Achieving the D-MG and DMD Tradeoffs of MIMO Fading Channels

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***Abstract*— An overview of our recent results relating to the explicit construction of space-time block codes achieving the D-MG tradeoff of the quasi-static fading channel is presented. The results include the explicit construction of D-MG optimal codes, generalization of perfect codes to any number of transmit antennas as well as optimal diversity-multiplexing-delay constructions for the MIMO ARQ Channel.**

I. Introduction

This results presented here represent joint work with Petros Elia, Hsiao-feng (Francis) Lu, K. Raj Kumar, Sameer Pawar and Bharath Sethuraman.

Quasi-Static Channel Model: The received signal over a quasi-static ST fading channel model with n_t transmit and n_r receive antennas is given by

$$Y \;=\; \theta H X + W \tag{1}$$

where θX is the $(n_t \times T)$ transmitted code matrix drawn from a ST code $\mathcal{X}$, H the $(n_r \times n_t)$ channel matrix and W the $(n_r \times T)$ noise matrix. The entries of W are assumed to be i.i.d., circularly symmetric complex Gaussian $\mathbb{C}\mathcal{N}(0,1)$ random variables. In the case of iid Rayleigh fading, the entries of H share an identical description. The scaling parameter θ ensures that the space-time code satisfies

$$\mathbb{E}(\| \,\theta X\, \|_F^2) \leq T \text{ SNR}. \tag{2}$$

D-MG Tradeoff: For large SNR, the ergodic capacity of the ST channel model in (1) is given by $C \approx \min\{n_t, n_r\} \log(\text{SNR})$. The ST code $\mathcal{X}$ transmits $R = r\log(\text{SNR}) = \frac{1}{T}\log(|\ \mathcal{X}\ |)$ bits per channel use and consequently has size $|\mathcal{X}| = \text{SNR}^{rT}$. It follows that the maximum achievable multiplexing gain equals $r = \min\{n_t, n_r\}$. Following [5], we will refer to r as the *multiplexing gain*. The *diversity gain* corresponding to a normalized rate r is defined by

$$d(r) \;=\; -\lim_{\text{SNR}\to\infty} \frac{\log(P_e)}{\log(\text{SNR})},$$

where P_e denotes the probability of codeword error. In the landmark result of Zheng and Tse [5], it is shown that in the case of iid Rayleigh fading, for a fixed integer multiplexing gain r, and $T \geq n_t + n_r - 1$, the maximum achievable diversity gain $d(r)$ is shown to be

$$d(r) \;=\; (n_t - r)(n_r - r). \tag{3}$$

The plot (see Fig. 1) for non-integral values is obtained through straight-line interpolation. For $T < n_t + n_r - 1$ only upper and lower bounds on the maximum possible $d(r)$ are available [5]. We will refer to a code that achieves the D-MG tradeoff with equality as an optimal code.

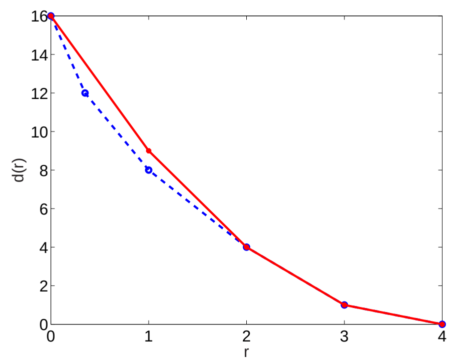

Fig. 1. *Upper and Lower Bounds on D-MG Tradeoff* $(n_t = n_r = 4)$.

A. Prior Work

a) Optimal 2 Antenna Codes: Yao and Wornell [2], [3] were the first to exhibit an optimal code and did so for the case $n_t = T = 2$. The diagonal and anti-diagonal threads of their (2×2) code matrix are unitary transformations of a pair of (2×1) vectors with QAM components. An appropriate choice of the unitary transformation caused the determinant of the difference code matrix to be bounded below irrespective of SNR and was shown to lead to D-MG optimality. Subsequent constructions by Dayal and Varanasi [4], Belfiore and Rekaya [14], Liao et. al. [13] and Oggier et. al. [16] also possess this property and hence are also D-MG optimal.

b) LAST Codes: In [7], El Gamal et. al. consider a lattice-based construction of space-time block codes termed LAST codes. Here, a code matrix X in the space-time code $\mathcal{C}$ is identified with a vector in $\mathbb{C}^{n_t T}$. The construction calls for a lattice Λ_c and a sublattice Λ_s. Message symbols are mapped onto coset representatives $\{\underline{c}\}$ of the subgroup Λ_s of Λ_c that

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lie within the fundamental region $\mathcal{V}_s$ of the sublattice Λ_s. Thus the fundamental region of the sublattice serves as a shaping region for the lattice. The transmitted vector $\underline{x}$ is then given by

$$\underline{x} \ = \ \underline{c} - \underline{u} \pmod{\Lambda_s},$$

where $\underline{u}$ is a pseudorandom "dither" vector known to the receiver and chosen with uniform probability from $\mathcal{V}_s$.The lattice pair Λ_s, Λ_c is drawn from an ensemble of lattices having good "covering" properties. It is shown that this ensemble of lattices contains a lattice such that the resultant space-time code, when suitably decoded using generalized minimum Euclidean distance lattice decoding, achieves the D-MG tradeoff for all $T \geq n_t + n_r - 1$. In actual code construction, a lattice drawn at random from the ensemble of lattices is used.

