \tilde{\mathbf{e}}, \Delta\mathbf{e} \in \mathcal{Q}^n$ (the numbers of non-zero components in these vectors) are bounded from above by given constants τ, t and Δt in such a way that

$$\mathrm{wt}(\mathbf{e}) \le \tau, \quad \mathrm{wt}(\tilde{\mathbf{e}}) \le t - \tau, \quad \mathrm{wt}(\Delta\mathbf{e}) \le t - \tau + \Delta t. \qquad (2)$$

Let User 1 decode the transmitted message as $\hat{\mathbf{u}} \in \mathcal{Q}^{nR}$ and send "a helping message" at the second stage of the protocol. This message is encoded by a vector $\mathbf{x}_1 \in \mathcal{Q}^{n_1}$, and it can be interpreted as "a color" associated with the pair $(\hat{\mathbf{u}}, \mathbf{y})$. We assume that the color received by User 2 is a vector $\mathbf{y}_1 = \mathbf{x}_1 \oplus \mathbf{e}_1$, where

$$\mathrm{wt}(\mathbf{e}_1) \le t_1. \qquad (3)$$

Require the correct decoding of the message of User 0 by User 2 under assumptions (2) and (3).

Let transmission of each q-ary symbol take one time instant. Then, by (1), the ratio of the q-ary logarithm of the number of messages to be delivered to User 2 and the transmission delay, $n + n_1$, is equal to $k/(n + n_1)$, which can be considered as an effective transmission rate. In the following considerations we assume that the pair (n, k) is fixed and maximize the effective transmission rate by selecting a protocol that requires the smallest value of the parameter n_1.

The scheme in Figure 1 is fixed in a way that the received vectors of both User 1 and User 2 contain a common error vector $\mathbf{e}$ caused by the noise in the cable. These vectors also contain individual error vectors $\tilde{\mathbf{e}}$ and $\Delta\mathbf{e}$ caused by the noise at the receivers. By changing the parameters τ, t, and Δt, we can change the relative

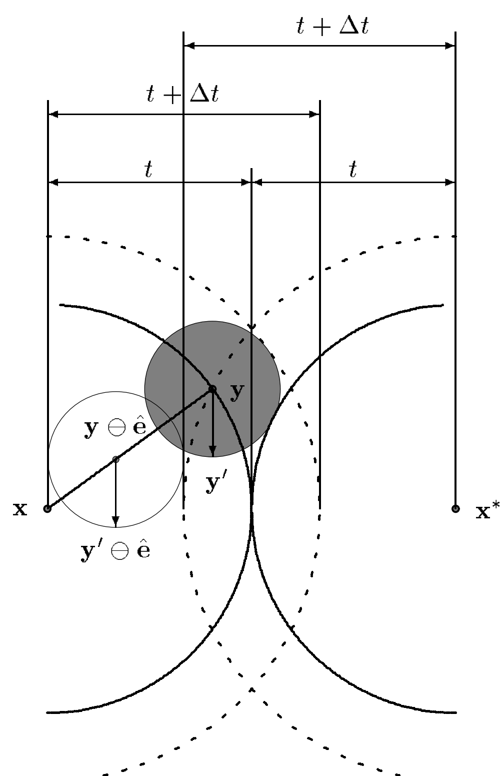

Fig. 2. The structure of decoding decision regions of the correct codeword $\mathbf{x}$ and incorrect codeword $\mathbf{x}^*$, where $\mathbf{y}$ and $\mathbf{y}'$ are possible vectors received by User 1 and User 2.

