# A Family of Distributed Space-Time Trellis Codes Achieving Full Diversity for Asynchronous Cooperative Communications

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***Abstract*— In current cooperative communication schemes, to achieve cooperative diversity, synchronization between terminals is usually assumed, which may not be practical since each terminal has its own local oscillator. In this paper, we first present a necessary and sufficient condition for the space-time trellis codes based on the stack construction proposed by Hammons and El Gamal to possess the full cooperative diversity order without the synchronization assumption. The condition is that the binary generator metrices of the trellis codes are shift full rank (SFR) matrices, i.e., have full row rank no matter the shifts of their row vectors, where a row corresponds to a terminal (or transmit antenna) and the length of a row vector is the memory size of the corresponding trellis code on the corresponding terminal. We then present a simple construction of SFR for any number of rows, whose number of columns, however, grows exponentially with the number of rows. We finally present some systematic SFR matrix constructions including shortest, i.e., square, SFR matrices.**

## I. Introduction

In wireless communication systems, to combat fading, multiple antennas may be equipped at the transmitter and/or the receiver, where multiple antennas may provide spatial diversity gain as well as multiplexing gain. However, in cellular systems or sensor networking systems, it may be hard for a mobile station or a sensor terminal to equip with multiple antennas because of their limited sizes and also the cost. By realizing that a cellular or sensor networking system usually has multiple users, the idea of making different users to communicate cooperatively to achieve the spatial diversity gain has been proposed in, for example, [1] [2], and such spatial diversity is called *cooperative diversity*.

In most existing schemes, for example [2]- [6], synchronization is assumed as an *a priori* condition. However, the cooperative diversity is provided by different antennas in different transceivers, so the cooperative diversity is asynchronous in nature. An approach to achieve the cooperative diversity where synchronization between relays is not a required condition was discussed in [7]. In [7], intentional delays are introduced in different terminals. At the destination receiver, minimum mean square error (MMSE) estimator is used to exploit the cooperative diversity. Although some diversity gain can be achieved in [7], full diversity order is not guaranteed, where full diversity order means that the diversity order equals to the number of relays involved. Other approaches for the asynchronous cooperative diversity can be found in for example [8]–[10]. The goal of this paper is to present a systematic construction of space-time trellis codes that achieve the full cooperative diversity order in symbol asynchronous cooperative communications for any number of relays involved.

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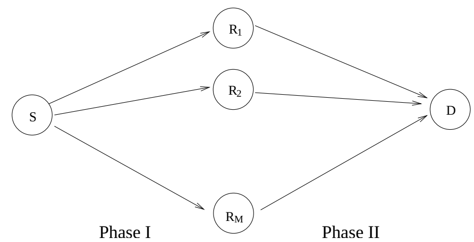


Fig. 1. System architecture.

This paper is organized as follows. In Section II, the system model is described and the problem of interest is formulated. In Section III, the space-time trellis code family is constructed. It is shown that the construction of such codes is equivalent to the construction of binary shift full rank (SFR) matrices. In this section, we also present a simple construction of a family of SFR matrices for any number of relays. In Section IV, we present other systematic constructions of SFR matrices including shortest SFR (SSFR) matrices that correspond to the space-time trellis codes with smallest memory sizes.

## II. System Model and Problem Formulation

Assume there are $M+2$ terminals that communicate cooperatively. The system is shown in Fig. 1. We assume that $S$ is the source terminal, $D$ is the destination terminal, and $R_i,\ i=1,2,\cdots,M$, are the potential relays. As in the analysis carried out in [2], we assume that there are two phases during the cooperative communication. In Phase I (Fig.1(a)), $S$ broadcasts its information to potential relays $R_i,\ i=1,2,\cdots,M$, and the destination $D$. In Phase II (Fig.1(b)), $S$ stops transmission, and potential relays start to transmit. There are two different transmission schemes for a potential relay [2]: one is *amplify-and-forward*; the other scheme is *decode-and-forward*. In this paper, we adopt *decode-and-forward*. During Phase I, each potential relay receives:

