Covering spheres with spheres

Ilya Dumer
University of California, Riverside
Email: dumer@ee.ucr.edu

***Abstract*— Given a sphere of radius $r>1$ in the n-dimensional Euclidean space, we study the coverings of this sphere with unit spheres. Our goal is to design a covering of the lowest *covering density*, which defines the average number of unit spheres covering a point in a bigger sphere. For growing n, we obtain the covering density of $(\mathbf{n}\ln\mathbf{n})/\mathbf{2}$. This new upper bound is half the order $\mathbf{n}\ln\mathbf{n}$ established in the classical Rogers bound.**

I. Introduction

Spherical coverings. Let $B_r^n(\mathbf{x})$ be the *ball* (solid sphere) centered at some point $\mathbf{x}=(x_1,\ldots,x_n)$ of an n-dimensional Euclidean space $\mathbf{R}^n$:

$$B_r^n(\mathbf{x})\overset{\text{def}}{=}\left\{\mathbf{z}\in\mathbf{R}^n\mid\sum_{i=1}^{n}(z_i-x_i)^2\le r^2\right\}.$$

We also use a simpler notation B_r^n if a ball is centered at the origin $\mathbf{x}=0$. For any subset $A\subset\mathbf{R}^n$, we say that a subset $\mathsf{Cov}(A,\varepsilon)\subseteq\mathbf{R}^n$ forms an ε*-covering* (an ε-net) of A if A is contained in the union of the balls of radius ε centered at points $\mathbf{x}\in\mathsf{Cov}(A,\varepsilon)$:

$$A\subseteq\bigcup_{\mathbf{x}\in\mathsf{Cov}(A,\varepsilon)}B_\varepsilon^n(\mathbf{x}).$$

By changing the scale in $\mathbf{R}^n$, we can always consider the rescaled set A/ε and the new covering $\mathsf{Cov}(A/\varepsilon,1)$ with unit balls $B_1^n(\mathbf{x})$. Without loss of generality, below we consider these (unit) coverings. One of the classical problems is to obtain tight bounds on the covering size $|\mathsf{Cov}(B_r^n,1)|$ for any ball B_r^n of radius r and dimension n.

Another related covering problem arises for a *sphere*

$$S_r^n\overset{\text{def}}{=}\left\{\mathbf{z}\in\mathbf{R}^{n+1}\mid\sum_{i=1}^{n+1}{z_i}^2=r^2\right\}.$$

Then the unit balls $B_1^{n+1}(\mathbf{x})$ cover this sphere with spherical caps

$$C_r^n(\rho,\mathbf{y})=S_r^n\cap B_1^{n+1}(\mathbf{x}).$$

In this notation, a cap $C_r^n(\rho,\mathbf{y})$ has some center $\mathbf{y}\in S_r^n$, half-chord $\rho\le1$, and the corresponding half-angle $\alpha=\arcsin\rho/r$. The maximum half-chord $\rho=1$ is obtained if the center $\mathbf{x}$ of the ball $B_1^{n+1}(\mathbf{x})$ has Euclidean weight

$$||\mathbf{x}||=\sqrt{r^2-1}. \tag{1}$$

To obtain a minimal covering, we shall consider spherical caps $C_r^n(1,\mathbf{y})$ assuming that all the balls $B_1^{n+1}(\mathbf{x})$ satisfy (1). Throughout the paper, all caps will be considered in S_r^n; therefore we use a shorter notation $C(\rho,\mathbf{y})$ instead of $C_r^n(\rho,\mathbf{y})$. Notation $C(\rho)$ will also be used when a specific center $\mathbf{y}$ is of no importance.

Spherical coverings often arise in multidimensional (vector) quantizers [5]. In particular, consider n-dimensional data points $\mathbf{z}\in\mathbf{R}^{n+1}$ that have the same energy r^2 and are uniformly distributed in the sphere S_r^n. A typical quantizer provides a covering $\{\mathbf{x}\}$ of the sphere S_r^n which limits the *mean* (squared) *rounding error* $(\mathbf{x}-\mathbf{z})^2$. Zador's theorem [5] shows that this error can be reduced per dimension by using quantization in higher dimensions n. In this case, given the *maximum rounding error* ε caused by a quantizer, we wish to find the thinnest ε-covering $\mathsf{Cov}(S_r^n)$ of the sphere S_r^n.