influence of these vectors. In particular, with the reference to power–line communications, the case when $\tau \approx 0$ is of interest for outdoor power–line communication, while $\tau \approx t$ is of interest for indoor power–line communication. The latter case is also of interest from a theoretical point of view because the helping codeword can be formed as a function of the message of User 0 and the common error vector. In other words, the problem can be formulated as follows : How much can we gain from the fact that the received vector $\mathbf{y}'$ is the degraded version of a known received vector $\mathbf{y}$? We will address this problem for $\tau = t$, i.e., there is no individual noise at the receiver of User 1. Then

$$\mathbf{y} = \mathbf{x} \oplus \mathbf{e}, \;\; \mathbf{y}' = \mathbf{x} \oplus \mathbf{e} \oplus \Delta\mathbf{e}$$

and

$$\mathrm{wt}(\mathbf{e}) \le t, \;\; \mathrm{wt}(\Delta\mathbf{e}) \le \Delta t. \tag{4}$$

This case is of interest itself, as it was mentioned above. Furthermore, the answers give guidelines to the organizing transmission protocols for a general scheme.

The statement of the problem can be presented as follows (see Figure 2). User 0 transmits a codeword $\mathbf{x}$ that is correctly decoded by User 1 since the minimum distance of the code is not less than $2t+1$ and t is the maximum number of errors in the vector $\mathbf{y}$. However, User 2 cannot always decode this codeword, since he receives a vector $\mathbf{y}'$, which is a degraded version of the vector $\mathbf{y}$. In particular, User 2 can also decide that an incorrect codeword $\mathbf{x}^*$ was transmitted. There are two kinds of help that can be delivered by User 1. The first kind of a help is transmission of a message that partitions the code into subcodes, and if the minimum distance of each subcode is not less than $2(t+\Delta t)+1$, then User 2 does not consider an incorrect vector $\mathbf{x}^*$ as a possible candidate for the codeword transmitted by User 0. The second kind of a help is transmission of an error vector $\hat{\mathbf{e}}$ that allows User 2 to replace the received vector $\mathbf{y}'$ with the vector $\mathbf{y}' \ominus \hat{\mathbf{e}}$ in such a way that any degraded version of this vector belongs to the decision region of the correct codeword, where $\ominus$ denotes the component-wise subtraction modulo q. There are also possibilities of combining these two types of a help, i.e., the message of User 1 can depend both on the message and the error vector $\mathbf{y} \ominus \mathbf{x}$. We will investigate these communication schemes and illustrate results for Reed–Solomon (RS) codes.

IV. Transmission rates achievable with Reed–Solomon codes

Suppose that RS codes over $GF(2^s)$ are used to transmit data. Thus, $q = 2^s$ and $n = 2^s - 1$. Let the parameters k, t, and Δt be assigned in such a way that

$$n - k \in \{2t, \ldots, 2(t+\Delta t) - 1\}. \tag{5}$$

Then User 1 can correctly decode the message of User 0 on the basis of the received vector $\mathbf{y}$, while User 2 is not capable to do so on the basis of the received vector $\mathbf{y}'$.

TABLE I

The smallest values of lengths of helping codewords for different communication protocols and an RS code having parameters $n = 63, k = 41$ when $t = (n-k)/2$ and (6) holds.

Δt	$n_1^{(\mathrm{m})}$	$n_1^{(\mathrm{e})}$	$n_1^{(\mathrm{m+e})}$	$n_1^{(\mathrm{e}_2)}$	$n_1^{(\mathrm{m+e}_2)}$
1	4	4		4	
2	8	8		8	
3	10	10		9	
4	14	14		10	
5	16	15		11	
6	20	17			14
7	22	20			16
8	26	22			20
9	28	25			22
10	32	27			26
11	34	28			28
12	38		32		32
13	40		34		34
14	44		38		38
15	48		40		40
16	50		44		44
17	53		48		48
18	56		50		50
19	60		54		54
20	62		56		56

We will assume that

$$t_1 = \lceil t n_1 / n \rceil, \tag{6}$$

where n_1 is the length of the codeword of User 1, and present possible protocols for data transmission where certain values of the parameter n_1 are assigned. Let User 2 have to decode the message $\mathbf{u}$ on the basis of the pair of received vectors $(\mathbf{y}', \mathbf{y}_1)$.

