Asymptotic Results for Star Circuit Switched Networks Using Occupancy Models

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***Abstract*— The (time) equilibrium distribution for a star circuit switched network is shown to be equivalent to the (conditional) outcome of a randomised occupancy experiment in which a Poisson number of balls of different colours are thrown. (In this way we cover the case when distinct numbers of circuits are needed on different arms by call.) The conditioning probability is the normalisation constant itself. We show that the large deviations exponent for the normalisation constant is determined by a Generalised Erlang Fixed Point (GEFP).**

I. Introduction

The Erlang fixed point (EFP) approximation, to loss probabilities in circuit switched networks [3], applies in two limiting regimes. In the first, the network topology is held fixed and the traffic and link capacities are increased together. In the second the link capacities are held fixed but the network "complexity" is increased. This paper is concerned with the latter, an example of which is a star network with n arms, C_S circuits per arm and $2\nu/(n-1)$ calls arriving per distinct pair of arms. The EFP for such star networks was demonstrated by Kelly and Ziedins [5] and Whitt [11]. Note that the star network is important not only as a communication model but also because it corresponds to any service system where demands require service from several facilities simultaneously.

In this paper we consider a generalisation of the above to the case where calls can be allocated more than a single circuit on each arm and further distinct numbers of circuits on distinct arms are also allowed. Rather than dealing exhaustively with all the cases covered by the following analysis, we concentrate on a single example and allow the reader to determine generalisations.

The central fact which we rely on is that the equilibrium statistics of such star networks coincide with that of a particular occupancy experiment in which there are coloured balls and conditional on no urn "overflowing". Each colour represents a different number of units of urn capacity and the urn overflows if the urn's capacity is exceeded. There are as many urns as arms in the star each with capacity C_S. The normalisation constant G_n of the equilibrium distribution appears as the conditioning probability. This equivalence can be exploited to obtain blocking performance in both asymptotic and non-asymptotic regimes, where the latter estimates may be obtained by importance sampling the occupancy experiments themselves. The connection between occupancy models and star networks was first observed in [4] where a large deviations principle is derived for the empirical occupation measure of the links. Our methods generalise these results and make the connection between occupancy models and star networks much more explicit.

Our analysis also uses this connection, as well as recent results for occupancy models detailed in [7]. These are for occupancy experiments with n urns, r balls and Maxwell-Boltzman (MB) statistics A sample path large deviations principle (LDP) has been shown to hold for such models, [9], [7] in which n is a scale parameter with r in fixed proportion; that is $n \to \infty$ with the fraction of balls per urn thrown $\lfloor m \rfloor / n \to \beta$. The LDP for the terminal distribution of balls in the urns then follows from the contraction principle. Both the extremals and the rate function for the terminal distribution are described in [6], [7].

In addition our results allow high accuracy approximations to blocking probabilities to be obtained for finite networks by using (change-of-measure) importance sampling. For example the normalisation constant itself may be estimated numerically, using a similar approach to that described in [10].

II. Coloured Occupancy Models

The following is an outline of the main results which are needed, in the interests of brevity formal definitions, proofs etc. have been ommitted. In the coloured case each occupancy experiment is indexed by n the number of urns and for each colour i a certain fixed number of balls are thrown $r_i = \lceil n\beta_i \rceil$ where β_i is a fixed constant. The order in which they are thrown does not matter. Balls enter urns uniformly at random and inpendently of any other throws, this is known as Maxwell-Boltzman (MB) statistics. Under very general assumptions about the (independent) colouring processes, it has been shown that a sample path LDP holds with a specific rate function see [8]. From the contraction principle, a corresponding terminal LDP holds as before.