$$y_{r_i}(n)=h_{s,r_i}(n)x_s(n)+w_{r_i}(n),$$

where we assume that the channel is quasi-static Rayleigh flat fading, $h_{s,r_i}(n)$ is the channel coefficient between $S$ and $R_i$ and is Rayleigh distributed with unit power. We also assume that $h_{s,r_i}(n)$ is known at the receiver. $w_{r_i}(n)$ is the AWGN at $R_i$ and has zero mean and variance $\sigma^2$ per real dimension. $x_s(n)$ is the transmitted symbol by $S$. During Phase II, firstly, $R_i$ demodulates the received signal and does CRC check to see whether the detected information is correct or not. We assume that those can pass the CRC check do not have any errors in their detected information. We use $\mathcal{R}_s$ to denote the set of potential relays that can detect successfully the source information during a packet/frame from $S$, and assume $M_s = |\mathcal{R}_s|$. Then, those $R_i \in \mathcal{R}_s$ will be enrolled in the transmission of Phase II. It is usually assumed that $M_s$ is a random variable [2]. As analyzed in [2], the protocol that relays transmit space-time coded signals on the overlapped channels performs better than the protocol that relays just repeat their detected information on the orthogonal channels. Therefore, in this paper, we assume that a space-time coded transmission is used during Phase II. In Phase II, if the enrolled relays are symbol synchronized, the destination receives:

$$y_d(n) = \sum_{1\le i\le M \text{ and } R_i\in\mathcal{R}_s} h_{r_i,d}(n)x_{r_i}(n) + w_d(n). \quad (1)$$

In the scenario of exploiting the user cooperative diversity, we usually assume that the channel is quasi-static. Assuming the packet/frame length is $L$, equation (1) can be written in matrix form as:

$$\mathbf{y}_d = \mathbf{h}_{r,d}X_r + \mathbf{w}_d, \quad (2)$$

where $\mathbf{y}_d \in \mathbb{C}^{1\times L}$, $\mathbf{h}_{r,d} \in \mathbb{C}^{1\times M_s}$, $\mathbf{w}_d \in \mathbb{C}^{1\times L}$, and $X_r \in \mathbb{C}^{M_s\times L}$ is the space-time coded signal matrix of dimension $M_s \times L$:

$$X_r = \begin{bmatrix} x_{r_1}(1) & x_{r_1}(2) & \cdots & x_{r_1}(L) \\ x_{r_2}(1) & x_{r_2}(2) & \cdots & x_{r_2}(L) \\ \vdots & \vdots & \ddots & \vdots \\ x_{r_{M_s}}(1) & x_{r_{M_s}}(2) & \cdots & x_{r_{M_s}}(L) \end{bmatrix},$$

and different rows in $X_r$ are transmitted by different relay terminals, and $\mathbb{C}$ is the set of all the complex numbers, i.e., the complex plane. There are two major differences between the conventional space-time codes and the space-time codes in the cooperative communication. One is that the row number $M_s$ in $X_r$ is a random variable instead of a constant in the conventional space-time codes which equals to the number of transmit antennas in the co-located antenna array. The other is that each row in matrix $X_r$ may not be symbol aligned, and the relative timing errors between different relays may be random. For example, $X_r$ can be equal to $X_r =$

$$\begin{bmatrix} \star & x_{r_1}(1) & \cdots & x_{r_1}(L) & \star & \star \\ \star & \star & x_{r_2}(1) & \cdots & x_{r_2}(L) & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{r_{M_s}}(1) & \cdots & x_{r_{M_s}}(L) & \star & \star & \star \end{bmatrix}.$$

In the following, we call $X_r^a$ as an asynchronous version of $X_r$. This is due to the asynchronous nature of the cooperative communication. In the above *asynchronous cooperative communication*, although the symbol synchronization is not required, we assume that each relay terminal is packet/frame synchronized, i.e., the start and the end of each packet/frame in different enrolled relays are aligned, which can be implemented by using network signaling. When a relay terminal is waiting for a packet/frame synchronizing flag, the dumb signal $\star$ is transmitted. We also assume that the relative timing errors between different relays are integers of the symbol duration and a fractional timing error can be absorbed in the channel dispersion. We further assume that these relative time errors are known at the receiver but not at the transmitter. The maximum relative timing error is assumed to be $L_e$. So the actual transmitted space-time code matrix is of dimension $M_s \times L'$, where $L \le L' \le L + L_e$. In each row, totally $L' - L$ dumb symbols $\star$ are padded to the beginning and/or the ending of a packet/frame transmission. Similar to the conventional space-time code design, to achieve good performance, we need to have the full *diversity order* and a good *diversity product*. But the difference now is that the rows in the space-time code matrix are not symbol-aligned. This is not easy to handle. For example, the delay diversity codes that are designed to ensure full diversity order in the conventional space-time codes [14] [11] do not have the full diversity property in the asynchronous cooperative communication. Also, the existing space-time block codes, for example, orthogonal space-time codes and lattice based space-time block codes, do not have the full diversity order property when the transmission is not synchronized. The objective of this paper is to design space-time codes with full diversity order in the asynchronous cooperative communication, i.e., $X_r^a$ in (II) has full diversity order for any symbolwise timing errors within a maximal range $L_e$.