A. Tandem–type protocol

Let User 1 use the same code as User 0 and re–transmit the codeword for the original message. Conditions (5), (6) guarantee that User 2 can correctly decode the message only on the basis of the received vector $\mathbf{y}_1$. In this case, $n_1^{(\mathrm{tnd})} = n$ is the length of the codeword transmitted by User 1, and the bus system is used $2n$ times.

B. Protocol (m) where the helping message depends only on the message of User 0

Let User 1 transmit k_1 symbols of the message by sending the corresponding codeword of a code of length $n_1^{(\mathrm{m})}$. User 2 has to decode these symbols correctly and decode the remaining $k - k_1$ information symbols from the vector $\mathbf{y}'$. The algorithm can be used if the parameters satisfy the inequalities

$$\begin{aligned} 2t_1 &\le n_1^{(\mathrm{m})} - k_1, \\ 2(t+\Delta t) &\le n - (k - k_1). \end{aligned}$$

C. Protocol (e) where the helping message depends only on the error vector of User 1

Suppose that

$$\Delta t \le t. \tag{7}$$

Let User 1 transmit a message representing the error vector $\hat{\mathbf{e}} \in \mathcal{Q}^n$ such that

$$\mathrm{wt}(\hat{\mathbf{e}}) = \Delta t \tag{8}$$

and

$$\hat{e}_i \neq 0 \;\Rightarrow\; \hat{e}_i = e_i, \text{ for all } i = 1, \ldots, n, \tag{9}$$

by sending a codeword of length $n_1^{(\mathrm{e})}$. In other words, User 1 informs User 2 about the content of ℓ error positions in the vector $\mathbf{y}$, where

$$\ell = \begin{cases} \mathrm{wt}(\mathbf{e}), & \text{if } \mathrm{wt}(\mathbf{e}) \leq \Delta t, \\ \Delta t, & \text{if } \mathrm{wt}(\mathbf{e}) > \Delta t. \end{cases}$$

If User 2 correctly decodes the transmitted data, then he subtracts $\hat{\mathbf{e}}$ from the vector $\mathbf{y}'$. The vector $\mathbf{y}' \ominus \hat{\mathbf{e}}$ has at most

$$\mathrm{wt}(\mathbf{e}) + \Delta t - \ell \leq \min\{\Delta t, \mathrm{wt}(\mathbf{e})\}$$

errors. By (4) and (7), $\min\{\Delta t, \mathrm{wt}(\mathbf{e})\} \leq t$, and this vector can be correctly decoded. Since, the total number of q-ary vectors satisfying (8) is bounded from above by $\binom{n}{\Delta t} q^{\Delta t}$, the algorithm can be used if

$$2t_1 \leq n_1^{(\mathrm{e})} - \left(\lceil \log_q \binom{n}{\Delta t} \rceil + \Delta t\right).$$

D. Protocol (m+e) where the helping message depends on the message of User 0 and the error vector of User 1

Suppose that

$$\Delta t > t. \tag{10}$$

Let User 1 transmit a message representing k_1 symbols of the message and the error vector $\hat{\mathbf{e}} \in \mathcal{Q}^n$ such that

$$\mathrm{wt}(\hat{\mathbf{e}}) = t \tag{11}$$

and (9) is valid. Denote the length of the codeword by $n_1^{(\mathrm{m+e})}$. If User 2 correctly decodes the transmitted data, then he subtracts $\mathbf{e}$ from the vector $\mathbf{y}'$ and decodes $k - k_1$ information symbols from the vector $\mathbf{y}' \ominus \mathbf{e}$. The algorithm can be used if the following parameters satisfy the inequalities

$$\begin{aligned} 2t_1 &\leq n_1^{(\mathrm{m+e})} - \left(k_1 + \lceil \log_q \binom{n}{t} \rceil + t\right), \\ 2\Delta t &\leq n - (k - k_1). \end{aligned}$$

The first inequality follows from the observation that there are at most $\binom{n}{t} q^t$ error vectors $\mathbf{e}$ satisfying (9) and (11). To check the second inequality we notice that $\mathbf{y}' \ominus \mathbf{e} = \mathbf{x} \oplus \Delta\mathbf{e}$ and use (4).

