What Does a 3 dB Buy in MIMO Channels?

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Abstract— **In this paper, an outage limited MIMO channel is considered. We build on Zheng and Tse's elegant formulation of the diversity-multiplexing tradeoff to develop a better understanding of the asymptotic relationship between probability of error, transmission rate and signal-to-noise ratio. We identify the limitation imposed by the notion of multiplexing gain and develop a new formulation for the throughput-reliability tradeoff that avoids this limitation. The new characterization is then used to shed more light on the asymptotic trends exhibited by the outage probability curves of MIMO channels.**

I. Problem Formulation

In an Additive White Gaussian Noise (AWGN) setting, it is well known that, a 3 dB increase in SNR translates into one extra bit in channel's capacity in the high SNR regime. The scenario considered in this paper, however, is more involved. We address an outage limited channel, where the randomness of the instantaneous mutual information results in a non-zero lower bound on the probability of error, for non-zero constant transmission rates. Hence, a fundamental tradeoff between the throughput, as quantified by the transmission rate, and reliability, as quantified by the so-called outage probability, arises. Our work explores this tradeoff in the high SNR regime.

In this paper, we consider a MIMO wireless communication system with m transmit and n receive antennas. We address a quasi-static flat-fading setup where the path gains remain constant over l consecutive symbol-intervals (*i.e.* a block), but change independently from one block to another. We further assume a coherent communication model implying the availability of channel state information (CSI) at the destination. Under these assumptions, the channel input-output relation is given by:

$$\mathbf{y} = \sqrt{\frac{\rho}{m}}\mathbf{H}\mathbf{x} + \mathbf{w}. \tag{1}$$

In (1), $\mathbf{y} \in \mathbb{C}^n$ has entries y_i representing the signal received at antenna $i \in \{1, \cdots, n\}$, $\mathbf{x} \in \mathbb{C}^m$ has entries x_j denoting the signal transmitted by antenna $j \in \{1, \cdots, m\}$, and $\mathbf{H} \in \mathbb{C}^{n\times m}$ has entries h_{ij} which represents the path gain from receive antenna $i \in \{1, \cdots, n\}$ to transmit antenna $j \in \{1, \cdots, m\}$. We model $\{h_{ij}\}$ as i.i.d unit-variance Rayleigh distributed random variables. $\mathbf{w} \in \mathbb{C}^n$ represents the unit-variance additive white Gaussian noise. Finally, ρ corresponds to the SNR at each receive antenna.

Our work builds on Zheng and Tse's formulation of the diversity-multiplexing tradeoff [1]. This formulation assumes a family of space-time codes $\{\mathcal{C}_\rho\}$ indexed by their operating SNR ρ, such that the code $\mathcal{C}_\rho$ has rate $R(\rho)$, in bits per channel use (bpcu), and error probability $P_e(\rho)$. For this family, the multiplexing gain r and the diversity gain d are defined by

$$r \triangleq \lim_{\rho\to\infty} \frac{R(\rho)}{\log_2 \rho} \quad \text{and} \quad d \triangleq -\lim_{\rho\to\infty} \frac{\log_2 P_e(\rho)}{\log_2 \rho}. \tag{2}$$

The optimal diversity-multiplexing tradeoff yields the maximum possible diversity gain for every value of r. The main result of [1] is summarized in the following theorem:

Theorem 1: The optimal diversity gain for the coherent block-fading MIMO channel with m transmit and n receive antennas, at multiplexing gain r, is given by $d(r) = f(r)$, where $f(\cdot)$ is the piecewise linear function joining the points $(k, (m-k)(n-k))$ for $k = 0, \ldots, \min\{m, n\}$. Moreover, there exists a code that achieves $d(r)$ for all block lengths $l \geq m+n-1$.