## III. Code Construction

Our space-time trellis code construction is based on the stack construction by Hammons and El Gamal [12], When the source information bits are detected in a relay $R_i$, $i = 1, 2, \cdots, M$, if they are correct during a packet/frame, they are passed through a tapped delay line with tapped coefficients $[g_{i,0}, g_{i,1}, \cdots, g_{i,\nu}]$, where $g_{i,d} \in \mathbb{F}_2 \triangleq \{0, 1\}$ for $d = 1, 2, \cdots, \nu$, and $\nu$ is the maximal delay. We denote $g_i(D) \triangleq g_{i,0} + g_{i,1}D + \cdots + g_{i,\nu}D^\nu$ and $G_M(D) = [g_1(D), g_2(D), \cdots, g_M(D)]$, where and in what follows $D$ denotes the delay symbol. The coefficient matrix of $G_M(D)$ is defined as:

$$G_M = \begin{bmatrix} g_{1,0} & g_{1,1} & \cdots & g_{1,\nu} \\ g_{2,0} & g_{2,1} & \cdots & g_{2,\nu} \\ \vdots & \vdots & \vdots & \vdots \\ g_{M,0} & g_{M,1} & \cdots & g_{M,\nu} \end{bmatrix}.$$

If the binary source information bits detected in the relays in one packet/frame is $\overline{u} \in \mathbb{F}_2^{1\times L_u}$, then the binary output of the tapped delay lines is the set $\mathcal{C} = \{C(\overline{u}) \in \mathbb{F}_2^{M\times(\nu+L_u)} \mid C(\overline{u}) = [c_1(\overline{u}), c_2(\overline{u}), \cdots, c_M(\overline{u})]^T, \overline{u} \in \mathbb{F}^{L_u}\}$, where $c_i(\overline{u}) =$

$$\overline{u} \times \begin{bmatrix} g_{i,0} & g_{i,1} & \cdots & g_{i,\nu} & 0 & \cdots & 0 \\ 0 & g_{i,0} & g_{i,1} & \cdots & g_{i,\nu} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & g_{i,0} & g_{i,1} & \cdots & g_{i,\nu} \end{bmatrix}_{L_u\times(L_u+\nu)}.$$

*Space-time code generated by* $G_M(D)$ is defined as the set $\mathcal{X} =$

$$\{X_r(\overline{u}) \in \mathbb{C}^{M\times(\nu+L_u)} \mid (X_r(\overline{u}))_{m,n} = (-1)^{(C(\overline{u}))_{m,n}}, C(\overline{u}) \in \mathcal{C}\}.$$

The above space-time codes have trellis structure. In this construction, if the maximum timing error range in one packet/frame is $L_e$ and BPSK modulation scheme is used, when the information bits in one packet/frame is $L_u$, the rate of the space-time code $\mathcal{X}$ generated from $G_M(D)$ is $L_u/(L_u+\nu+L_e)$ bits/sec/Hz. For long packet/frame, the rate approaches 1 bit/sec/Hz. The above construction in general is the same as the one obtained by Hammons and El Gamal [12]. In the following, we investigate conditions on the generating matrix $G_M$ for achieving the full diversity order in asynchronous cooperative communications.

Assuming that the relative timing error of relay $R_i$ is $k_i$, i.e., $X_r^a$ in $\mathcal{X}^a$, the asynchronous version of $\mathcal{X}$, $k_i$ dumb symbols $\star$ are padded to the left of the $i$th row of the matrix $X$ in $\mathcal{X}$. If dumb symbol $\star = 1 = (-1)^0$, then it is equivalent to that $k_i$ many 0's are padded to the left of the $i$th row of binary matrix $C$ in $\mathcal{C}$. These matrices can be generated by $G^a(D) = [g_1^a(D),\ g_2^a(D),\ \cdots, g_M^a(D)]$, where $g_i^a(D) = D^{k_i}g_i(D)$. Correspondingly, to ensure the full diversity order in the asynchronous cooperative communication, there are requirements for the tapped coefficients $g_{i,d}$ for $i = 1, 2, \cdots, M$ and $d = 1, 2, \cdots, \nu$, which are stated in Theorem 1 and the proof is based on the stack construction in [12].

