Relay Networks with Delays

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***Abstract*— The paper considers the effect of link delays on the capacity of a class of discrete memoryless relay networks. Sufficient conditions for two relay networks with the same topology and probabilistic structure but different link delays to have the same capacity are established. A general cut-set upper bound on capacity, which takes link delays into consideration is established. Examples including the classical relay, relay-without-delay, and a two-relay network are discussed.**

I. Introduction

In recent work [1], [2], it was shown that the capacity of the classical relay channel can be increased if the relay transmission at any instant of time is allowed to depend on its current received symbol in addition to past received symbols. The resulting channel was referred to as the *relay-without-delay* (RWD). It was also shown that the capacity of the relay-without-delay can exceed the well-known cut-set bound [3], and a new cut-set bound involving an auxiliary random variable in place of the relay sender random variable was established.

The work in this paper is motivated by the observation that the classical relay channel can be obtained from the relay-without-delay and vice versa by appropriately adding transmission delays to the links. Specifically, the classical relay channel can be obtained from the relay-without-delay by adding a delay of 1 to the link from the sender to the relay (instead of adding relay coding delay of 1). Thus, from the results in [1], [2], we conclude that adding link delay can reduce capacity. Conversely, the relay-without-delay can be obtained from the classical delay channel by adding a delay of 1 to the link from the sender to the receiver (instead of subtracting off the coding delay). Thus adding link delay can also increase capacity.

In this paper, we investigate the effect of link delays on capacity for the general class of relay networks with no feedback, which we refer to as *relay network with delays*. We define a discrete memoryless (DM) relay network with delays to consist of: (i) a directed acyclic graph (DAG) $(\mathcal{N},\mathcal{E})$, $\mathcal{N}=\{1,2,\ldots,K\}$, where node 1 is the sender and node K is the receiver and the rest are relay nodes, (ii) a set of random variables associated with each node, where $X^1\in\mathcal{X}^1$ is associated with the sender node 1, $Y^K\in\mathcal{Y}^K$ is associated with the receiver node K, and $(X^i,Y^i)\in\mathcal{X}^i\times\mathcal{Y}^i$ is associated with relay sender-receiver pair i along with a family of conditional probability mass functions (pmfs) $\{p(y^i|x^{\mathcal{N}_i}),\ i=2,3,\ldots,K\}$, where $\mathcal{N}_i=\{j\in\mathcal{N}:\ (j,i)\in\mathcal{E}\}$, i.e., the set of nodes with edges incident on i, and (iii) a set of edge weights (delays), where the weight of an edge (i,j) is $d(i,j)\in\{0,1,2,\ldots\}$.

Let $d(i)$ and $D(i)$ be the minimum and the maximum path delay from node 1 to node i, $i\in\mathcal{N}$, respectively. The network is memoryless in the following sense. For any block length $n\geq 1$,

$$\begin{aligned}p\Big(\Big\{y^i_{d(i)+1:D(i)+n}:i\in\mathcal{N}-\{1\}\Big\}\Big|&\\ \Big\{x^j_{d(j)+1:D(j)+n}:j\in\mathcal{N}-\{K\}\Big\}\Big)&\\ =\prod_{i=2}^{K}\prod_{t=d(i)+1}^{n+D(i)}p\left(y^i_t\Big|\{x^j_{t-d(j,i)}:j\in\mathcal{N}_i\}\right),&\end{aligned}$$

where $x^j_{t-d(j,i)}$ is an arbitrary symbol in $\mathcal{X}^j$ when the subscript is not positive.

A $(2^{nR},n)$ code for the DM relay network with delays consists of: (i) a set of messages $\{1,2,\ldots,2^{nR}\}$, (ii) an encoding function that maps each message w into a codeword $x^1_{1:n}(w)$ of length n, (iii) relay encoding functions $x^i_t=f^i_t(y^i_{d(i)+1:t})$ for $i=2,\ldots,K-1$ and $t\geq d(i)+1$, and (iv) a decoding function that maps each received sequence $y^K_{d(K)+1:D(K)+n}$ into an estimate $\hat{w}(y^K_{d(K)+1:D(K)+n})$. A rate R is achievable if there exists a sequence of $(2^{nR},n)$ codes with $P_e^{(n)}=\mathrm{P}\{\hat{\mathrm{W}}\neq\mathrm{W}\}\to 0$, as $n\to\infty$. The network capacity C is defined as the supremum over the set of achievable rates.