In the sequel, we will use the notation $d_{max} = mn$ and $r_{max} = \min\{m, n\}$. To understand the motivation behind our work, let us use the diversity-multiplexing tradeoff to make a first attempt in answering our central question on the utility of a 3 dB SNR gain in MIMO channels. Using **only** the extreme points of the tradeoff curve, i.e., $(0, d_{max})$ and $(r_{max}, 0)$, we start with a *motivating* conjecture to shed some light on the answer for our question

1) At high enough SNRs, one can fix the transmission rate and obtain d_{max} orders of decay in the outage probability (on a log-log scale), for every 10 dB gain in SNR.
2) At high enough SNRs, one can fix the outage probability and obtain a rate increase of r_{max} bpcu, for every 3 dB gain in SNR.

Fig. 1 and Fig. 2 examine the validity of this conjecture in a 2×2 MIMO channel[1]. In these figures, the transmission rates and SNR ranges are carefully chosen to better illustrate the following points.

1) The slope of the outage probability curves in Fig. 1 is shown to approach the asymptotic value of $d_{max} = 4$, on the log-log scale, as predicted by the first part of our conjecture. The *surprising* observation, however, is that for a

[1]Throughout the paper, we have used the following formula to generate the outage probability curves: $P_o(R, \rho) = \Pr\{\log_2 \det(I + \rho HH^H) < R\}$.

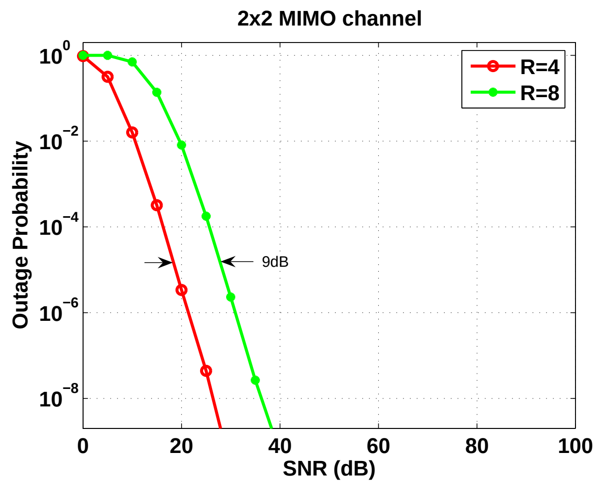

Fig. 1. Outage curves corresponding to $R = 4, 8$ bpcu, for a 2×2 MIMO channel.

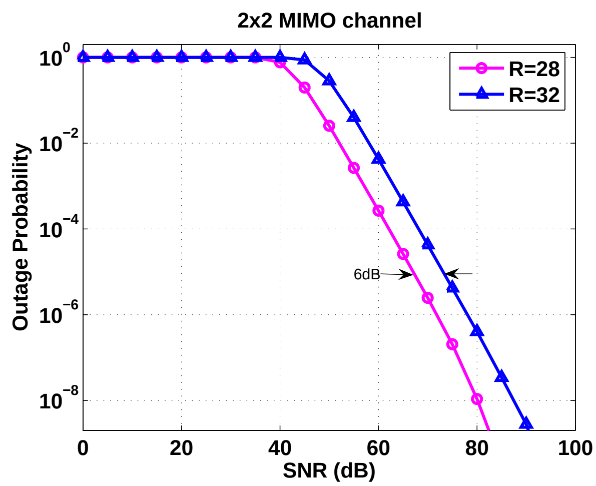

Fig. 2. Outage curves corresponding to $R = 28, 32$ bpcu, for a 2×2 MIMO channel.

constant outage probability, a 4.5 dB gain in SNR is needed to obtain an $r_{max} = 2$ bpcu increase in the throughput. This is in contrast with the second part of our conjecture, which predicts a need for only 3 dB. More interestingly, this 4.5 dB *horizontal spacing* seems to persists as the SNR increases.