Preliminaries. Given a set A of volume $|A|$, the two important quantities are the minimal covering size $|\mathsf{Cov}(A,1)|$ and the covering density

$$\delta(A)=\sum_{\mathbf{x}\in\mathsf{Cov}(A,1)}\frac{|B_1^n(\mathbf{x})\cap A|}{|A|}.$$

Minimal coverings have been long studied for both the sphere S_r^n and the ball B_r^n. In particular, we mention the following results. The Coxeter-Few-Rogers "simplex" bound shows that for any sphere S_r^n of radius $r\ge\left(1+\frac{1}{n}\right)^{1/2}$,

$$\delta(S_r^n)\ge c_1n.$$

Here and below c_i denote some universal constants. Various upper bounds on the minimum covering density are obtained for B_r^n and S_r^n in the renowned papers [1] and [2]. In particular, a sufficiently large ball B_r^n can be covered with the density

$$\delta(B_r^n)\le\left(1+\frac{\ln\ln n}{\ln n}+\frac{5}{\ln n}\right)n\ln n. \tag{2}$$

For smaller radii r, recent advances are obtained in [3] and [4]. In particular, it is proven in [3] (see Corollary 1.2 and Remark 5.1) that for any radius r,

$$\delta(S_r^n)\le\left(1+c_2\frac{\ln\ln n}{\ln n}\right)n\ln n. \tag{3}$$

Our main result is as follows.

Theorem 1: For any $\beta>\frac{1}{2}$, a sphere S_r^n of any radius r and growing dimension $n\to\infty$ can be covered with spherical caps $C(1,\mathbf{y})$ of half-chord 1 with density

$$\delta(S_r^n)\le\beta n\ln n. \tag{4}$$

II. Basic approach

Notation. Below we consider a sphere S_r^n for growing dimension $n\rightarrow\infty$ and any given radius $r>1$. Consider any cap $C(\rho)$ of half-chord $\rho\leq 1$ and half-angle $\alpha=\arcsin\frac{\rho}{r}$. Let

$$\Omega_\rho=\frac{|C(\rho)|}{|S_r^n|}$$

denote the fraction of the surface of the sphere S_r^n covered by a cap $C(\rho)$. For any $\rho<1$, we extensively use two inequalities (see [3]):

$$\Omega_\rho<\frac{1}{\sqrt{2\pi n}}\frac{\sin^n\alpha}{\cos\alpha} \tag{5}$$

$$\frac{|\Omega_1|}{|\Omega_\rho|}\leq\rho^{-n}. \tag{6}$$

Given any $d\in(0,r)$, we say that the two caps, $C(\rho_1,\mathbf{y})$ and $C(\rho_2,\mathbf{z})$ are *d-close* if (vectors) $\mathbf{y}$ and $\mathbf{z}$ have angle

$$\measuredangle(\mathbf{y},\mathbf{z})\leq\arcsin d/r.$$

Finally, we use a shorter notation $\mathsf{Cov}(\rho)$ for any covering of the sphere S_r^n with spherical caps of half-chord ρ.

A simplified algorithm. Let

$$\varepsilon=\tfrac{1}{n\ln n},\quad \rho=1-\varepsilon. \tag{7}$$

To design a covering $\mathsf{Cov}(1)$, we shall also use another covering

$$\mathsf{Cov}(\varepsilon):\ S_r^n\subseteq\bigcup_{\mathbf{u}\in\mathsf{Cov}(\varepsilon)}C(\varepsilon,\mathbf{u}) \tag{8}$$

with smaller caps $C(\varepsilon,\mathbf{u})$. In general, $\mathsf{Cov}(\varepsilon)$ with density δ has size