In the case without colours, the rate function for the terminal LDP was shown to be

$$D\left(\Gamma \mid\mid \mathcal{P}(\beta)\right)$$

where Γ_i is the fraction of urns with i balls $\Gamma_i, i = 0, 1, 2, \cdots$ and it is required that the mean number of balls per urn is conserved, $\sum_i i\Gamma_i = \beta$. $\mathcal{P}(\beta)$ denotes the Poisson process mean β. This result is natural in that the strong law distribution of balls in urns under this scaling is Poisson. No direct proof of the terminal LDP with the above rate function has been obtained however, without going through the sample path LDP.

In the coloured case a similar result can be shown to hold. In fact letting green correspond to 1 and red to 2 and supposing $\beta_i, i = 1, 2$ balls per urn of each colour being thrown the rate function for this case can be seen to be

$$D\left(\Gamma \mid\mid \mathcal{P}(\beta_1) \times \mathcal{P}(\beta_2)\right)$$

where Γ_{ij} is the fraction of urns with i green and j red. Thus the probability of a rare event taking place in a set $\mathcal{A}$ where the event is defined in terms of some maximum count for each colour say I, J, satisfies

$$-\frac{1}{n}\log \mathbb{P}\{\Gamma_n \in \mathcal{A}\} \approx \inf_{\Gamma \in \mathcal{A}} D\left(\Gamma \mid\mid \mathcal{P}(\beta_1) \times \mathcal{P}(\beta_2)\right)$$

Here Γ_n is the fraction of urns with each possible pair of colour counts up to their maximum with an additional aggregate count where the maximum for that colour is exceeded. This is all we need to describe the results for the star network. Moreover in this case the counts above maximum can and will be taken 0 as we will work to the urn capacity. This being the case determination of the rate function involves optimisation over a finite number of variables.

III. The Equilibrium as an Occupancy Experiment

Consider the n arm star network shown in figure 1. There are two call classes. In class 1 calls are made using any two arms $i, j, i \neq j$, and require a single circuit from the i-arm and from the j-arm. Each arm has C_S circuits and calls hold their circuits for a random period with unit mean, independently of other calls and of arrivals. Class 1 calls arriving for node pairs

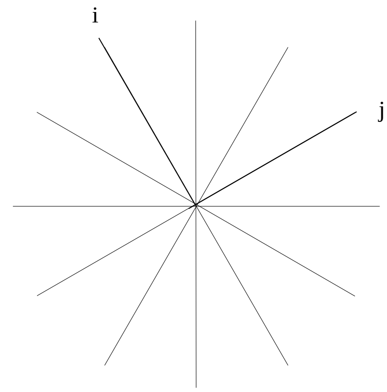

Fig. 1. A Star Network with n Arms.

(i, j) arrive at rate $2\nu_1/(n-1)$ and are accepted only if there is a free circuit on each of their arms otherwise they are lost and cleared. Hence if there were an infinite number of circuits on each arm the number of calls carried would be $n\nu_1$. Class 2 calls are similar but require two circuits from one of the two arms. Class 2 calls arrive at a rate $\nu_2/(n-1)$ per arm, so that there would be $n\nu_2$ calls carried.

The system state is the number of calls up for each class. That is $(\mathbf{r}_1, \mathbf{r}_2)$ where for class 2 $\mathbf{r}_2 = (r^{(2)}_{(i,j)})_{i \neq j}$, where 2 circuits are required from the i arm and one from the j arm. The class 1 state vector is similarly defined.

The equilibrium distribution [2] may be written

$$\begin{aligned} G_n(C_S)\pi((\mathbf{r}_1, \mathbf{r}_2)) &= e^{-n(\nu_1+\nu_2)} F_1(\mathbf{r}_1) \cdot F_2(\mathbf{r}_2) \\ F_1(\mathbf{r}_1) &= \prod_{(i,j)} \frac{(2\nu_1)^{r^{(1)}_{(i,j)}}}{(n-1)^{r^{(1)}_{(i,j)}} \cdot r^{(1)}_{(i,j)}!} \\ F_2(\mathbf{r}_2) &= \prod_{(i,j)} \frac{\nu_2^{r^{(2)}_{(i,j)}}}{(n-1)^{r^{(2)}_{(i,j)}} \cdot r^{(2)}_{(i,j)}!} \end{aligned}$$