2) Fig. 2, on the other hand, comes in close agreement with the second part of our conjecture. Here, the horizontal spacing, for a 2 bpcu increase in throughput, is seen to be 3 dB. The disagreement in this case, however, is exhibited in the fact that the slope of the outage probability curves, corresponding to fixed rates, seems to *stabilize* for a wide range of SNRs at a value of 2 (instead of 4).
3) Repeating the experiment for different values of m and n reveals the same trends, i.e., 1) Our motivating conjecture seems to offer *partially* accurate predictions in certain *operating regions*[2] and 2) Except for the 1×1 channel, the predictions for outage probability rate of decay and horizontal spacing are *not simultaneously* accurate.
4) Overall, these disagreements clearly disprove our *naive* conjecture. However, it seems that the conjecture is *not completely* false as it offers some accurate predictions, at least in certain operating regions.

Inspired by these observations, this paper aims at developing a better understanding of the fundamental throughput-reliability tradeoff in outage limited MIMO channels. It turns out that such an understanding requires a more general formulation which is not limited by the multiplexing gain as defined in (2). The multiplexing gain notion limits the scenarios of interests to *asymptotic lines* on the $R - \log_2 \rho$. Our formulation allows for investigating more general scenarios by relaxing this constraint. Specifically, we shed more light on the relationship between the three quantities $(R, \log_2 \rho, P_e(R, \rho))$, in the asymptotic limit of large ρ, when

$$\limsup_{\rho \to \infty} \frac{R}{\log_2 \rho} \neq \liminf_{\rho \to \infty} \frac{R}{\log_2 \rho}. \tag{3}$$

It is clear that (3) allows for investigating scenarios defined by arbitrary asymptotic trajectories in the $R - \log_2 \rho$ plane, where the multiplexing gain is not defined. It will be argued in the sequel that this *freedom* of walking along arbitrary trajectories on the $R - \log_2 \rho$ plane is the key to obtaining accurate predictions for the outage probability slopes and horizontal spacings in different operating regions. While our characterization is *rigorous* only in the asymptotic scenario where SNR grows to infinity (i.e., $\rho \to \infty$), we will show in the sequel that it yields very accurate predictions for practically relevant values of SNR.

The rest of the paper is organized as follows. In Section II, we state our main result formulating the throughput-reliability tradeoff (TRT) for the point-to-point coherent MIMO channel, along with numerical results and intuitive arguments that demonstrate the utility of our results in predicting the *behavior* of outage probability curves in the high SNR regime. We offer a few concluding remarks in Section III. Because of the space limitations the reader is referred to [2] for a detailed proof of Theorem 2.

II. The Throughput-Reliability Tradeoff

An outage is defined as the event that the realized mutual information does not support the intended rate, i.e.

$$O_{p(\mathbf{x})} \triangleq \{H \in \mathbb{C}^{n \times m} | I(\mathbf{x}; \mathbf{y} | \mathbf{H} = H) < R\}.$$

Notice that the mutual information depends on both, the realized channel H and the input distribution $p(\mathbf{x})$.

[2] A more formal definition of an operating region is presented in the sequel.

For this channel, the outage probability $P_o(R,\rho)$ is defined as

$$P_o(R,\rho) = \inf_{p(\mathbf{x})} \Pr\{O_{p(\mathbf{x})}\}.$$

The following theorem characterizes the relationship between R, ρ, and $P_o(R,\rho)$. Due to space limitations, though, we save establishing the operational significance of this theorem for [2], where we show that one can replace the outage probability $P_o(R,\rho)$ in Theorem 2 by the coding scheme's Maximum Likelihood (ML) error probability $P_e(R,\rho)$, provided that the code-length l satisfies $l \geq m+n-1$.

Theorem 2: For the $m\times n$ MIMO channel described by (1),

$$\lim_{\substack{\rho\to\infty \\ R\in\mathcal{R}(k)}} \frac{\log P_o(R,\rho) - c(k)R}{\log\rho} = -g(k), \quad (4)$$

where $P_o(R,\rho)$ denotes the outage probability at rate R and SNR ρ. $\mathcal{R}(k)$ is defined by

$$\mathcal{R}(k) \triangleq \{R | k+1 > \frac{R}{\log\rho} > k\},$$
$$\text{for} \quad k\in\mathbb{Z}, \min\{m,n\} > k \geq 0. \quad (5)$$

In (4), $c(k)$ and $g(k)$ are given by

$$c(k) \triangleq m+n-(2k+1), \quad (6)$$

and

$$g(k) \triangleq mn - k(k+1). \quad (7)$$

Moreover, in the degenerate case $R > \min\{m,n\}\log\rho$, $\lim_{\rho\to\infty}\log P_o(R,\rho)/\log\rho = 0$.