*Theorem 1:* [15] The space-time code generated by $G_M(D) = [g_1(D), g_2(D), \cdots, g_M(D)]$ has full diversity order in the asynchronous cooperative communication if and only if the coefficient matrix $G_M^a$ of any asynchronous version $G_M^a(D) = [g_1^a(D), g_2^a(D), \cdots, g_M^a(D)] = [D^{k_1}g_1(D), D^{k_2}g_2(D), \cdots, D^{k_M}g_M(D)]$ of $G_M(D)$:

$$G_M^a = \begin{bmatrix} \overline{g}_1^a \\ \overline{g}_2^a \\ \vdots \\ \overline{g}_M^a \end{bmatrix} = \begin{bmatrix} g_{1,0}^a & g_{1,1}^a & \cdots & g_{1,\nu+L_e}^a \\ g_{2,0}^a & g_{2,1}^a & \cdots & g_{2,\nu+L_e}^a \\ \vdots & \vdots & \vdots & \vdots \\ g_{M,0}^a & g_{M,1}^a & \cdots & g_{M,\nu+L_e}^a \end{bmatrix}$$

has full rank, $M$, in the binary field $\mathbb{F}$ for arbitrary $k_1, k_2, \cdots, k_M$, where $L_e = \max_{1\le i\le M} k_i$.

*Proof:* See [15] ∎

Clearly, in Theorem 1, $G_M^a$ is a row-shifted version of $G_M$, and for each row $i$, $i = 1, 2, \cdots, M$, the shift amount $k_i$ is arbitrary. The importance of Theorem 1 is that, in order to construct space-time code $\mathcal{X}$ generated by $G_M(D)$ with full diversity order in asynchronous cooperative communication, we need to and only need to construct the generating matrix $G_M(D)$ such that any row-shifted version $G_M^a$ of its coefficient matrix $G_M$ has full rank. The difficulty of constructing such a matrix $G_M$ lies in the fact that even when $G_M$ satisfies the above requirement, i.e., its any row shifted version $G_M^a$ is full row rank, adding a column to $G_M$, the resulting $G_M'$ may not satisfy the requirement anymore, which is different from the conventional full row rank matrices. We next present some constructions of such $G_M(D)$.

The following theorem gives a sufficient condition for $G_M$ to ensure the full diversity order of the corresponding space-time code in the asynchronous cooperative communication for a general $M$, i.e., a general number of relays involved.

*Theorem 2:* [15] Let $w(\overline{v})$ denote the weight of binary vector $\overline{v}$, i.e., the number of 1's in $\overline{v}$. Assume that the binary matrix

$$G_M = \begin{bmatrix} \overline{g}_1 \\ \overline{g}_2 \\ \vdots \\ \overline{g}_M \end{bmatrix} = \begin{bmatrix} g_{1,0} & g_{1,1} & \cdots & g_{1,\nu} \\ g_{2,0} & g_{2,1} & \cdots & g_{2,\nu} \\ \vdots & \vdots & \vdots & \vdots \\ g_{M,0} & g_{M,1} & \cdots & g_{M,\nu} \end{bmatrix}.$$

are row permuted such that $w(\overline{g}_1) \le w(\overline{g}_2) \le \cdots \le w(\overline{g}_M)$, where $\overline{g}_i$ is the $i$-th row vector of matrix $G_M$. If

$$w(\overline{g}_k) > \sum_{i=1}^{k-1} w(\overline{g}_i) \tag{3}$$

for $k = 2, 3, \cdots, M$, and $G_M$ is used as the coefficient matrix of $G_M(D)$ to construct the space-time code $\mathcal{X}$, then $\mathcal{X}$ has the full diversity order in the asynchronous cooperative communication. In contrary, if

$$w(\overline{g}_k) \le \sum_{i=1}^{k-1} w(\overline{g}_i),\ \ k = 3, 4, \cdots, M, \tag{4}$$

with $M \ge 3$, then there exists $G_M$ such that the space-time code generated by $G_M(D)$ with coefficient matrix $G_M$ does not have full diversity order in the asynchronous cooperative communication.