$$|\mathsf{Cov}(\varepsilon)|=\frac{\delta}{\Omega_\varepsilon}. \tag{9}$$

Here we allow to employ any redundant covering, whose density δ can be much greater than $n\ln n$. Then we try to directly cover all the centers $\mathbf{u}\in\mathsf{Cov}(\varepsilon)$ by the randomly chosen caps $C(\rho,\mathbf{y})$. We use positive parameter γ and stop after performing

$$N=\frac{\gamma n\ln n}{\Omega_\rho} \tag{10}$$

trials (here we assume that N is an integer). We consider all uncovered centers $\mathbf{u}'\in\mathsf{Cov}(\varepsilon)$ and form the extended set

$$\bar{Y}=\{\mathbf{y}\}\cup\{\mathbf{u}'\}.$$

Obviously, this set forms a covering $\mathsf{Cov}(1)$ of the sphere S_r^n with the bigger caps $C(1,\mathbf{y})$.

Lemma 2: Let $n\geq 3$ and

$$\gamma\geq 1+\frac{\ln\ln n}{\ln n}+\frac{\ln\delta}{n\ln n}. \tag{11}$$

Then the sphere S_r^n has a covering $\mathsf{Cov}(1)$ of size

$$|\bar{Y}|\leq\frac{n\ln n}{\Omega_1}\cdot\gamma\left(1+\frac{4}{\ln n}+\frac{1}{n\ln n}\right) \tag{12}$$

and density

$$\delta\left(S_r^n\right)\leq\gamma n\ln n\left(1+\frac{4}{\ln n}+\frac{1}{n\ln n}\right). \tag{13}$$

Proof. We use inequalities

$$1-\varepsilon<e^{-\varepsilon}<1-\varepsilon/2,\quad e^{\varepsilon}<1+2\varepsilon, \tag{14}$$

valid for any $\varepsilon\in(0,1)$. Also, according to (6),

$$\Omega_1/\Omega_\varepsilon\leq\varepsilon^{-n}.$$

Any point $\mathbf{u}$ is covered by some cap $C(\rho,\mathbf{y})$ iff $\mathbf{y}$ and $\mathbf{u}$ are ρ-close. The latter occurs with probability $p=\Omega_\rho$. Note that

$$pN=\gamma n\ln n.$$

Now we use (10) to find the expected number $\mathbb{E}\ |\{\mathbf{u}'\}|$ of uncovered centers

$$\begin{aligned}\mathbb{E}\ |\{\mathbf{u}'\}|&=(1-p)^N\cdot|\mathsf{Cov}(\varepsilon)|\leq e^{-pN}\delta/\Omega_\varepsilon\\&\leq e^{-\gamma n\ln n}\cdot\varepsilon^{-n}\cdot\delta/\Omega_1\\&\leq e^{-(\gamma-1)n\ln n}e^{n\ln\ln n}\cdot\delta/\Omega_1\leq 1/\Omega_1,\end{aligned}$$

where we used a substitution from (11) in the last inequality. Then we estimate N. From (6) and (7), we deduce that

$$\Omega_1/\Omega_\rho\leq\left(1-\tfrac{1}{n\ln n}\right)^{-n}<1+\tfrac{4}{\ln n}. \tag{15}$$

Here the second inequality can be numerically verified for any $3\leq n\leq 7$, while for $n\geq 8$ it follows from (14). Therefore

$$N<\frac{n\ln n}{\Omega_1}\cdot\gamma\left(1+\frac{4}{\ln n}\right).$$

Then our covering $\bar{Y}$ of size $N+\mathbb{E}\,|\{\mathbf{u}'\}|$ gives estimates (12) and (13). □