for *admissible* states $(\mathbf{r}_1, \mathbf{r}_2)$,

$$\sum_{\{(j,k), j=i, k=i\}} r^{(1)}_{(j,k)} + 2\sum_{j \neq i} r^{(2)}_{(i,j)} + \sum_{j \neq i} r^{(2)}_{(j,i)} \leq C_S, \forall i \quad (3.1)$$

This follows immediately from the detailed balance conditions for each class in the case of exponential holding times. We thus in principle may obtain the normalisation constant $G_n(C_S))$ by adding over states at which point the probability a call is lost and cleared may be calculated.

Unfortunately in most cases of interest this approach is numerically too complex to be of any practical value. To circumvent this problem the Generalised Erlang Fixed Point (GEFP) approximation was developed. This supposes that the number of used circuits on each arm, proceeds independently; further across classes at the arm, the arrivals themselves are independent. Thus given the arrival rates the probability that a circuit or group of circuits is available can be determined. It remains to specify the arrivals rates. But these are simply the original arrival rate to the arm "thinned" by the supposed independent blockings at the other arms involved in each call.

In terms of our example there are two blockings B_1, B_2 for single and double circuit calls respectively. These rates are $2\nu_1(1-B_1)$ for single circuits class 1, $\nu_2(1-B_2)$ for single circuit calls class 2 and $\nu_2(1-B_1)$ for double circuit calls class 2. The fixed point arises in equating the blockings given the thinned arrival rates. We thus obtain,

$$\begin{aligned} B_1 &= E_1(2\nu_1(1-B_1)+\nu_2(1-B_2), \nu_2(1-B_1), C_S) \\ B_2 &= E_2(2\nu_1(1-B_1)+\nu_2(1-B_2), \nu_2(1-B_1), C_S) \end{aligned} \tag{3.2}$$

where E_i are defined by

$$E_1(\nu_1, \nu_2, C_S) := D^{-1} \sum_{i+2j=C_S} \frac{\nu_1^i}{i!}\frac{\nu_2^j}{j!} \tag{3.3}$$

$$E_2(\nu_1, \nu_2, C_S) := D^{-1} \sum_{C_S \geq i+2j \geq C_S-1} \frac{\nu_2^i}{i!}\frac{\nu_2^j}{j!} \tag{3.4}$$

and $D = \sum_{i+2j \leq C} \frac{\nu_2^i}{i!}\frac{\nu_2^j}{j!}$. The loss probabilities $L_i, i = 1, 2$ are determined as

$$\begin{aligned} 1 - L_1 &= (1-B_1)^2 \\ 1 - L_2 &= (1-B_1)(1-B_2) \end{aligned}$$

The blockings turn out to be unique. The loss probabilities so calculated turn out to be asymptotically exact as $n \to \infty$.

It turns out that these results emerge naturally by obtaining a representation of the star network equilibrium as an occupancy model with a random number of balls thrown. We now proceed to discuss this.

Corresponding to their being r_1, r_2 calls of each class, suppose we throw r_1 pairs of green balls into n urns, labeled $1, 1, \cdots, r_1, r_1$ and r_2 pairs of green and red balls labeled $1, 1, 2, 2, \cdots, r_2, r_2$ using MB statistics, all throws independent. Actually we will not use MB statistics but a slightly modified version which we call $\mathbb{Q}$. Under these statistics, the first ball in each pair enters an urn uniformly at random and the second ball also uniformly but into any urn *except* the one in which the first ball fell. Let $\mathbb{Q}(\mathbf{r}_i), i = 1, 2$ be the probability that we get the given occupancy $i = 1, 2$. By counting the number of ways to succeed,

$$\mathbb{Q}(\mathbf{r}_1)\,[n(n-1)]^{r_1} = 2^{r_1} \cdot \frac{r_1!}{\prod_{(i,j)} r_{(i,j)}^{(1)}!} \tag{3.5}$$

$$\mathbb{Q}(\mathbf{r}_2)\,[n(n-1)]^{r_2} = \frac{r_2!}{\prod_{(i,j)} r_{(i,j)}^{(2)}!} \tag{3.6}$$

