# Generalized Nonlinear Impulse Response and Nonlinear Convolution in a Reproducing Kernel Hilbert Space $F$

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***Abstract*— Let the input-output relation for a general time-invariant dynamical system be expressed by $y[n] = f(x)$ where $f : H \to C$ is a functional on a separable Hilbert space $H$ mapping the input $x \in H$ to the complex value $y[n]$ of the output at time $n$. If the system is linear, its *impulse response* (with its elements in reverse order) $\tilde{h}$, also belonging to $H$, is the representer of $f$, viewed according to Riesz's Theorem, as a point evaluation functional in $H$; and *convolution* is the scalar product, in $H$, between such $\tilde{h}$ and $x$. We generalize and extend these two concepts to nonlinear dynamical systems by allowing $f$ to belong to a Reproducing Kernel Hilbert Space $F(H)$ of analytic functionals (and hence representable by abstract power series) on $H$. The space $F(H)$, denoted simply by $F$, is a Generalized (Weighted) Fock Space, a generalization of the conventional Symmetric Fock Space, the state space of non-self interacting Boson fields in quantum field theory. $F$ was introduced in the late 1970's by the author, in collaboration with T.A.W. Dwyer III, and L.V. Zyla, to represent the input-output maps of large-scale nonlinear dynamical systems. As we show in the present paper, the reproducing kernel of $F$ permits an elegant abstract representation of the impulse response and convolution for nonlinear dynamical systems. Furthermore, by using a best approximation of $f$ in $F$, given a finite number of nonlinear "training" observations (represented by linear functionals in $F$), then this approximation, interpreted as a best approximation of the abstract nonlinear impulse response of the system in $F$, is realized in the form of an optimal interpolating or smoothing neural network (OINN or OSNN)), optimally interpolating or smoothing the given observation outputs at their respective inputs. Thus the results presented can be used as powerful tools in the analysis and design of nonlinear (not necessarily linear) physical and information theoretic dynamical systems.***

## I. INTRODUCTION

In the past century, the concepts of *Impulse Response* (also known as Point Spread Function) and *Convolution* (also known as Superposition Integral or Summation) have played a major role in the analysis and design of linear dynamical systems. In the present paper, we rigorously generalize and extend these concepts to *nonlinear dynamical systems through a space* $F$. This space, at times denoted more specifically as $F(H)$, is a Reproducing Kernel Hilbert Space (RKHS) of analytic (nonlinear) functionals (representable by abstract power series) on a separable Hilbert Space . It was introduced in late 1970's by the author [1], in collaboration with T.A.W. Dwyer III, and L.V. Zyla, to represent the input-output maps of large-scale nonlinear dynamical systems. $F$ is a *Weighted Fock Space*, a generalization of the conventional Symmetric Fock Space, the state space of non-self interacting Boson fields in quantum field theory. Let $S$ denote a (causal) dynamical system with the input $x$ belonging to a separable Hilbert Space $H$ over the field of complex numbers $C$, with the scalar product between any two elements $u$ and $v$ of denoted by $< u, v >_H$. For simplicity in presentation, assume that $S$ is a single-input/single-output (SISO) discrete-time dynamical system with the input-output map, corresponding to its zero-state response, denoted by

$$y[n] = f(x) \tag{1}$$

where $f$ is a *functional* on $H$ mapping the input $x$ to the value $y[n]$ of the output at time $n$. Thus if $x$ is a finite string $(x(0), x(1), \cdots, x(n))$, $H$ will be an Euclidian space, and otherwise, $H = l_2$, the space of square-sumable strings on $[0.\infty]$. Let $h$ denote the impulse response of $S$, that is,

$$h[n] = f(1) \tag{2}$$

where $1 = \delta(n) \in H$ is the string of length $n + 1$ with unit pulse at the origin and 0 elsewhere. Finally, let $\tilde{h}$ denote the string obtained from $h$ with its elements in reverse order.

## II. LINEAR DYNAMICAL SYSTEMS

If $S$ is a *linear time-invariant dynamical system*, $\tilde{h}$ can be assumed to belong to the same space $H$ as the input, because the dual space of $H$ is $H$. In fact, it follows from (1) and (2) that $\tilde{h}$ can serve as the *representer* of $f$ in $F$. This allows us to represent (1) as the scalar product in :

$$y[n] =< \tilde{h}, x >_H \tag{3}$$

This equation represents the process of *convolution of the impulse response* $h$ *of the system with the input* $x$ *as a scalar product in* $H$. *For a nonlinear dynamical system, the representer of* $f$ *can no longer be in* $H$. A new Hilbert Space $F$ needs to be created where $f$ may reside. Furthermore, $x$ has to be raised from the space $H$ where it inhabits to the higher level space $F$ inhabited by $f$. All of this can be achieved by our formulation briefly described below. Mathematical details, especially on the construction of the reproducing kernel of $F$ can be found in the author's papers [1,2,5] and references therein contained.