To help understand these definitions, consider the following simple examples:

1) Relay-without-delay: Here $\mathcal{N}=\{1,2,3\}$ and $\mathcal{E}=\{(1,2),(2,3),(1,3)\}$ as in Figure 1. The delays are $d(1,2)=d(2,3)=d(1,3)=0$ and thus $d(2)=d(3)=0$. The conditional pmfs are of the form
$$p(y^2_t y^3_t|x^1_t x^2_t)=p(y^2_t|x^1_t)p(y^3_t|x^1_t x^2_t).$$
2) Classical relay channel. Here $\mathcal{N}=\{1,2,3\}$ and $\mathcal{E}=\{(1,2),(2,3),(1,3)\}$ as in Figure 1. The delays are $d(1,2)=1$, $d(2,3)=d(1,3)=0$ and thus $d(2)=1$ and $d(3)=0$. The conditional pmfs are of the form
$$p(y^2_{t+1}y^3_t|x^1_t,x^2_t)=p(y^2_{t+1}|x^1_t)p(y^3_t|x^1_t x^2_t).$$
3) Two-relay network (see Figure 2). Here $\mathcal{N}=\{1,2,3,4\}$ and $\mathcal{E}=\{(1,2),(1,3),(2,4),(3,4)\}$. The delays are $d(1,2)=2$, $d(3,4)=1$, $d(1,3)=d(2,4)=0$ and

thus $d(2)=2$, $d(4)=1$ and $d(3)=0$. The conditional pmfs are of the form

$$p(y^2_{t+2}y^3_t y^4_{t+1}|x^1_t x^2_{t+1}x^3_t) = p(y^2_{t+2}|x^1_t)p(y^3_t|x^1_t)p(y^4_{t+1}|x^2_{t+1}x^3_t).$$

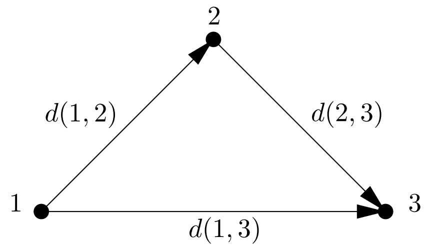

Fig. 1. A network with a single relay node.

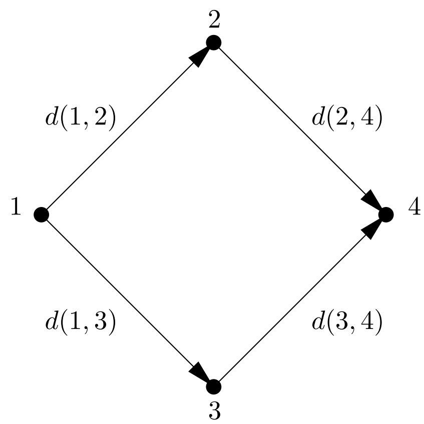

Fig. 2. A network with two relay nodes.

In this paper we present two preliminary results. The first result concerns the question of when delay does not change the capacity of a network. The second result provides a new cut-set upper bound on the capacity of relay networks with delays. This bound involves auxiliary random variables and coincides with known results in special cases such as the classical cut-set bound for the relay channel and the cut-set bound for the relay-without-delay.

Theorem 1: Consider two relay networks with the same directed acyclic graph $(\mathcal{N},\mathcal{E})$ and the same set of associated random variables and conditional probability mass functions but different link delays $\{d_1(i,j)\}$ and $\{d_2(i,j)\}$. Let $w_i(p)$, $i=1,2$, be the delay of path p from the sender to the receiver, $i=1,2$. If there exists an integer m such that $w_1(p)-w_2(p)=m$ for every path p, i.e., if all paths in both networks have the same relative delays, then the two networks have the same capacity.

It can be shown that the condition in the theorem can be checked in $O(K+|\mathcal{E}|)$ time, although the theorem requires checking *all* path delays from node 1 to node K.

We now show the implications of the above theorem for some simple networks.

1) Consider the relay channel in Figure 1 with $d(1,2)=d(1,3)=1, d(2,3)=0$. Here there are two paths from the sender to the receiver with delays 1 and 1. The relative delays are the same as that for the relay-without-delay, which has two paths with relative delays 0 and 0. As a result the above result implies that both these networks have the same capacity.
2) Consider two 2-relay networks $\mathcal{N}_1$ and $\mathcal{N}_2$ with the same DAG as shown in Figure 2 and the same associated random variables. Let the delays for $\mathcal{N}_1$ be $d(1,2)=0, d(1,3)=1, d(2,4)=0, d(3,4)=2$, and for $\mathcal{N}_2$ be $d(1,2)=1, d(1,3)=4, d(2,4)=1, d(3,4)=1$. Both networks have two paths from the sender to the receiver with path delays $(0,3)$ in $\mathcal{N}_1$ and $(2,5)$ in $\mathcal{N}_2$, respectively. Using the above result, it follows that both $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same capacity since the relative delays in both networks are the same.