Discussion. Assume that we use a covering $\mathsf{Cov}(\varepsilon)$ of polynomial density $\delta=n^c$. Then it is readily verified that for large n, bound (11) gives an estimate

$$\delta\left(S_r^n\right)\leq n\ln n\left(1+\frac{\ln\ln n}{\ln n}+\frac{5}{\ln n}\right) \tag{16}$$

which is similar to the estimate of [3] and the above bound (2). Note, however, that (16) is valid even if we use an exponential density $\delta=e^{cn}$ in (9). Indeed, in this case we obtain $\mathsf{Cov}(1)$ with density of order $\delta'=(c+1)n\ln n$ for any sphere S_R^n. Due to the scaling property of $\mathbf{R}^{n+1}$, this $\mathsf{Cov}(1)$ also gives the same density δ' for the rescaled covering $\mathsf{Cov}(\varepsilon)$ on the sphere $S_{R\varepsilon}^n$. Thus, we can take $r=R\varepsilon$ and begin designing $\mathsf{Cov}(1)$ using the new covering $\mathsf{Cov}(\varepsilon)$ of density δ'. In turn, this again gives estimate (16).

Note, however, that this approach is inefficient for any $\gamma<1$. First, using a better covering $\mathsf{Cov}(\varepsilon)$ (even with density less than $n\ln n$) still results in the same density order of $n\ln n$ in (13), after this method is applied. Nor can we increase parameter ε. In particular, let $\varepsilon'=n^{-\beta}$, where $\beta\in(0,1)$. Then the ratio Ω_1/Ω_ρ has an exponential order of $e^{(1-\beta)n}$ in (15), which in turn yields an exponential density in (13), even in spite of a reduced size $|\mathsf{Cov}(\varepsilon)|$.

Therefore, in the next section we modify our approach. We indeed use the larger radius $n^{-\beta}$ and design the covering $\mathsf{Cov}(n^{-\beta})$. However, it will only work due to the fact that a typical cap $C(n^{-\beta},\mathbf{z})$ intersects multiple caps $C(\rho,\mathbf{y})$. Also, even these multiple intersections will be allowed to leave some small holes in the caps $C(n^{-\beta},\mathbf{z})$. This approach is described below.

III. New covering algorithm for a sphere S_r^n.

Let $\beta\in(\frac{1}{2},1)$ and $\gamma\in(\beta,1)$. We consider the same parameters ε and ρ as defined in (7). Let

$$\mu=n^{-\beta},\quad d=1-2\varepsilon-\mu^2. \tag{17}$$

Below we obtain a covering $\mathsf{Cov}(1)$ of the sphere S_r^n with density $\beta n\ln n$ using three steps. In the first - preliminary - step, we assume that the same sphere S_r^n is covered by two different coverings $\mathsf{Cov}(\mu)$ and $\mathsf{Cov}(\varepsilon)$. In the second step, we try to cover the caps $C(\mu,\mathbf{z})\in\mathsf{Cov}(\mu)$ with the bigger caps $C(\rho,\mathbf{y})$. The trials occur so many times that typical caps $C(\mu,\mathbf{z})$ are left with the holes of radius ε or less. All other (atypical) caps $C(\mu,\mathbf{z})$ are then added to the former set $\{C(\rho,\mathbf{y})\}$. Finally, we expand the caps $C(\rho,\mathbf{y})$ to the full size $C(1,\mathbf{y})$ and cover the entire sphere. More precisely, we perform the following.

1. Consider two coverings of the sphere S_r^n :

$$\mathsf{Cov}(\mu):\ S_r^n\subseteq\bigcup_{\mathbf{z}\in\mathsf{Cov}(\mu)}C(\mu,\mathbf{z})$$

$$\mathsf{Cov}(\varepsilon):\ S_r^n\subseteq\bigcup_{\mathbf{u}\in\mathsf{Cov}(\varepsilon)}C(\varepsilon,\mathbf{u})$$

We assume that both coverings have density $\delta\leq n^2$. Then $\mathsf{Cov}(\varepsilon)$ has size (9) and

$$|\mathsf{Cov}(\mu)|\leq\frac{\delta}{\Omega_\mu}.$$

2. Randomly choose a subset $\{\mathbf{y}\}$ that includes

$$N=\frac{\gamma n\ln n}{\Omega_d} \tag{18}$$

points $\mathbf{y}\in S_r^n$. Consider N spherical caps $C(\rho,\mathbf{y})$. Then we examine the centers $\mathbf{u}$ of the covering $\mathsf{Cov}(\varepsilon)$ as follows.