*Proof:* See [15] ∎

From Theorem 2, we can give a construction $G_M$ easily. For example, we can construct $G_2$ as:

$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{5}$$

$G_3$ as:

$$G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{6}$$

For the construction from Theorem 2, the minimum requirement of the weights is:

$$w(\overline{g}_k) = \sum_{i=1}^{k-1} w(\overline{g}_i) + 1,$$

for $k = 2, \cdots, M$, from which we can easily calculate that the minimum required number of columns of $G_M$ is $2^{M-1}$, which corresponds to the memory size of the corresponding space-time trellis code. We next present some constructions of such matrices $G_M$ with smaller and/or smallest sizes obtained in [16], [17]. For notational convenience, in the following we use $n$ instead of $M$ to denote the number of rows.

## IV. Shift Full Rank Matrices

In order to systematically construct the above matrices $G_n$ with smaller numbers of columns, we first inroduce some notations and concepts. Although the results below are over the binary field, they also hold over a general communtative integral domain.

### A. Notations and Definitions

In what follows, we use small case bold font letters to denote vectors over the binary field $\mathbb{F}_2$ and small case letters to denote scalars. For convenience, a horizontal coordinate system is used to characterize the shifts of binary vectors in which the position of the component with that a dot is underneath denotes the origin and the right to the origin is the positive direction, such as $\mathbf{v} = 1\underset{\cdot}{1}001$. Furthermore, $(\mathbf{v})_k$ denotes the component of $\mathbf{v}$ on the coordinate $k$, $\mathbf{0}$ denotes the all-zero vector and $\mathbf{1}$ denotes the single-component vector $\underset{\cdot}{1}$. For example, for the preceding $\mathbf{v}$, $(\mathbf{v})_{-1} = 1$, $(\mathbf{v})_0 = 1$ and $(\mathbf{v})_1 = 0$, etc.

*Definition 1:* The length $l(\mathbf{v})$ of a binary row vector $\mathbf{v}$ is defined as the number of components between the most left and the most right 1's in $\mathbf{v}$, including the two 1's themselves. In particular, let $l(\mathbf{0}) = 0$ and the length of a vector with only one non-zero component is defined as 1. The weight $w(\mathbf{v})$ of $\mathbf{v}$ is defined as the number of 1's in $\mathbf{v}$ as usual.

Since padding any number of 0's to the two ends of a vector doesn't affect its properties in the following discussions, we do not differentiate them. For example, we treat vector $1\underset{\cdot}{1}001$ and vector $001\underset{\cdot}{1}0010$ the same.

*Definition 2:* For any vector $\mathbf{v}$, $\mathbf{v}^{R_j}$ denotes the row vector resulted from the $j$ bits (coordinate positions) right shift of every component of $\mathbf{v}$ and simultaneously padding 0's to its two ends if needed, where, when $j$ is negative, it means the $|j|$ bits left shift of $\mathbf{v}$.

*Definition 3:* For two vectors $\mathbf{u}$ and $\mathbf{v}$, their convolution is denoted by $\mathbf{u} \circ \mathbf{v}$. For $\mathbf{v}$, its $i$th power is defined as $\mathbf{v}^i = \underbrace{\mathbf{v} \circ \mathbf{v} \circ \cdots \circ \mathbf{v}}_{i}$ for $i > 0$ and we define $\mathbf{v}^0 = \mathbf{1}$.

As we know, convolution of two vectors is equivalent to product of the corresponding polynomials. Similar to the concepts of polynomials, we define divisions over binary vectors. We say $\mathbf{u}$ divides $\mathbf{v}$, denoted by $\mathbf{u}|\mathbf{v}$, if there exists an $\mathbf{x}$ such that $\mathbf{v} = \mathbf{u} \circ \mathbf{x}$. Otherwise, $\mathbf{u} \nmid \mathbf{v}$. Obviously, for any non-zero vector $\mathbf{v}$, $\mathbf{v} \circ \mathbf{1} = \mathbf{v}$ and $\mathbf{v} \nmid \mathbf{1}$ unless $\mathbf{v} = \mathbf{1}$. Also, for given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can directly check whether $\mathbf{u}|\mathbf{v}$ or $\mathbf{u} \nmid \mathbf{v}$ by using their corresponding polynomials.