Thus the normalisation constant is,

$$G_n(C_S) = \sum_{r_1, r_2=0}^{\infty} \mathcal{P}_{r_1}(n\nu_1)\mathcal{P}_{r_2}(n\nu_2)\mathbb{Q}_{r_1, r_2}(T_{C_S}), \tag{3.7}$$

adding over exclusive, exhaustive events. T_{C_S} is the event that the balls correspond to an admissible state, where the statistics $\mathbb{Q}_{r_1, r_2}$ are obtained as the product over each class. A state is now admissible provided $i_l + 2j_l \leq C_S$ for each urn $l = 1, \cdots, n$ where i_l, j_l are the numbers of green and red balls.

The above discussion shows that the equilibrium distribution of the star network is *identical* to a random occupancy experiment in which a Poisson number of balls are thrown conditional on the event that the urn experiment corresponds to an actual state. This latter probability is given by the normalisation constant and corresponds to a rare (exponentially small in n) event.

IV. Large Deviations Analysis

A. Minimising Distribution for the Carried Traffic

As mentioned in section II the exponent for a rare event is determined by the relative entropy with respect to a product Poisson. Hence if we have $2\beta_1 + \beta_2$ green balls thrown (single circuits used) per urn (arm) and β_2 red balls thrown (double circuits used) then the exponent $H(\beta_1, \beta_2)$ for an admissible state is given as follows,

$$\min_{\Gamma_{GR}} D\left(\Gamma^{GR} \mid\mid \mathcal{P}_G(2\beta_1+\beta_2)) \times \mathcal{P}_R(2\beta_2)\right)$$

where the minimum is taken over all occupancy distributions over admissible states and subject to the constraints for normalisation and the total number of balls of each colour so that $\Gamma_{i,j} > 0$ only if $i + 2j \leq C_S$. We have omitted the GR superscript.

A standard application of Lagrange multipliers shows the solution is product Poisson,

$$\Gamma_{i,j} = K^{-1}\mathcal{P}_i((2\beta_1+\beta_2)\rho_1)\mathcal{P}_j(\beta_2\rho_2) \tag{4.8}$$

We have taken $y \geq 0$ to be multiplier corresponding to the constraint

$$2\beta_1 + \beta_2 = \sum_{i,j} i\Gamma_{i,j}$$

and $v \geq 0$ the one corresponding to

$$\beta_2 = \sum_{i,j} j\Gamma_{i,j}$$

In terms of these multipliers $\log\rho_1 = y, \log\rho_2 = v, \rho_i \geq 1$. These latter can be obtained from the conservation equations for green and red balls which after some elementary simplification are

$$K(1-\frac{1}{\rho_1}) = \sum_{i+2j=C_S} \mathcal{P}_i((2\beta_1+\beta_2)\rho_1)\mathcal{P}_j(\beta_2\rho_2))$$
$$K(1-\frac{1}{\rho_2}) = \sum_{C_S \geq i+2j \geq C_S-1} \mathcal{P}_i(\beta_1\rho_1)\mathcal{P}_j(\beta_2\rho_2))$$

where

$$K = \sum_{i+2j \leq C_S} \mathcal{P}_i((2\beta_1+\beta_2)\rho_1)\mathcal{P}_j(\beta_2\rho_2). \qquad (4.9)$$