Let $C(\mu,\check{\mathbf{z}})$ be any cap of the covering $\mathsf{Cov}(\mu)$ that contains at least one center $\mathbf{u}\in\mathsf{Cov}(\varepsilon)$ yet uncovered by the bigger caps $C(\rho,\mathbf{y})$. We consider the entire set $\{\check{\mathbf{z}}\}$ of such centers.

3. Finally, we extend the subset $\{\mathbf{y}\}$ and consider the set

$$\bar{Y}=\{\mathbf{y}\}\cup\{\check{\mathbf{z}}\}.$$

Note that any point on the sphere S_r^n falls within distance ε from the union of the caps $C(\rho,\mathbf{y})$ with centers $\mathbf{y}\in\bar{Y}$. Therefore, the extended caps $C(1,\mathbf{y})$ form a covering

$$\mathsf{Cov}(1):\ S_r^n\subseteq\bigcup_{\mathbf{y}\in\bar{Y}}C(1,\mathbf{y}). \tag{19}$$

To prove Theorem 1, we shall prove the following.

Claim 3: Given two parameters β and γ,

$$\frac{1}{2}<\beta<\gamma<1,$$

and a sufficiently large dimension n, subset $\bar{Y}$ has size

$$\left|\bar{Y}\right|\leq\Omega_d^{-1}\cdot\gamma n\ln n\,(1+o(1))\,. \tag{20}$$

We begin with the following important lemma.

Lemma 4: Consider two d-close caps, $C(\mu,\mathbf{Z})$ and $C(\rho,\mathbf{Y})$ on a sphere S_r^n,

$$\sin\measuredangle(\mathbf{Y},\mathbf{Z})\leq d/r, \tag{21}$$

where d is defined in (17). Then the bigger cap $C(\rho,\mathbf{Y})$ covers the smaller cap $C(\mu,\mathbf{Z})$, with an exception of its fraction

$$\theta=\frac{|C(\mu,\mathbf{z})\setminus C(\rho,\mathbf{Y})|}{|C(\mu,\mathbf{Z})|}\leq\exp\left\{-\frac{n^{2\beta-1}}{4\ln^2 n}\right\}.$$

Sketch of the proof. In Fig. 1, we represent the two caps $C(\mu,\mathbf{Z})$ and $C(\rho,\mathbf{Y})$, with centers $\mathbf{Z}$ and $\mathbf{Y}$. These two caps have bases $PQRS$ and $PMRT$, which form the balls $B_\mu^n(\mathbf{A})$ and $B_\rho^n(\mathbf{B})$. The bigger cap $C(\rho,\mathbf{Y})$ covers the base $B_\mu^n(\mathbf{A})$ of the smaller cap, with an exception of the part PQR. Note that the boundary of the base $B_\mu^n(\mathbf{A})$ is the sphere $S_\mu^{n-1}(\mathbf{A})$. In turn, the boundary of the uncovered base PQR forms a cap on $S_\mu^{n-1}(\mathbf{A})$ with center Q and half-angle $\alpha=\measuredangle PAQ$. We first estimate this half-angle α.

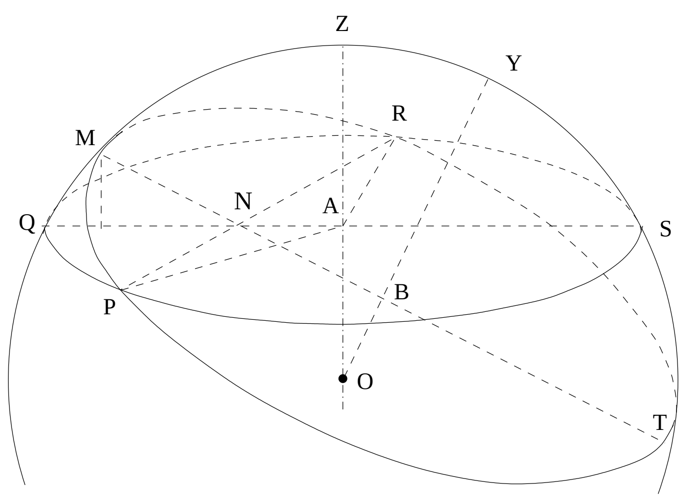

Fig. 1. Two intersecting caps $C(\mu,\mathbf{Z})$ and $C(\rho,\mathbf{Y})$ with bases $PQRS$ and $PMRT$.

