Grassmannian Packings from Multidimensional Second Order Reed-Muller Codes

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I. Extended abstract

Let Q_1, Q_2 be two M-dimensional subspaces of $\mathbb{C}^T, T \geq M$ and P_1, P_2 be the corresponding orthogonal projector operators.

Definition 1: The chordal distance between Q_1 and Q_2 is defined as

$$d(Q_1, Q_2) = M - \mathrm{Tr}\,(P_1 P_2).$$

We will construct and analyze Grassmannian packings with respect to this distance metric. It is sometimes more convenient to work directly with orthogonal projector operators (orthogonal projectors).

Definition 2: An (T, M, L, d) Grassmannian packing is a collection of subspaces $Q_1, \ldots, Q_L$ of dimension M of the space $\mathbb{C}^T$ such that

$$d(Q_i, Q_j) \geq d, \text{ for } 1 \leq i < j \leq L.$$

Sometimes it is more convenient to use the following equivalent definition.

Definition 3: An (T, M, L, d) Grassmannian packing is a collection of orthogonal projectors $P_1, \ldots, P_L$ acting on $\mathbb{C}^T$ and such that

$$\begin{aligned} \mathrm{Tr}\,(P_i) &= M,\ 1 \leq i \leq L; \\ d(P_i, P_j) &= M - \mathrm{Tr}\,(P_i P_j) \geq d, \text{ for } 1 \leq i < j \leq L. \end{aligned}$$

The parameter

$$d = \min_{1 \leq i < j \leq L} M - \mathrm{Tr}\,(P_i P_j)$$

is the minimum distance of the packing.

We are motivated by two important applications. First, Grassmannian packings are highly structured examples of dictionaries, and the problem of finding sparse representations in non-orthogonal dictionaries has been considered by a number of authors, see for example [2] and references over there. The minimum distance of a packing determines the coherence of the dictionary, which is critical to the design of fast algorithms. The second application is to noncoherent wireless communication with multiple antennas.

Binary Reed-Muller codes were used [1] to construct a large family of Grassmannian packings, denoted by $ST\text{-}RM(r, m)$. There is an injective map between orthogonal projectors $ST\text{-}RM(r, m)$ and codewords of $RM(r, 2m)$, fast algorithms for decoding binary Reed-Muller codes can then be used to distinguish the different subspaces in the Grassmannian packing.

This paper enlarges the size of the Grassmannian packing $ST\text{-}RM(2, 2m)$ without decreasing the minimum distance. We denote the new packings by $G\text{-}RM(2, m)$. The mapping from orthogonal projectors to Reed-Muller codewords extends to the large packing and facilitates fast decoding.

Next we consider application of the packings $G\text{-}RM(2, m)$ to information transmission through the noncoherent MIMO channel. We show that at low and moderate SNR the constructed packings allow archiving the noncoherent MIMO channel capacity (under the constraint that only isotropically distributed unitary matrices are used for information transmission) . The case of two transmitting antennas $M = 2$ and the coherence interval $T = 4$ is presented on the following picture.

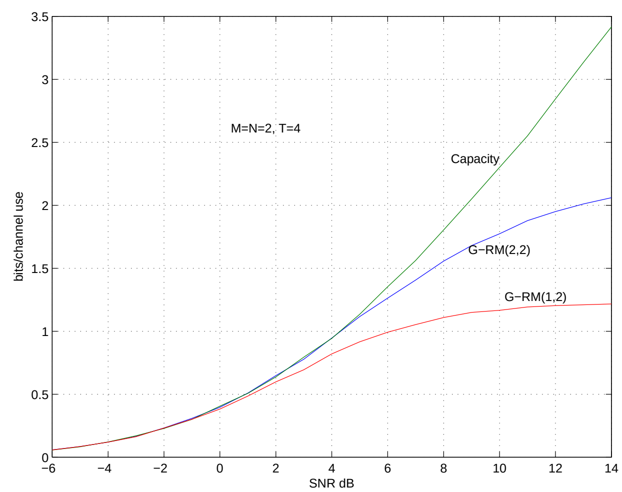

Fig. 1. Mutual Information of ML decoders of $ST\text{-}RM(1, 2)$ and $G\text{-}RM(2, 2)$

References

[1] A. Ashikhmin and A. R. Calderbank, “Space-Time Reed-Muller Codes for Noncoherent MIMO Transmission,” *International Symposium on Information Theory*, Adelaide, Australia, 2005, pp.1952–1956.
[2] J. A. Tropp, “Greed is Good: Algorithmic Results for Sparse Approximation,”*IEEE Trans. on Inform.Theory,* **50**, (2004), pp. 2231–2242.