E. Protocol (e_2) where the helping message depends only on the positions of errors in the error vector of User 1

Suppose that

$$t + 2\Delta t \leq n - k \tag{12}$$

and denote

$$w \triangleq 2(t + \Delta t) - (n - k). \tag{13}$$

Let User 1 inform User 2 about the location of w positions chosen in such a way that ℓ_2 of them are error positions in the vector $\mathbf{y}$, where

$$\ell_2 = \begin{cases} \mathrm{wt}(\mathbf{e}), & \text{if } \mathrm{wt}(\mathbf{e}) \leq w, \\ w, & \text{if } \mathrm{wt}(\mathbf{e}) > w, \end{cases}$$

by sending a codeword of length $n_1^{(\mathrm{e}_2)}$. If User 2 correctly decodes the transmitted data, then he erases these positions in the vector $\mathbf{y}'$. The remaining vector has length $n - w$ and it contains $\mathrm{wt}(\mathbf{e}) + \Delta t - \ell_2$ errors.

The algorithm can be used if

$$2t_1 \;\leq\; n_1^{(\mathrm{e}_2)} - \lceil \log_q \binom{n}{2(t + \Delta t) - (n - k)} \rceil,$$

i.e., if User 2 can correctly decode one of $\binom{n}{2(t+\Delta t)-(n-k)}$ binary vectors transmitted by User 1. Let us check that the correct decoding of the message from the $n - w$ positions of the vector $\mathbf{y}$ is always possible.

1). Let $\mathrm{wt}(\mathbf{e}) \leq w$. Then the number of errors is at most

$$\mathrm{wt}(\mathbf{e}) + \Delta t - \mathrm{wt}(\mathbf{e}) = \Delta t,$$

and the correct decoding is possible since (12) and (13) imply $2\Delta t \leq (n - w) - k$. 2). Let $\mathrm{wt}(\mathbf{e}) > w$. Then the number of errors is at most

$$\mathrm{wt}(\mathbf{e}) + \Delta t - w \leq t + \Delta t - w,$$

and the correct decoding is possible since (13) implies $2(t + \Delta t - w) \leq (n - w) - k$.

F. Protocol (m+e_2) where the helping message depends on the message of User 0 and positions of errors in the error vector of User 1

Suppose that $t + 2\Delta t > n - k$. Let User 1 transmit k_1 symbols of the message and a binary vector $\hat{\mathbf{e}}_2$ such that $\mathrm{wt}(\hat{\mathbf{e}}_2) = t$ and

$$e_i \neq 0 \;\Rightarrow\; \hat{e}_{2,i} = 1, \text{ for all } i = 1, \ldots, n$$

by sending a codeword of length $n_1^{(\mathrm{m+e}_2)}$. If User 2 correctly decodes the transmitted data, then he erases t positions in the vector $\mathbf{y}'$ and decodes $k - k_1$ information symbols using the remaining $n - t$ positions. Similarly to the previous considerations one can check that the algorithm can be used if the following parameters satisfy the inequalities

$$\begin{aligned} 2t_1 &\leq n_1^{(\mathrm{m+e}_2)} - \left(k_1 + \lceil \log_q \binom{n}{t} \rceil\right), \\ 2\Delta t &\leq (n - t) - (k - k_1). \end{aligned}$$

V. Numerical illustration and asymptotic evaluation of the performances of communication protocols

Let $s = 6$. Thus, User 0 and User 1 transmit data by using an RS code over $GF(64)$, and the codeword for the message of User 0 has length $n = 63$. The values of parameters determined above are given in Table I for $k = 41$.