Let $d(\mathbf{H},\mathbf{G})$ denote the distance between any two points $\mathbf{H}$ and $\mathbf{G}$. Also, let $\sigma(\mathbf{H})$ be the distance from a point $\mathbf{H}$ to the line $\mathbf{OBY}$ that connects the origin $\mathbf{O}$ of the sphere S_r^n to the base center $\mathbf{B}$ and then to the cap center $\mathbf{Y}$. Then

$$\sigma(\mathbf{A})\leq\sigma(\mathbf{Z})\leq d,$$

where we use (21) for the second inequality. On the other hand,

$$\begin{aligned}d(\mathbf{B},\mathbf{N})&=\sqrt{d^2(\mathbf{B},\mathbf{P})-d^2(\mathbf{N},\mathbf{P})}\\&\geq\sqrt{d^2(\mathbf{B},\mathbf{P})-\mu^2}=\sqrt{\rho^2-\mu^2}.\end{aligned}$$

Finally, note that by definition of the center $\mathbf{B}$, $d(\mathbf{B},\mathbf{N}) = \sigma(\mathbf{N})$ and

$$d(\mathbf{A},\mathbf{N}) \geq \sigma(\mathbf{N}) - \sigma(\mathbf{A}) \\ \geq \sqrt{\rho^2-\mu^2} - d \geq \varepsilon.$$

Therefore, we consider the right triangle $\mathbf{ANP}$ and deduce that

$$\cos\alpha = d(\mathbf{A},\mathbf{N})/d(\mathbf{A},\mathbf{P}) \geq \varepsilon/\mu = n^{\beta-1}/\ln n.$$

Now we can calculate the fraction Ω that the cap PQR occupies on the sphere $S_{\mu}^{n-1}(\mathbf{A})$. This fraction Ω is defined in (5) by its angle α, which gives

$$\Omega \leq (2\pi(n-1))^{-1/2}\frac{\sin^{n-1}\alpha}{\cos\alpha} \leq \frac{\ln n}{n^{\beta-1/2}}\left(1-\frac{n^{2\beta-2}}{\ln^2 n}\right)^{\frac{n-1}{2}} \\ \leq \exp\left\{-\frac{n^{2\beta-2}}{\ln^2 n}\cdot\frac{n-1}{2}\right\} \leq \exp\left\{-\frac{n^{2\beta-1}}{4\ln^2 n}\right\}.$$

Finally, it can be easily proven that the uncovered fraction θ of the entire cap $C(\mu,\mathbf{Z})$ is upper-bounded by the uncovered fraction Ω of its base. □

Remark. The choice of d in (21) is critical for Lemma 4. In fact, about half the cap $C(\mu,\mathbf{Z})$ becomes uncovered if the two caps used in Lemma 4 are $(d+\varepsilon)$-close. If this distance is further increased to ρ, then $C(\mu,\mathbf{Z})$ becomes almost uncovered.

Our next goal is to prove that most centers $\mathbf{z}$ have sufficiently many d-close caps $C(\rho,\mathbf{y})$ after N trials. Namely, let N caps $C(\rho,\mathbf{y})$ be randomly chosen. We separate all spherical caps $C(\mu,\mathbf{z})$ into two subsets as follows. We say that a cap $C(\mu,\mathbf{z})$ is bad and denote it $C'(\mu,\mathbf{z})$ if it has only

$$l = n^{3/2-\beta}$$

or less d-close caps $C(\rho,\mathbf{y})$. Let $\{C'(\mu,\mathbf{z})\}$ be the subset of all bad caps, whereas $\{C''(\mu,\mathbf{z})\}$ includes the remaining (good) caps. We first prove that the mean number N' of bad caps is negligible even to the number of random trials N.