## III. NONLINEAR DYNAMICAL SYSTEMS

In the *nonlinear* (not necessarily linear) case, with very little loss of generality, we assume that $f$ is an analytic functional on $H$. [Parenthetically, it may be stated that some of the most powerful general results on linear dynamical systems, established by R. E. A. C. Paley and N. Wiener are based on the analycity assumption]. Therefore it can be represented by an *abstract power series* (abstract Volterra functional series) in elements of $H$

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} f_m(x) \tag{4}$$

where $f_m$ are (abstract) homogeneous Hilbert-Schmidt (H-S) polynomials of degree $m$ in elements $x$ of $H$. For conditions under which (4) holds and detailed expressions for (4) corresponding to various cases, see [1,2,5]. In our definition of $F$, we introduce a vector

$$\lambda = (\lambda_0, \lambda_1, ...) \tag{5}$$

of non-negative weights $\lambda_m$ representing respectively the prior *uncertainty* in the terms $f_m$ of the power series in (4). The scalar product between any two elements $f$ and $g$ of $F$ is

$$< f, g >_F = \sum_{m=0}^{\infty} (\frac{1}{m!})(\frac{1}{\lambda_m}) < f_m, g_m >_m \tag{6}$$

where $< .,. >_m$ denotes the scalar product in the appropriate Hilbert space of $m^{th}$ degree homogeneous HS polynomials in elements of $H$. Note that the above scalar product depends on the uncertainty parameter vector $\lambda$. Also, the Reproducing Kernel for such a $F$ was derived by us in the form

$$K(u,v) = \sum_{m=0}^{\infty} (\frac{1}{m!})(\lambda_m)(< u, v >_H)^m = \phi(\lambda; < u, v >_H) \tag{7}$$

where $\phi(\lambda; s)$ is a scalar-valued analytic function of the vector parameter $\lambda$ and a scalar variable, say $s$, and is defined by

$$\phi(\lambda; s) = \sum_{m=0}^{\infty} (\frac{\lambda_m}{m!}) s^m \tag{8}$$

In the special case that $\lambda_n = \lambda_0^n$, $\varphi$ is an exponential function, and thus $K(x,z)$ becomes

$$K(x,z) = \exp(\lambda_0 \langle x, z \rangle) \tag{9}$$

Now, in the expression (1) for $f \in F$, there is, according Riesz's Theorem, a *representer* $\tilde{h}$ of $f$ in $F$. Then (1), for this general nonlinear case, can be written as

$$y[n] = < \tilde{h}, K(., x) >_F \tag{10}$$

We now see that *the above scalar product (10) in $F$ plays the same role in the analysis of nonlinear dynamical systems as does (2) for linear dynamical systems*. In particular, $\tilde{h}$ in (10) is the representer of $f$ in $F$ and may be viewed as an abstract representation of the impulse response of the system. Therefore **we call the $h$ in $F$ corresponding to such $\tilde{h}$ the Generalized Nonlinear Impulse Response of the nonlinear dynamical system $S$; and we call the process (10) Generalized Nonlinear Convolution of $h$ with the input $x$** achieved via the Reproducing Kernel $K$ of the space $F$. For this reason, we will refer to $f$ and $\tilde{h}$ interchangeably. There is the following best approximation result [1,2,5] for $f$ (and hence for $\tilde{h}$ ). Suppose we are given a $f$ in $F$ and a set of $N$ exemplary input-output observation (training) pairs

$$\{(x^i \in H, y^i[n] \in C) : i = 1, 2...., N\} \tag{11}$$

Then a best approximation $\hat{f}$ of $f$ subject to the constraints (11) is given by

$$\hat{y}[n] = \hat{f}(x) = \sum_{i=1}^{N} c_i K(x^i, x) = \sum_{i=1}^{N} c_i \phi(\lambda; < x^i, x >_H) \tag{12}$$

where the constants $c_i$ are obtained by requiring that (11) be satisfied. Expression (12) is realized by a two-hidden-layer neural network which we called an *Optimal Interpolative Neural Network (OINN)*. If the exemplary observation outputs $y^i$ are noisy, we get a corresponding expression for an *Optimal Smoothing Neural Network (OSNN)*.

## IV. CONCLUSION

In this paper, we have generalized the definitions and concepts of impulse response (point spread function) and convolution used in linear dynamical system theory to nonlinear dynamical systems, through the Reproducing Kernel Hilbert Space $F$ introduced by us in [1]. This naturally leads to nonlinear dynamical system models in the form of neural networks, and does, in fact provide a mathematical justification for why the human brain has the structure of a neural network to perform the nonlinear processing of the data that it receives through its natural (visual, auditory, olfactory, taste, and tactile) sensors.

## REFERENCES


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