We refer to $g(k)$ as the *reliability gain coefficient* and $t(k) \triangleq g(k)/c(k)$ as the *throughput gain coefficient*. It is immediate to check that if the multiplexing gain is well defined, i.e., if

$$r = \lim_{\rho\to\infty} \frac{R(\rho)}{\log\rho}$$

exists, then Theorem 2 reduces to Theorem 1. In the more general case, however, Theorem 2 replaces the restrictive multiplexing gain notion with the new concept of operating regions $\mathcal{R}(k)$. It is worth noting that every operating region corresponds to a line segment in the diversity-multiplexing tradeoff. In fact, this correspondence inspires the following observation

$$g(k) = d(k) - kd'(k^+), \quad (8)$$
$$c(k) = -d'(k^+), \quad (9)$$

where $d'(k^+)$ is the slope of the line segment connecting $d(k)$ and $d(k+1)$.

We are now ready to investigate the asymptotic trends of the throughput-reliability tradeoff using Theorem 2. The following discussion hinges on the intuitive interpretation of equation (4).

$$\log P_o(R,\rho) \approx c(k)R - g(k)\log\rho, \quad (10)$$

where the approximation in (10) becomes progressively more accurate as ρ increases. Equation (10) implies that the slope of the outage curve, for a constant rate, is given by $g(k)$, while the horizontal spacing in dB between two outage curves with a ΔR rate difference is given by $3\Delta R/t(k)$. The key observation here is that in order to fix the transmission rate (or outage probability), while staying in the same operating region, one **must** deviate from the linear trajectory imposed by the multiplexing gain notion. The following *heuristic* derivation of (8) and (9) further illustrates this point. To derive (8), we start from two approximate relationships obtained from the diversity-multiplexing tradeoff

$$\log P_o\left(R, 2^{\log\rho}\right) \approx -d\left(\frac{R}{\log\rho}\right)\log\rho, \quad (11)$$

$$\log P_o\left(R, 2^{\log\rho+\Delta\log\rho}\right) \approx$$
$$-d\left(\frac{R}{\log\rho+\Delta\log\rho}\right)(\log\rho+\Delta\log\rho). \quad (12)$$

We further approximate $d\left(\frac{R}{\log\rho+\Delta\log\rho}\right)$ with the first two terms of its Taylor series expansion, i.e.,

$$d\left(\frac{R}{\log\rho+\Delta\log\rho}\right) \approx d\left(\frac{R}{\log\rho}\right) - \frac{R\times\Delta\log\rho}{\log\rho(\log\rho+\Delta\log\rho)}d'\left(\frac{R}{\log\rho}\right), \quad (13)$$

Now (13), together with (11) and (12), gives

$$\frac{\log P_o(R,2^{\log\rho}) - \log P_o(R,2^{\log\rho+\Delta\log\rho})}{\Delta\log\rho} \approx$$
$$d\left(\frac{R}{\log\rho}\right) - \frac{R}{\log\rho}d'\left(\frac{R}{\log\rho}\right). \quad (14)$$

Realizing that the left-hand side of (14) gives the slope of $P_o(R,\rho)$ with respect to ρ, i.e. $g(k)$, and that $d(r) - rd'(r)$ remains constant over the line segments of $d(r)$, we get (8). Deriving (9) follows the same lines. In particular, we first compute the horizontal spacing, $\Delta\log\rho$, between the outage curves corresponding to rates R and $R+\Delta R$. For this purpose, we use (11) to write

$$\log P_o\left(R+\Delta R, 2^{\log\rho+\Delta\log\rho}\right) \approx$$
$$-d\left(\frac{R+\Delta R}{\log\rho+\Delta\log\rho}\right)(\log\rho+\Delta\log\rho) \quad (15)$$