We now proceed to establish an upper bound on the capacity of relay networks with delays. We begin with some needed notation. Let $\mathbb{S}$ be a set of subsets defined as $\mathbb{S}=\{S\subset\mathcal{N}: 1\in S,\ K\in S^c\}$. Given $S\in\mathbb{S}$, define $V_1(S)=\{i:\ (i,j)\in\mathcal{E},\ i\in S,\ j\in S^c\}$. That is, $V_1(S)$ is the set of nodes in S with outgoing edges to nodes in S^c. Similarly define $V_2(S)=\{j: (i,j)\in\mathcal{E},\ i\in S,\ j\in S^c\}$. That is, $V_2(S)$ is the set of nodes in S^c with incoming edges from nodes in S. For $i\neq 1$, let $A_{ij}=1$ if $(i,j)\in\mathcal{E}$ is on a shortest path from node 1 to node j, and $i\in V_2(S)$ and $j\in S^c$ for some $S\in\mathbb{S}$; otherwise let $A_{ij}=0$. Define the set $A=\{i:\ A_{ij}=1,\ \text{for some } j\in\mathcal{N}\}$. It turns out that the nodes in A have corresponding auxiliary random variables in the expression for the upper bound. Let $D^*=\max\{d(i,j)-d(j)+1:\ \ (i,j)\in\mathcal{E}\}$. This is used to ensure that all subscripts (time indices) are positive in the definitions that follow. We define the following sets of random variables for a given $S\in\mathbb{S}$:

$$\begin{aligned}
X(S) &= \{X^i_{d(j)-d(i,j)+D^*}:\ i\in V_1(S),\ j\in V_2(S),\ A_{ij}=0\},\\
U(S) &= \{U^i_{d(i)+D^*}:\ i\in V_1(S),\ j\in V_2(S),\ A_{ij}=1\},\\
X(S^c) &= \{X^i_{d(j)-d(i,j)+D^*}:\ i\in A^c\cap V_2(S),\ j\in S^c\},\\
U(S^c) &= \{U^i_{d(i)+D^*}:\ i\in A\cap V_2(S),\ \exists\ j\in S^c\ni A_{ij}=1\},\\
Y(S^c) &= \{Y^j_{d(j)+D^*}:\ j\in V_2(S)\}.
\end{aligned}$$

The following theorem provides a "single-letter" upper bound on the capacity of the relay network with delays.

Theorem 2: The capacity of a discrete memoryless relay network with delays is upper bounded by

$$C\leq\sup\min_{S\in\mathbb{S}}\{I(X(S),U(S);Y(S^c)|X(S^c),U(S^c)\},$$

where the supremum is over all joint distributions of the random variables constituting $X(S),U(S),X(S^c)$, and $U(S^c)$ for all $S\in\mathbb{S}$, and over all functions such that $x^i_{d(i)+D^*}=f(u^i_{d(i)+D^*},y^i_{d(i)+D^*})$ for all i such that $A_{ij}=1$ for some j.

We now apply the above theorem to obtain an upper bound on the capacity of some networks.

1) Classical relay channel: In this case $D^* = 1$, $\mathbb{S} = \{\{1\},\{1,2\}\}$, and $A_{12} = A_{13} = A_{23} = 0$, and $A = \emptyset$. For $S = \{1\}$, we obtain $V_1(S) = \{1\}$ and $V_2(S) = \{2,3\}$. Hence

$$X(S) = \{X_1^1\},\ X(S^c) = \{X_1^2\},\ U(S) = U(S^c) = \emptyset,$$
$$Y(S^c) = \{Y_2^2, Y_1^3\}.$$

Similarly for $S = \{1,2\}$, we obtain $V_1(S) = \{1,2\}, V_2(S) = \{3\}$ and hence

$$X(S) = \{X_1^1, X_1^2\},\ X(S^c) = \emptyset,\ U(S) = U(S^c) = \emptyset,$$
$$Y(S^c) = \{Y_1^3\}.$$

As a result the upper bound on capacity is given by

$$\sup_{p(x_1^1, x_1^2)} \min\{I(X_1^1 : Y_2^2, Y_1^3 | X_1^2), I(X_1^1, X_1^2 : Y_3^1)\}.$$

This is exactly the same as the upper bound for the classical relay channel in [3], although the notation is different. Note that $A_{ij} = 0$ for all (i,j) and hence there are no auxiliary random variables, i.e., $U(S) = U(S^c) = \emptyset$.

