# Variations On The Multiple Access Problem

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***Abstract*—It is well known that the standard capacity region of a multiple-access channel does not provide fundamental bounds on communication performance when the underlying messages are dependent. The same is true for the scenario where the decoder only wants to recover a merged "summary" or function of all the messages together, even if the latter are independent. Various aspects of this insight are discussed. This may be of interest to sensor networks and network coding.**

## I. Preliminaries

The standard multiple-access problem (see e.g. [3, p.388]) considers multiple transmitters, each with an independent data stream. Via interfering transmissions, these streams need to be communicated to a base station. The base station desires to reliably recover each of the data streams. This gives rise to the standard notion of the "capacity region" of the multiple-access channel (MAC) (see e.g. [3, Thm.14.3.1]).

A key question is whether this capacity region has the same universal significance as the capacity of a point-to-point channel, in the sense that it represents the fundamental limit on communication performance. For example, consider the transmission of independent source streams with respect to (separate) distortion criteria, such as average bit error probability or mean-squared error. It is not hard to show (see e.g. [5, Thm.1.9]) that a "separation theorem" applies, in the following sense: Communication satisfying the imposed fidelity requirement is feasible if and only if the rate-distortion region for the source coding problem (subject to the same fidelity requirement) intersects the capacity region of the MAC, implying that separately designed source and channel codes are sufficient to attain optimum performance.

Hence, in this special problem, the capacity region *does* characterize the ultimate limit on communication performance. However, it is well known that for more general communication problems, this does not apply. In this abstract, we discuss two facets of this fact, primarily via two examples. The first example is well-known and concerns the situation where the underlying sources are *dependent*. In the second example, the sources are assumed to be independent, but we consider "dependent criteria:" Instead of separate recovery, the decoder is interested in a merged summary or *function* of the data streams (potentially subject to a fidelity requirement). An example is the bit-wise modulo-2 sum of all the messages, which may be of interest to sensor networks and network coding.

## II. Two Examples

### A. MAC with Dependent Sources

The simplest example illustrating the issue of multiple access and dependent sources was given in [2], as follows: Consider two *dependent* binary sources, $(S_1, S_2)$, where $(0, 1)$ never occurs and the other three possibilities are equally likely, and suppose that the base station is interested in the entire sequence of values that each source takes on. The MAC under study takes binary (0 or 1) inputs, and its output is their (real) sum (0, 1, or 2), without any noise. It is immediately clear that we can simply transmit the two sources without further coding, and always meet our goal. The more interesting question, however, is: can we also conclude this by looking at the capacity region of the channel?

The answer to this question is negative: The capacity region of this channel can be shown not to admit a sum rate that exceeds $1.5$ bits per channel use, but the joint source entropy is $\log_2 3 \approx 1.58$ bits per source sample, *more* than what the MAC can support.

One way in which this has been interpreted is that there is a "performance gain" for dependent sources on the multiple-access channels, where the gain is in comparison to a scheme in which the sources are first optimally compressed in a distributed fashion [9], and the resulting messages are communicated via a "standard" multiple-access code [3, p.393].

Further treatments along these lines may be found, e.g., in [1], [4], [10].

### B. MAC with Dependent Criteria

By "dependent criteria", we mean that the destination reconstructs a *joint function* of all the sources (possibly with respect to a fidelity criterion).

The simplest example illustrating this can be phrased as follows: Consider two *independent* binary (Bernoulli$(p)$) sources, $S_1$ and $S_2$, and suppose that the base station is interested in the modulo-2 sum of these sources. The MAC under study takes binary inputs, and it outputs their modulo-2 sum, without any noise. It is immediately clear that we can simply transmit the two sources without further coding, and always meet our goal. The more interesting question, however, is: can we also conclude this by looking at the capacity region of the channel?

The answer to this question is negative: The capacity region of this channel can be shown not to admit a sum rate that exceeds $1$ bit per channel use, but the rate required to compress the modulo-2 sum of two independent binary sources in a distributed fashion is $2H_b(p)$, where $H_b(\cdot)$ denotes the binary entropy function (see [7]). When $0.11 < p < 0.89$, this is *more* than what the MAC can support.

One way in which this can be interpreted is that there is a "performance gain" for dependent criteria, where the gain is in comparison to a scheme in which the sources are first optimally compressed in a distributed fashion with respect to the desired "dependent criterion" (in extension of [7]), and the resulting messages are communicated via a "standard" multiple-access code [3, p.393].

## III. Independent or Dual Aspects?

By comparing the MAC with dependent sources and the MAC with dependent criteria, it may be tempting to suspect that the two effects are due to the same underlying structural property. The resulting *duality* could be one under which a given dependence structure between the sources (subject to separate recovery of each source) would be equivalent to an appropriately chosen "dependent criterion" (for independent sources).