Then we expand $d(\frac{R+\Delta R}{\log\rho+\Delta\log\rho})$ in a way similar to (13) and equate (11) with (15) to get

$$\Delta\log\rho \approx \frac{-d'(\frac{R}{\log\rho})}{d(\frac{R}{\log\rho}) - \frac{R}{\log\rho}d'(\frac{R}{\log\rho})}\Delta R,$$
$$\Delta\log\rho \approx \frac{-d'(\frac{R}{\log\rho})}{g(k)}\Delta R. \quad (16)$$

Realizing that $\Delta\log\rho = \frac{c(k)}{g(k)}\Delta R$, and that $d'(r)$ remains constant over the line segments of $d(r)$, we get (9).

Revisiting our naive conjecture, we can now see that $g(0) = mn = d_{max}$ which agrees with the first part, while $t(\min\{n, m\} - 1) = \min\{m, n\} = r_{max}$ agrees with the second part. This explains the *partial* correctness of the conjecture and the fact that, except for the 1×1 MIMO channel, the two parts are **never** simultaneously accurate, since they correspond to different operating regions. We also observe that both the reliability and throughput gain coefficients exhibit a staircase behavior. Moreover, it is easy to see that $g(k)$ is a decreasing function of k, while $t(k)$ is an increasing function of k. This implies that, at a fixed rate R and for sufficiently large SNRs, as $R/\log\rho$ increases (i.e., ρ decreases), the decay rate of $P_o(R, \rho)$ decreases and the horizontal spacing between the outage curves corresponding to a fixed rate difference shrinks. In the following, we present numerical results that validate this observation.

Before proceeding to the numerical results, we need the following rule of thump for determining the operating regions, for large but finite values of ρ and R, such that the approximation in (10) is accurate. The operating point (R, ρ) is in operating region $\mathcal{R}(k)$ if and only if

$$\frac{\rho^k}{2^R} \leq \delta, \quad \text{and} \quad \frac{2^R}{\rho^{k+1}} \leq \delta,$$

where δ is a **small** value which determines the accuracy of the approximation. It is now straightforward to see that the high SNR segment of Fig. 1 falls in the region $\mathcal{R}(0)$ and, indeed, the 4 levels of diversity and 4.5 dB spacing (for every 2 bpcu throughput increase) in this figure correspond precisely to $g(0) = 4$ and $t(0) = 4/3$. Similarly, the high SNR segment of Fig. 2 falls in $\mathcal{R}(1)$ and, again, the 2 levels of diversity and 3 dB spacing, for every 2 bpcu throughput increase, agree with $g(1) = 2$ and $t(1) = 2$. Fig. 4 and Fig. 5 are different from the previous two figures in that the high SNR segments of the outage curves fall within **both** of the two regions. As a result, the slope of the curves and the spacing between them change as the operating point leaves one operating region and enters another. Again, the values of the slopes, spacings, and operating points at which the change occurs (which can be read from Fig. 3) match nicely with the predictions of the TRT as formulated by Theorem 2. Fig. 6 through Fig. 9 correspond to a 3×3 MIMO system. As can be seen from Fig. 6, the solid segment of the curve corresponding to $R = 10$ bpcu, falls in $\mathcal{R}(1)$ and, as predicted, we observe a slope of $g(1) = 7$. It should be noted, however, that the tail of the curve corresponding to $R = 4$ bpcu is leaving $\mathcal{R}(1)$ and entering $\mathcal{R}(0)$ and thus the slope of this curve is larger than 7 (about 7.7). For the same reason, the horizontal spacing between the two curves (almost 10 dB) is larger than the value predicted for $k = 1$ (i.e., 7.7 dB). The solid segments of the outage curves corresponding to $R = 58$ and 64 bpcu in Fig. 7 fall in $\mathcal{R}(2)$, and therefore, we observe 3 levels of diversity and 3 dB of spacing, for every 3 bpcu throughput increase, which correspond precisely to $g(2) = 3$ and $t(2) = 3$. Fig. 8 and Fig. 9 depict the case where the high SNR segments fall within two different regions ($k = 2$ and $k = 1$). Again the slopes and spacings are in agreement with the predictions of the TRT. Finally, Fig. 10 and Fig. 11 demonstrate how closely the *piecewise linear* approximation suggested by the TRT (recall (10)) follows the actual outage probability curves derived through Monte-Carlo simulations.