These equations may be rewritten as,

$$B_1 = E_1((1-B_1)^{-1}(2\beta_1+\beta_2),(1-B_2)^{-1}\beta_2,C_S)$$
$$B_2 = E_2((1-B_1)^{-1}(2\beta_1+\beta_2),(1-B_2)^{-1}\beta_2,C_S)$$

where we have used the definition (3.3) for $E_i, i = 1, 2$.

$$E_1(\nu_1,\nu_2,C_S) := D^{-1} \sum_{i+2j=C_S} \frac{\nu_1^i}{i!}\frac{\nu_2^j}{j!}$$
$$E_2(\nu_1,\nu_2,C_S) := D^{-1} \sum_{C_S \geq i+2j \geq C_S-1} \frac{\nu_2^i}{i!}\frac{\nu_2^j}{j!}$$

and $D = \sum_{i+2j \leq C} \frac{\nu_2^i}{i!}\frac{\nu_2^j}{j!}$. $B_i \in [0,1], i = 1,2$ are the *blockings* and satisfy $B_i = 1 - 1/\rho_i$.

B. Exponent for the Normalising Constant

(3.7) shows that the normalisation constant is equal to the probability that an admissible state is produced in a random urn experiment with the number of coloured balls thrown being determined by Poisson variates, with rates ν_1, ν_2. As we discussed this is a rare event as we are throwing a large number of balls into a large number of urns so that in almost every case some urn would have too many balls. The large deviations exponent is a tradeoff between the exponent for throwing fewer balls than average and the exponent for their distribution which latter we discussed in section IV-A.

The large deviations exponent to produce a mean β from a Poisson with mean ν is $\nu - \beta + \beta \log(\beta/\nu)$. Hence the exponent for the normalising constant $F(\nu_1, \nu_2, C_S)$ is

$$\min_{\beta_1,\beta_2} \nu_1 - \beta_1 + \beta_1 \log(\beta_1/\nu_1) + \nu_2 - \beta_2 + \beta_2 \log(\beta_2/\nu_2) + H(\beta_1,\beta_2)$$

It is convenient to minimise with respect to $2\beta_1 + \beta_2$ and β_2 as these formed the constraints earlier. Taking derivatives we obtain the following pair of conditions,

$$0 = \frac{1}{2}\log\beta_1/\nu_1 + y$$
$$0 = -\frac{1}{2}\log\beta_1/\nu_1 + \log\beta_2/\nu_2 + v$$

Substituting for $y = \log\rho_1$ and $v = \log\rho_2$,

$$\beta_1 = \nu_1(1-B_1)^2 \qquad (4.10)$$
$$\beta_2 = \nu_2(1-B_1)(1-B_2)$$

which determine the call loss probabilities in terms of the blockings. These may be obtained by eliminating the carried traffic to obtain the GEFP (3.2).

To summarise, we have just shown that the GEFP determines the minimising exponent for the normalisation constant which is the rare event probability in the occupancy experiment that the state is admissible. That the solutions for B_1, B_2 are unique can be shown exploiting this relationship. A detailed theoretical argument yields

$$\lim -\frac{1}{n}\log G_n(C_S) = F(\nu_1,\nu_2,C_S)$$

V. Importance Sampling

The above discussion may be used as a basis to shows that the LDP for the empirical occupancy distribution of used circuits, see [4] holds in these more general circumstances and consequently the GEFP remains asymptotically exact. This allows us to explore rare events within the star network. We move to a brief discussion of importance sampling.

For the star network we may obtain accurate performance estimates by using change of measure importance sampling to obtain the normalisation constant. The tilted parameters are obtainable from the GEFP, which in the case of our example are the $\beta_i, B_i, i = 1,2$ for determining the number of balls to throw and the terminal distribution of the number of single and double circuit requirements on each arm. Having obtained the number of balls to throw, the balls themselves are thrown according to a change of measure obtained from a polynomial extremal whose coefficients are determined by the terminal distribution itself. Examples of such polynomial extremals for the non-coloured case are described in [7].

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