### A. Duality for a simple class

Such a duality can be established at least for a limited class of multiple access problems. Consider the multiple access structure sketched in Figure 1. Denote the source sequences by $\{S_{m,i}\}_{i>0}$, for $m = 1, \ldots, M$, and let $\mathbf{S}_i = (S_{1,i}, \ldots, S_{M,i})$ and $\mathbf{S}^n = (\mathbf{S}_1, \ldots, \mathbf{S}_n)$. We assume that $\mathbf{S}^n$ is a sequence of independent and identically distributed random vectors, according to a fixed distribution $p(\mathbf{s})$. Suppose that the source sequences need to be reconstructed with respect to a fidelity criterion of the form

$$E\left[d(\mathbf{S}^k, \hat{\mathbf{S}}^k)\right] \quad \leq \quad D. \tag{1}$$

We consider block encoders $f_1, f_2, \ldots, f_M$ that each map $k$ source symbols onto $n$ channel input symbols, and the block decoder $g$ that maps $n$ channel output symbols onto the reconstruction sequences. The MAC is assumed to be memoryless and characterized by a conditional distribution $p_{Y|\mathbf{X}}$. We define the input constraint over a block of $n$ channel inputs as

$$E\left[\rho(\mathbf{X}^n)\right] \quad \leq \quad P. \tag{2}$$

Hence, from this particular (and somewhat limited) perspective, a multiple access problem is characterized by the source distribution $p(\mathbf{s})$ with the corresponding distortion measure $d$ as well as the channel conditional distribution $p(y|\mathbf{x})$ with the corresponding cost function $\rho$.

The following theorem provides a *sufficient* condition under which a code $(f_1, \ldots, f_M, g)$ is optimal for a multiple access problem. Note, however, that the condition is by no means necessary.

*Theorem 1:* If the code $(f_1, \ldots, f_M, g)$ satisfies[1]

$$d(\mathbf{s}^k, \hat{\mathbf{s}}^k) \ = \ -c_0(\log_2 p(\hat{\mathbf{s}}^k|\mathbf{s}^k) - \log_2 p(\hat{\mathbf{s}}^k)) + d_0(\mathbf{s}^k) \tag{3}$$

$$\rho(\mathbf{x}^n) \ = \ c_1 D(p_{Y^n|\mathbf{X}^n}(\cdot|\mathbf{x}^n)||p_{Y^n}(\cdot)) + \rho_0, \tag{4}$$

[1]Here, $D(p(\cdot)||q(\cdot))$ denotes the Kullback-Leibler divergence between the distributions $p(\cdot)$ and $q(\cdot)$.

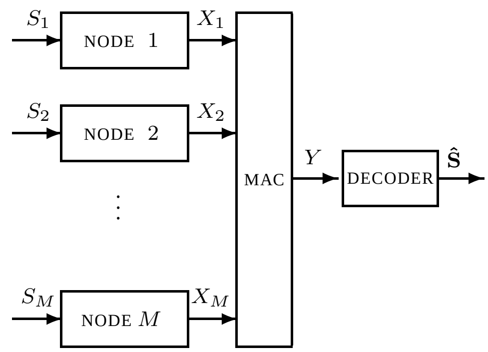


Fig. 1. The multiple access problem considered in this paper: There are $M$ potentially dependent source streams, to be encoded separately and transmitted over a MAC. The decoder produces a vector of estimates $\hat{\mathbf{S}}$ which may attempt to separately reconstruct each source *or* to reproduce functions of the source streams ("dependent criteria").

where $c_0 > 0, c_1 > 0$, $\rho_0$ is an arbitrary constant, and $d_0(\mathbf{s}^k)$ an arbitrary function of $\mathbf{s}^k$, then it is an optimal code for the multiple access problem.

This theorem follows directly from the results in [5]. It can also rather straightforwardly be extended to the case of multiple cost and/or distortion constraints.

To interpret the theorem, note that a *fixed* code will be optimal for a number of multiple access problems, according to the dependence structure of the source. For each such structure (characterized by $p(\mathbf{s})$), a different distortion measure will be imposed, according to Equation (3) in the above theorem. Generally, the resulting distortion measure will be a "dependent criterion," i.e., it will *not* be possible to split it into separate fidelity criteria for each source. In this sense, one may identify a duality between dependence and distortion measure.

### B. Separation theorems

A separate (and perhaps more interesting) question is that of *gains* over separately designed source and channel codes. Here, one may initially suspect that gains due to dependent sources are possible if and only if they are also possible due to dependent criteria. This, however, does not apply, as can be verified easily by reconsidering the example presented in Section II-B.

Specifically, reconsider the MAC that takes binary inputs and whose output is their modulo-2 sum, without any noise. For this channel, it is easy to show that a separation theorem applies whenever the sources need to be reconstructed separately: Consider two dependent binary sources with joint entropy $H(S_1, S_2)$. Then, from the data processing inequality, we infer that (denoting, as in Section III-A, the channel inputs by $X_1$ and $X_2$, and its output by $Y$)

$$H(S_1, S_2) \leq \max_{p(x_1,x_2)} I(X_1, X_2; Y) \leq 1, \tag{5}$$

where the last inequality follows since $Y$ is binary. Hence, we can see that $S_1$ and $S_2$ can only be recovered if their joint entropy is no larger than one. However, in this case, separately designed source and channel codes will work, too. Hence, in this example, dependent messages do not enable any gains. By contrast, as explained in Section II-B, there is a gain for dependent criteria.

This argument straightforwardly extends to larger alphabets, and similar insights apply for the Gaussian MAC (as defined in [3, p.378]) but under a constraint on the *received* power, see [6].

## IV. Computation over MACs

The most interesting perspective on the insights developed in this paper is that in the multiple access problem where the decoder is only interested in a "summary" of the original source data, joint source-channel coding techniques result in a gain over separately designed source and channel codes. However, the coding techniques that permit to harvest this gain are significantly different from the codes that exploit dependent messages, such as developed in [2]. Specifically, the gains enabled by dependent criteria hinge on a *structural similarity* between the multiple-access channel at hand and the desired criterion, which can be thought of as "computing a function" of the original source information. Therefore, these gains typically cannot be harvested via generic random codes, but require structured codes. An initial study appears in [8].


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