*Definition 4:* A *shift linear combination* of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_n$ is defined as

$$a_1 \cdot \mathbf{v}_1^{R_{j_1}} + a_2 \cdot \mathbf{v}_2^{R_{j_2}} + \cdots + a_n \cdot \mathbf{v}_n^{R_{j_n}},$$

where $a_i \in \mathbb{F}_2$ and $j_i \in \mathbb{Z}$, $i = 1, 2, \cdots, n$. Furthermore, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are called *shift linearly independent* if

$$a_1 \cdot \mathbf{v}_1^{R_{j_1}} + a_2 \cdot \mathbf{v}_2^{R_{j_2}} + \cdots + a_n \cdot \mathbf{v}_n^{R_{j_n}} = \mathbf{0}$$

if and only if $a_1 = a_2 = \cdots = a_n = 0$; otherwise, they are called *shift linearly dependent*.

*Definition 5:* A matrix formed as $G = [\mathbf{r}_1^T, \mathbf{r}_2^T, \cdots, \mathbf{r}_n^T]^T$ is called shift full rank (SFR) if all of its row vectors are shift linearly independent.

Similar to vectors, since padding any number of all-zero columns to the two ends of a matrix doesn't affect its SFR property, we treat them the same and will always focus on those matrices without all-zero columns at the two ends. Furthermore, because the shift linear independence/dependence of $n$ vectors implies the same property for their shifted versions and the row permutation in a matrix does not affect its SFR property, matrices

$$G_3' = \begin{bmatrix} \underset{\cdot}{1} & 0 & 0 \\ \underset{\cdot}{1} & 1 & 0 \\ \underset{\cdot}{1} & 0 & 1 \end{bmatrix} \text{ and } G_3'' = \begin{bmatrix} \underset{\cdot}{0} & 0 & 1 & 0 & 1 \\ \underset{\cdot}{1} & 0 & 0 & 0 & 0 \\ \underset{\cdot}{0} & 1 & 1 & 0 & 0 \end{bmatrix}$$

are both SFR or neither is. With this observation, we give the definition of *shift equivalence*.

*Definition 6:* Two matrices $A = [\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T]^T$ and $B = [\mathbf{b}_1^T, \mathbf{b}_2^T, \cdots, \mathbf{b}_n^T]^T$ are called shift equivalent if there exist integers $j_1, j_2, \cdots, j_n$ such that $\mathbf{b}_{\sigma(i)} = \mathbf{a}_i^{R_{j_i}}$ for $i = 1, 2, \cdots, n$, where $\sigma$ is a permutation on the set $\{1, 2, \cdots, n\}$. We denote this relationship by $A \backsim B$.

Let $\mathcal{T}$ denote the set of all the vectors with their most left 1's located at the origin, i.e., $\mathcal{T} = \{\cdots 0\underset{\cdot}{1}abc \cdots \mid a, b, c, \cdots \in \mathbb{F}_2\}$. Obviously, $\mathbf{0} \notin \mathcal{T}$ and $(\mathcal{T}, \circ)$ is a commutative semi-group with the identity $\mathbf{1}$.

*Definition 7:* We call a matrix $G$ as of *standard form* if all of its row vectors belong to $\mathcal{T}$.

It is easy to notice that any binary matrix without any all-zero row is shift equivalent to a matrix of standard form. Thus, without loss of generality, we only need to study matrices of standard form for the shift full rankness. Given a vector $\mathbf{v}$ and a matrix $G = [\mathbf{r}_1^T, \mathbf{r}_2^T, \cdots, \mathbf{r}_n^T]^T$, we define $\mathbf{v} \circ G = [(\mathbf{v} \circ \mathbf{r}_1)^T, (\mathbf{v} \circ \mathbf{r}_2)^T, \cdots, (\mathbf{v} \circ \mathbf{r}_n)^T]^T$.

### B. Shift Full Rank Matrices: General Construction

Given any initial SFR matrix $G_0$ with $n_0$ rows, we construct a matrix with $n_0 + n - 1$ rows by

$$G_{n_0+n-1}^{G_0, \mathbf{v}_i, \bar{\mathbf{v}}_i} = \begin{bmatrix} \bar{\mathbf{v}}_{n-1} & \\ \mathbf{v}_{n-1} \circ \bar{\mathbf{v}}_{n-2} & \\ \vdots & \ddots \\ \mathbf{v}_{n-1} \circ \mathbf{v}_{n-2} \circ \cdots \circ \bar{\mathbf{v}}_2 & \\ \mathbf{v}_{n-1} \circ \mathbf{v}_{n-2} \circ \cdots \circ \mathbf{v}_2 \circ \bar{\mathbf{v}}_1 & \\ \mathbf{v}_{n-1} \circ \mathbf{v}_{n-2} \circ \cdots \circ \mathbf{v}_2 \circ \mathbf{v}_1 \circ G_0 & \end{bmatrix} \quad (7)$$