Lemma 5: For $n\to\infty$, any cap $C(\mu,\mathbf{z})$ is bad with a probability

$$P \leq e^{-(\gamma+o(1))n\ln n}. \tag{22}$$

The expected number of bad caps is

$$\mathbb{E}(N') = o'(N).$$

where $o'(N)/N \to 0$.

Proof. Given any center $\mathbf{z}$, a randomly chosen cap $C(\rho,\mathbf{y})$ is d-close to $\mathbf{z}$ with the probability $p=\Omega_d$. Thus, for N of (18), the expected number of d-close caps $C(\rho,\mathbf{y})$ is

$$pN = \gamma n \ln n$$

Then for $n\to\infty$, the probability P that $l=o(n)$ or less caps are d-close is

$$P \leq \sum_{i=0}^{l}\tbinom{N}{i}p^i(1-p)^{N-i} \leq N^l p^l e^{-p(N-l)} \\ \leq (\gamma n\ln n)^l e^{pl}e^{-pN} \leq (en\ln n)^l e^{-pN} \\ = e^{l\ln(en\ln n)}e^{-\gamma n\ln n} = e^{-(\gamma+o(1))n\ln n}.$$

In our last equality, we again use the fact that $l=n^{3/2-\beta}$ is sublinear in n (in fact, the above function $o(1)$ has the order of $n^{1/2-\beta}$). Thus, given the covering size $|\mathsf{Cov}(\mu)| = \delta/\Omega_\mu$, we calculate the expected number of bad caps,

$$\mathbb{E}(N') = \frac{\delta P}{\Omega_\mu} \leq \frac{n^2}{\Omega_d}\left(\frac{d}{\mu}\right)^{n-1}P \\ \leq \frac{n^2}{\Omega_d}\cdot n^{\beta n}e^{-(\gamma+o(1))n\ln n} \leq nNe^{-(\gamma-\beta+o(1))n\ln n},$$

which is an exponentially small fraction $o'(N)$ of N for any $\gamma>\beta$. □

The second important fact is that most remaining caps $C''(\mu,\mathbf{z})$ are covered almost entirely. Namely, let N'' be the total number of centers $\mathbf{u}'' \in\mathsf{Cov}(\varepsilon)$ left uncovered in all the caps $C''(\mu,\mathbf{z})$.

Lemma 6: For $n\to\infty$, the number of uncovered centers $\mathbf{u}'' \in\mathsf{Cov}(\varepsilon)$ has expectation

$$\mathbb{E}\ (N'') = o''(N), \tag{23}$$

where $o''(N)/N \to 0$.

Proof. There exist at most $|\mathsf{Cov}(\varepsilon)|$ centers $\mathbf{u}$ within all caps $C''(\mu,\mathbf{z})$. Any cap $C''(\mu,\mathbf{z})$ is covered at least l times. Each time, a random fraction θ of its surface is left uncovered. Therefore any point $\mathbf{u}''\in C''(\mu,\mathbf{z})$ is left uncovered with probability θ^l, according to Lemma 4. Thus,

$$\mathbb{E}\ (N'') \leq |\mathsf{Cov}(\varepsilon)|\cdot\theta^l \\ \leq \frac{n^2}{\Omega_d}\cdot\left(\frac{d}{\varepsilon}\right)^n\cdot\exp\left\{-\frac{n^{2\beta-1}}{4\ln^2 n}\cdot n^{3/2-\beta}\right\} \tag{24} \\ \leq N\,(n\ln n)^{n+1}\exp\left\{-\frac{n^{\beta+1/2}}{4\ln^2 n}\right\} < N\exp\left\{-\frac{n^{\beta+1/2}}{8\ln^2 n}\right\}.$$

Here in the last inequality we use the fact that $n\ln n = o(n^{\beta+1/2})$ for any $\beta>1/2$ and sufficiently large n, which gives a vanishing fraction (4). □