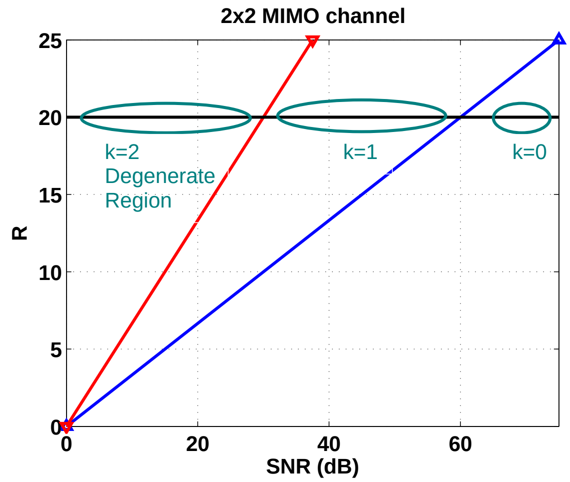

Fig. 3. The constant rate trajectory with $R = 20$ bpcu passes through different operating regions in a 2×2 MIMO system

III. Conclusions

We have developed a new asymptotic relationship between P_e, ρ, and R in outage limited MIMO channels. By relaxing the restriction imposed by the multiplexing gain notion, our characterization sheds more light on the throughput-reliability tradeoff in the high SNR regime. We presented numerical results which validate our claim that the throughput-reliability tradeoff offers accurate predictions on the *worth* of a 3 dB SNR gain in a MIMO wireless system operating in the high SNR regime.

IV. Acknowledgment

The authors would like to thank Profs. G. Caire and M. O. Damen for inspiring discussions.

References

[1] L. Zheng and D. N. C. Tse, "Diversity and Multiplexing: A Fundamental Tradeoff in Multiple Antenna Channels," *IEEE Trans. Info. Theory*, 49:1073 –1096, May 2003.

[2] K. Azarian and H. El Gamal, "The Throughput-Reliability Tradeoff in MIMO Channels," *Accepted for publication, IEEE trans. Info. Theory*.

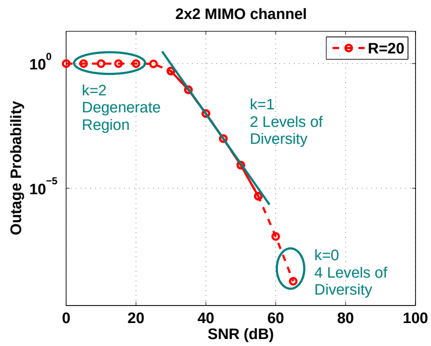

Fig. 4. Outage curves corresponding to $R = 20$ bpcu for a 2×2 MIMO channel.

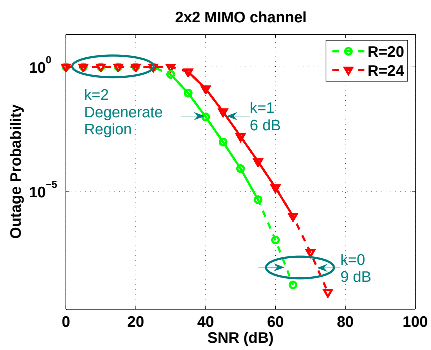

Fig. 5. Outage curves corresponding to $R = 20, 24$ bpcu for a 2×2 MIMO channel.