for $\forall n \geq 1$, where $\mathbf{v}_i$ and $\bar{\mathbf{v}}_i$ can be any non-zero vectors such that $\mathbf{v}_i \nmid \bar{\mathbf{v}}_i$, $i = 1, 2, \cdots, n-1$. Thus, $\mathbf{v}_i \neq \mathbf{1}$. In particular, since any non-zero vector is an SFR matrix by itself, when $G_0 = \mathbf{v}_0 \neq \mathbf{0}$, we have $n_0 = 1$, the matrix in (7) is then denoted by $G_n^{\mathbf{v}_0, \mathbf{v}_i, \bar{\mathbf{v}}_i}$. The following lemma tells us that for the construction (7), we only need to consider the case when $\mathbf{v}_i, \bar{\mathbf{v}}_i \in \mathcal{T}$ and $G_0$ is of standard form.

*Theorem 3:* [16] Any matrix $G_{n_0+n-1}^{G_0, \mathbf{v}_i, \bar{\mathbf{v}}_i}$ constructed in (7) with an initial SFR matrix $G_0$ of standard form and $\mathbf{v}_i, \bar{\mathbf{v}}_i \in \mathcal{T}$ such that $\mathbf{v}_i \nmid \bar{\mathbf{v}}_i$, $i = 1, 2, \cdots, n-1$, is an SFR matrix of standard form.

The basic idea of this construction is that from one known SFR matrix $G_0$, we can generate a new SFR matrix with one more row by first "convoluting" $G_0$ by a non-zero vector $\mathbf{v}$ and then adding a non-zero row vector $\bar{\mathbf{v}}$ such that $\mathbf{v} \nmid \bar{\mathbf{v}}$ into the matrix, i.e., the basic building block has the form[1]

$$\begin{bmatrix} \bar{\mathbf{v}} \\ \mathbf{v} \circ G_0 \end{bmatrix}.$$

[1]For its generalization to a commutative integral domain $\mathcal{D}$, the condition $\mathbf{v} \nmid \bar{\mathbf{v}}$ is changed to $\mathbf{v} \nmid a \cdot \bar{\mathbf{v}}$ for any $0 \neq a \in \mathcal{D}$, when $\mathcal{D}$ is not a field. If $\mathcal{D}$ is a field, the condition $\mathbf{v} \nmid \bar{\mathbf{v}}$ does not need to be changed.

Let us see a special case of the construction (7). Let $G_0 = \bar{\mathbf{v}}_i = \mathbf{1}$ and $\mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_{n-1} = \mathbf{v} \neq \mathbf{1} \in \mathcal{T}$, then $\mathbf{v}_i \nmid \bar{\mathbf{v}}_i$ and we denote the resulting matrices by $G_n^{\mathbf{v}}$, i.e.,

$$G_n^{\mathbf{v}} = \begin{bmatrix} \mathbf{v}^0 \\ \mathbf{v}^1 \\ \vdots \\ \mathbf{v}^{n-1} \end{bmatrix}. \tag{8}$$

*Corollary 1:* [16] For any vector $\mathbf{v} \neq \mathbf{1} \in \mathcal{T}$ and any positive integer $n$, the matrix $G_n^{\mathbf{v}}$ in (8) is an SFR matrix of standard form and has size $n \times [(n-1)l(\mathbf{v}) - n + 2]$.

*Theorem 4:* [16] There exists at least a binary SFR matrix of standard form with size $n \times m$ if and only if $m \geq n$.