2) Relay-without-delay: In this case $\mathbb{S}$ and D^* are the same as for the classical relay channel discussed above and $A_{13} = A_{12} = 0, A_{23} = 1$, and $A = \{2\}$. For $S = \{1\}$, we obtain $V_1(S) = \{1\}$ and $V_2(S) = \{2,3\}$. Hence

$$X(S) = \{X_1^1\},\ U(S) = X(S^c) = \emptyset,\ U(S^c) = \{U_1^2\},$$
$$Y(S^c) = \{Y_1^2, Y_1^3\}.$$

Similarly for $S = \{1,2\}$, we obtain $V_1(S) = \{1,2\}$ and $V_2(S) = \{3\}$ and hence

$$X(S) = \{X_1^1\},\ U(S) = \{U_1^2\},\ X(S^c) = U(S^c) = \emptyset,$$
$$Y(S^c) = \{Y_1^3\}.$$

As a result, the upper bound on capacity is given by

$$\sup_{p(x_1^1, u_1^2), f} \min\{I(X_1^1 : Y_2^2, Y_1^3 | U_1^2), I(X_1^1, U_1^2 : Y_1^3)\},$$

where $x_1^2 = f(y_1^2, u_1^2)$. This is exactly the same as the upper bound for the RWD channel in [2], but with different notation.

3) A 2-relay network (see Figure 3): In this case $\mathbb{S} = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}$, $D^* = 2$, $A_{12} = A_{13} = A_{34} = 0, A_{23} = A_{24} = 1$, and $A = \{2\}$. The four elements of $\mathbb{S}$ correspond to the four cuts $\mathcal{C}_1, \ldots, \mathcal{C}_4$ shown in Figure 3. For $S = \{1\}$, we have $V_1(S) = \{1\}$ and $V_2(S) = \{2,3\}$. Hence

$$X(S) = \{X_1^1, X_2^1\},\ U(S) = \emptyset,\ X(S^c) = \{X_3^3\},$$
$$U(S^c) = \{U_2^2\},\ Y(S^c) = \{Y_2^2, Y_3^3\}.$$

For $S = \{1,2\}$, we have $V_1(S) = \{1,2\}$ and $V_2(S) = \{3,4\}$. Hence

$$X(S) = \{X_1^1\},\ U(S) = \{U_2^2\},\ X(S^c) = \{X_1^3\},$$
$$U(S^c) = \emptyset,\ Y(S^c) = \{Y_3^3, Y_2^4\}.$$

For $S = \{1,3\}$, we have $V_1(S) = \{1,3\}$ and $V_2(S) = \{2,4\}$. Hence

$$X(S) = \{X_2^1, X_1^3\},\ U(S) = \emptyset,\ X(S^c) = \emptyset,$$
$$U(S^c) = \{U_2^2\},\ Y(S^c) = \{Y_2^2, Y_2^4\}.$$

For $S = \{1,2,3\}$, we have $V_1(S) = \{2,3\}$ and $V_2(S) = \{4\}$. Hence

$$X(S) = \{X_1^3\},\ U(S) = \{U_2^2\},\ X(S^c) = \emptyset,$$
$$U(S^c) = \emptyset,\ Y(S^c) = \{Y_2^4\}.$$

As a result the upper bound on capacity is given by

$$\sup_{p(x_1^1, x_2^1, x_1^3, u_2^2), f} \min\{I_1, I_2, I_3, I_4\},$$

where $x_2^2 = f(y_2^2, u_2^2)$ and

$$I_1 = I(X_1^1, X_2^1; Y_2^2, Y_3^3 | U_2^2, X_3^3),$$
$$I_2 = I(X_1^1, U_2^2; Y_3^3, Y_2^4 | X_1^3),$$
$$I_3 = I(X_2^1, X_1^3; Y_2^2, Y_2^4 | U_2^2),$$
$$I_4 = I(X_1^3, U_2^2; Y_2^4).$$

In this example note that variables X_1^1, X_2^1 corresponding to the same sender (node 1) with different time indices appeared in the upper bound.

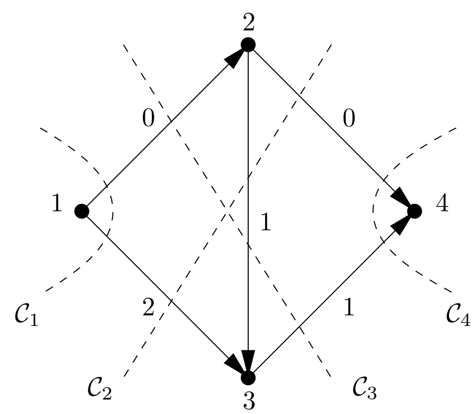

Fig. 3. A network with two relays.

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