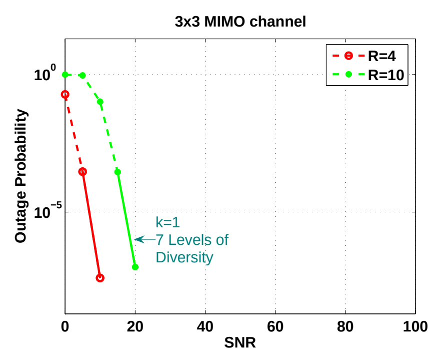

Fig. 6. Outage curves corresponding to $R = 4, 10$ bpcu, for a 3×3 MIMO channel.

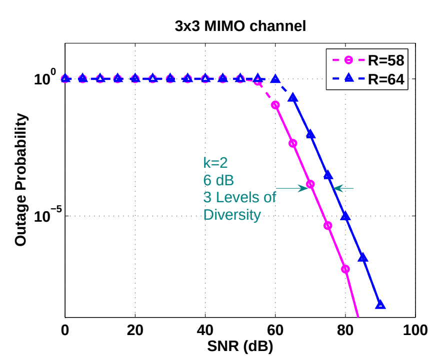

Fig. 7. Outage curves corresponding to $R = 58, 64$ bpcu, for a 3×3 MIMO channel.

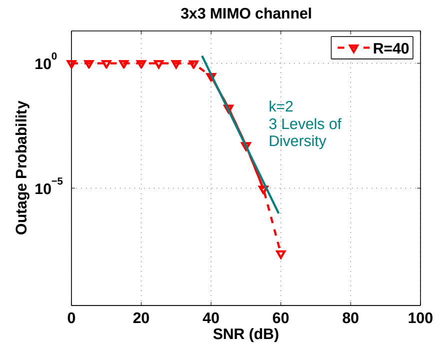

Fig. 8. Outage curves corresponding to $R = 40$ bpcu, for a 3×3 MIMO channel.

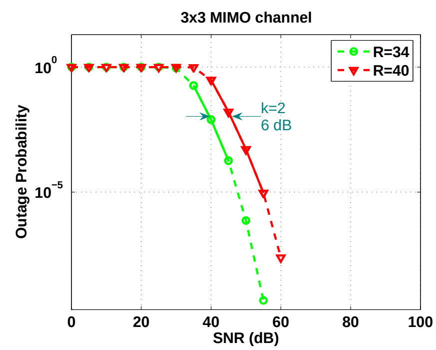

Fig. 9. Outage curves corresponding to $R = 34, 40$ bpcu, for a 3×3 MIMO channel.

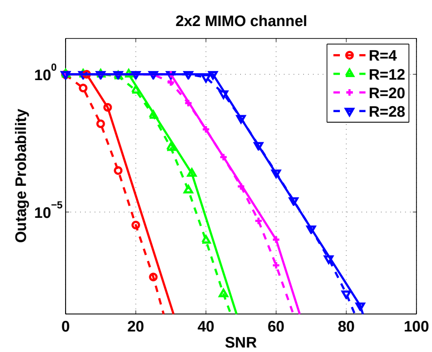

Fig. 10. Outage probability curves for 2×2 MIMO channel (dashed), along with the piecewise linear approximation (solid) suggested by TRT.

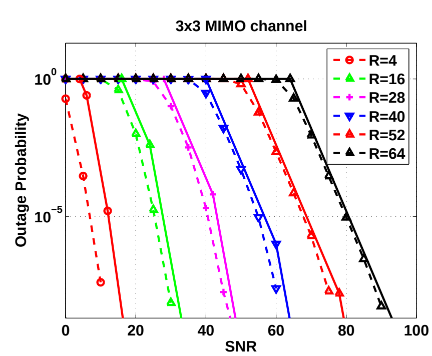

Fig. 11. Outage probability curves for 3×3 MIMO channel (dashed), along with the piecewise linear approximation (solid) suggested by TRT.