Proof of Claim 3 and Theorem 1. According to Lemmas 5 and 6, there exists a set of N caps $C(\rho,\mathbf{y})$ that leaves only $N'=o'(N)$ bad caps $C'(\mu,\mathbf{z}')$ and $N''=o''(N)$ uncovered centers $\mathbf{u}'' \in\mathsf{Cov}(\varepsilon)$. Let $C''(\mu,\mathbf{z}'')$ be any cap within the set C'' with an uncovered center $\mathbf{u}''$. Now we take the set of additional centers

$$\{\check{\mathbf{z}}\} = \{\mathbf{z}'\}\cup\{\mathbf{z}''\},$$

and surround all of them with the caps $C(\rho,\check{\mathbf{z}})$. The total number of these new caps is $o(N)$, which gives Claim *3*.

The extended set of caps $C(\rho, \mathbf{y}) \cup C(\rho, \check{\mathbf{z}})$ with centers

$$\bar{Y} = \{\mathbf{y}\} \cup \{\check{\mathbf{z}}\}$$

has size $\bar{N} = N + o(N)$. By extending all these caps to radius 1, we obtain a 1-covering of density

$$\delta = \Omega_1 \bar{N} = \frac{\Omega_1}{\Omega_d} \gamma n \ln n (1 + o(1)).$$

Finally, we prove that $\Omega_d \sim \Omega_1$ for sufficiently large n. Similarly to the proof of Lemma 2, we see that for $n \to \infty$

$$\frac{\Omega_1}{\Omega_d} \leq d^{-n} = \left(1 - \tfrac{1}{n \ln n} - n^{-2\beta}\right)^{-n} < 1 + \tfrac{5}{\ln n}.$$

This gives the density

$$\delta \leq \gamma n \ln n (1 + o(1)).$$

Since both parameters β and γ can be chosen arbitrarily close to $1/2$, this proves (4) and completes the proof of Theorem 1. □

Remark. The latter two lemmas also explain our choice of parameter l. First, for any $\gamma > \beta$, it gives only an exponentially small fraction $o'(N)$ of bad caps in Lemma 5. On the other hand, it gives a vanishing fraction $o''(N)$ of uncovered centers in (24) for any $\beta > 1/2$.

The following Theorem 7 can be proven in a similar way, by combining the technique of Theorem 1 with multilayered design described in [2] and [4].

Theorem 7: For any $\beta > \frac{1}{2}$, a ball B_r^n of growing radius $r \to \infty$ and dimension $n \to \infty$ can be covered with unit balls $B_1^n(\mathbf{x})$ with density

$$\delta(B_r^n) \leq \beta n \ln n. \tag{25}$$

Corollary 8: Euclidean spaces $\mathbf{R}^n$ of growing dimension $n \to \infty$ can be covered with unit balls $B_1^n(\mathbf{x})$ with density

$$\delta(\mathbf{R}^n) \leq (\tfrac{1}{2} + o(1)) n \ln n. \tag{26}$$

REFERENCES

[1] C.A. Rogers, "A note on coverings,"*Mathematika,* v. 4, pp. 1–6, 1957.

[2] C.A. Rogers, "Covering a sphere with spheres,"*Mathematika,* v. 10, pp. 157–164, 1963.

[3] K. Böröczky, Jr. and. G. Wintsche, "Covering the sphere with equal spherical balls," In: *Discrete Comput. Geometry - The Goldman-Pollack Festschrift,* B. Aronov, S. Bazú, M. Sharir, J. Pach (eds.), Springer, 2003, 237-253.

[4] J.-L. Verger-Gaugry, "Covering a ball with smaller equal balls in $\mathbf{R}^n$, " *Discrete Comput. Geometry,* v. 33, pp. 143–155, 2005.

[5] J.H. Conway and N.J.A. Sloane, *"Sphere packings, lattices and groups"*, New York: Springer-Verlag, 1988.

[6] G.F. Toth, "New results in the theory of packing and covering" in *Convexity and its Applications*, P.M. Gruber and J.M. Wills, Eds., pp. 318–359, 1983.