Let us evaluate the asymptotic performance of the transmission protocols. If we omit rounding to integers and replace the inequalities for the lengths of codewords for helping messages with equalities, then, assuming that

$$t_1/n_1^{(\cdot)} = t/n$$

we have:

$$\begin{aligned}
(1-2t/n)n_1^{(\mathrm{m})} &= 2(t+\Delta t)-(n-k);\\
(1-2t/n)n_1^{(\mathrm{e})} &= \log_q \binom{n}{\Delta t} + \Delta t;\\
(1-2t/n)n_1^{(\mathrm{m+e})} &= t+2\Delta t+\log_q \binom{n}{t} - (n-k);\\
(1-2t/n)n_1^{(\mathrm{e_2})} &= \log_q \binom{n}{2(t+\Delta t)-(n-k)};\\
(1-2t/n)n_1^{(\mathrm{m+e_2})} &= t+2\Delta t+\log_q \binom{n}{t} - (n-k).
\end{aligned}$$

Denote

$$\alpha \triangleq t/n,\ \Delta\alpha \triangleq \Delta t/n$$

and suppose that these parameters are constants, while $n \to \infty$. Moreover, let

$$2t/n = 1 - k/n.$$

Since $q = n+1$, we have $(1/n)\log_q 2^n \to 0$. Therefore:

$$\begin{aligned}
(1-2\alpha)n_1^{(\mathrm{m})}/n &\to 2\Delta\alpha;\\
(1-2\alpha)n_1^{(\mathrm{e})}/n &\to \Delta\alpha,\ \text{ if } \Delta\alpha \le \alpha;\\
(1-2\alpha)n_1^{(\mathrm{m+e})}/n &\to 2\Delta\alpha-\alpha,\ \text{ if } \Delta\alpha > \alpha;\\
(1-2\alpha)n_1^{(\mathrm{e_2})}/n &\to 0,\ \text{ if } \Delta\alpha \le \alpha/2;\\
(1-2\alpha)n_1^{(\mathrm{m+e_2})}/n &\to 2\Delta\alpha-\alpha,\ \text{ if } \Delta\alpha > \alpha/2.
\end{aligned}$$

Thus, an asymptotically optimum communication protocol chosen from the protocols presented above can be specified as follows:

$$\begin{aligned}
\Delta\alpha \le \alpha/2 &\Rightarrow (\mathrm{e_2});\\
\Delta\alpha \in (\alpha/2, \alpha) &\Rightarrow (\mathrm{m+e_2});\\
\Delta\alpha \ge \alpha &\Rightarrow (\mathrm{m+e_2}) \text{ or } (\mathrm{m+e}).
\end{aligned}$$

The optimum protocol never depends only on the message, i.e., the knowledge of the error vector $\mathbf{e}$ is very important for User 1 while constructing the helping message. Moreover, not the values of errors, but their locations are important, and the algorithm can be fixed as either ($\mathrm{e_2}$) or ($\mathrm{m+e_2}$).

VI. Conclusion

Coding methods, as it was demonstrated, bring essential improvement in the performance of communication schemes with a bus system over the performance of schemes where coding is not used. These methods are based on the point that the code of User 0 has large minimum distance. A proper help of User 1 allows User 2 to change the received vector in such a way that the decoder correctly recovers the message transmitted by User 0. Thus, the knowledge of the common noise can be very important for User 1 to form a shortest helping sequence that allows User 2 to decode the message of User 0. This conclusion can be also extended to arbitrary error–correcting codes. However, one can show that conclusions derived for random block codes are different : the help should be formed only on the basis of the message of User 0 from the random coding argument point of view. Such a phenomenon is interesting, since it shows the importance of a large minimum distance of the code for multi–user systems even when the code rate is close to the capacity. Moreover, it shows that random coding arguments should be applied very carefully if one wants to obtain an asymptotically optimum transmission protocol.

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