### C. Shortest Shift Full Rank Matrices (SSFR)

For the general construction (7) from an initial SFR matrix $G_0$ of standard form with size $n_0 \times m_0$ and $\mathbf{v}_i, \bar{\mathbf{v}}_i \in \mathcal{T}$ such that $\mathbf{v}_i \nmid \bar{\mathbf{v}}_i$ for $i = 1, 2, \cdots, n-1$, let $\mathbf{r}_i$ be the $i$th row of $G_{n_0+n-1}^{G_0,\mathbf{v}_i,\bar{\mathbf{v}}_i}$, then we have

$$\begin{aligned} &\max\{l(\mathbf{r}_1), l(\mathbf{r}_2), \cdots, l(\mathbf{r}_{n_0+n-1})\} \geq \\ &\qquad l(\mathbf{v}_{n-1} \circ \mathbf{v}_{n-2} \circ \cdots \circ \mathbf{v}_1) + m_0 - 1 \geq n + m_0 - 1 \end{aligned}$$

since $\mathbf{v}_i \neq \mathbf{1}$. On the other hand, the number of rows equals to the number of columns for $G_{n_0+n-1}^{G_0,\mathbf{v}_i,\bar{\mathbf{v}}_i}$ means that

$$\max\{l(\mathbf{r}_1), l(\mathbf{r}_2), \cdots, l(\mathbf{r}_{n_0+n-1})\} = n + n_0 - 1 \leq n + m_0 - 1.$$

Hence, $n_0 = m_0$ and $\mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_{n-1} = \dot{1}1 \triangleq \mathbf{e}$. Thus, for a matrix $G_{n_0+n-1}^{G_0,\mathbf{v}_i,\bar{\mathbf{v}}_i}$ in (7) to be SSFR, it must have the following form: $\mathbf{v}_i = \mathbf{e}$ and hence

$$G_{n_0+n-1}^{G_0,\mathbf{v}_i,\bar{\mathbf{v}}_i} = \begin{bmatrix} \mathbf{e}^0 \circ \bar{\mathbf{v}}_{n-1} \\ \mathbf{e}^1 \circ \bar{\mathbf{v}}_{n-2} \\ \vdots \\ \mathbf{e}^{n-2} \circ \bar{\mathbf{v}}_1 \\ \mathbf{e}^{n-1} \circ G_0 \end{bmatrix} \triangleq G_{n_0+n-1}^{G_0,\mathbf{e},\bar{\mathbf{v}}_i}, \tag{9}$$

where $G_0$ is an SSFR matrix of standard form and $\bar{\mathbf{v}}_i \in \mathcal{T}$ such that $\mathbf{e} \nmid \bar{\mathbf{v}}_i$.

*Theorem 5:* [16] The matrix $G_{n_0+n-1}^{G_0,\mathbf{v}_i,\bar{\mathbf{v}}_i}$ in (7) is an SSFR matrix of standard form if and only if it has the form of $G_{n_0+n-1}^{G_0,\mathbf{e},\bar{\mathbf{v}}_i}$ in (9), where $G_0$ is an SSFR matrix of standard form with size $n_0 \times n_0$, and $\bar{\mathbf{v}}_i \in \mathcal{T}$ with odd $w(\bar{\mathbf{v}}_i)$ and $l(\bar{\mathbf{v}}_i) \leq n_0 + i$ for $i = 1, 2, \cdots, n-1$.

As an example, Let initial SSFR matrix $G_0 = \mathbf{1}$, i.e., the single-entry matrix $\begin{bmatrix} 1 \end{bmatrix}$, in this example. If we let $\bar{\mathbf{v}}_i = \mathbf{1}$ in (9), then $G_n^{G_0,\mathbf{e},\bar{\mathbf{v}}_i} = G_n^{\mathbf{e}}$. When $n = 1, 2$, $G_1^{\mathbf{e}} = \mathbf{1}$ and $G_2^{\mathbf{e}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and when $n = 3, 4$,

$$G_3^{\mathbf{e}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } G_4^{\mathbf{e}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We can easily verify that the SFR constructed in Theorem 2 is just a submatrix of $G_{2^{n-1}}^{\mathbf{e}}$ since the $i$th row of $G_n$ equals to the $2^{i-1}$th row of $G_{2^{n-1}}^{\mathbf{e}}$ for $1 \leq i \leq n$. More results on all classifications and properties of SFR and SSFR can be found in [16], [17].

## V. Conclusion

In this paper, to consider the asynchronous nature of the cooperative diversity, we constructed a family of space-time trellis codes that have full diversity order without the symbol synchronization requirement. It is shown that to construct such a family of codes is equivalent to construct binary shift full rank (SFR) matrices. We then presented several systematic constructions of SFR matrices and shortest SFR (SSFR) matrices that correspond to the space-time codes with the smallest memory sizes. Although due to the space limitation, the code family in this paper is for BPSK signals, it can be easily extended [15] to general PSK and QAM signals by using the unified construction proposed by Lu and Kumar in [13].

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