c) ST Codes from Division Algebras: Space-time code construction from division algebras was first proposed by Sethuraman and Rajan [18]–[21] and independently by Belfiore and Rekaya [14]. In [20], the authors consider the construction of ST codes from field extensions as well as division algebras (DA). It is shown how Alamouti's code arises as a special instances of the DA construction. A method of constructing cyclic division algebras (CDA) using transcendental elements is given and the capacity of the corresponding STBCs studied. A second construction of CDA due to Brauer is also applied to construct ST codes.

d) Non-Vanishing Determinant: In [14], the notion of a non-vanishing determinant (NVD) is introduced by Belfiore and Rekaya. The coding gain of a space-time code as determined by pairwise error probability considerations, is a function of the determinant of the difference code matrix. It is therefore of interest to maximize the value of this determinant. The authors of [14] note that while many constructions of space-time codes have the property of having a non-zero determinant, this determinant often vanishes as the SNR increases and the size of the signal constellation is accordingly increased. In [14], the authors describe an approach for constructing CDA-based square ST codes with the NVD property for $n_t = T = 2^r$ and $n_t = T = 3 \cdot 2^r$.

e) Perfect Codes: (see Section IV).

f) Constructions with NVD: In [25], square ST codes with the NVD property are constructed by Kiran and Rajan for $n_t = T = 2k, 3.2k, \ 2.3k$ or $n_t = T = q^k(q-1)/2$, where $q = 4k+3$ is a prime. Also contained in this paper, is a Lemma that simplifies CDA-based NVD code construction.

g) Approximate Universality: In [9], [11], Tavildar and Vishwanath consider the general case when the channel matrix H has an arbitrary statistical description. They show the existence of permutation codes that achieve the D-MG tradeoff for the parallel (diagonal H) channel. It is noted that by using D-BLAST in conjunction with a permutation code, one can achieve the D-MG tradeoff of the general correlated fading channel in the limit as the delay parameter $T \to \infty$. A sufficient condition for a ST code to be approximately universal, i.e., be D-MG optimal for every correlated MIMO fading channel is provided in terms of the product of the squared-singular values of the difference code matrix.

B. Results of Present Paper

First, a complete solution to the problem of explicitly constructing ST codes that achieve the D-MG tradeoff is presented here for all (n_t, n_r) and all $T \geq n_t$ including both minimum-delay $T = n_t$ and rectangular $T > n_t$ cases.

Both the square and rectangular constructions achieve the same tradeoff as do the constructions in the $T \geq n_t + n_r - 1$ case for which the D-MG tradeoff is exactly known from [5]. Apart from establishing the optimality of the rectangular constructions, this also extends the range of values of T for which the D-MG tradeoff is exactly known.

Perfect codes introduced by Oggier et. al. [16] were previously known to exist only for $n_t = 2, 3, 4, 6$. Here we present a general construction of perfect codes, valid for all n_t. Finally we consider the problem of code design under the single-bit-feedback ARQ signaling scheme introduced by El Gamal et. al. [7] and provide optimal constructions for many situations.

II. General Construction for a D-MG Optimal Code

For M even, let $\mathcal{A}_{\text{QAM}}$ denote the M^2-QAM constellation:

$$\mathcal{A}_{\text{QAM}} = \{a + \imath b \ \mid \ -M+1 \leq a, b \leq M-1, a, b \quad \text{odd } \}.$$

Theorem 1 (Optimal Code Construction [22]): Let $n_t, T, \ T \ \geq \ n_t$ be given. Let $\mathbb{L}$ be a cyclic (Galois) extension of $\mathbb{Q}(\imath)$ of degree T such that if $\mathcal{O}_L$ denotes the integral closure of $\mathbb{Z}[\imath]$ in $\mathbb{L}$, the extension $\mathcal{O}_L/\mathbb{Z}[\imath]$ contains a prime ideal $\mathfrak{p}$ that remains inert in the extension. Let $\{\beta_i \mid i = 1, 2, \cdots, T\}$ be an integral basis for $\mathbb{L}/\mathbb{Q}$, σ denote a generator of $\text{Gal}(\mathbb{L}/\mathbb{Q}(\imath))$ and $\gamma \in \mathfrak{p}^2 \setminus \mathfrak{p}$. Then the $(n_t \times T)$ CDA based space-time code

$$\mathcal{X} = \left\{ \theta \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{T-1}) & \dots & \gamma\sigma^{T-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \dots & \gamma\sigma^{T-1}(\ell_2) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{T-1} & \sigma(\ell_{T-2}) & \dots & \sigma^{T-1}(\ell_0) \end{bmatrix}_{n_t} \mid \ell_i \in \mathcal{L} \right\},$$

where

- the subscript n_t on the matrix indicates that we take the submatrix comprised of the first n_t rows
- $M^2 = \text{SNR}^{\frac{r}{T}}$, $\theta^2 = \text{SNR}^{1-\frac{r}{T}}$,
- and where the signal constellation

$$\mathcal{L} \ = \ \left\{ \sum_{j=1}^{T} c_j \beta_j \mid c_j \in \mathcal{A}_{\text{QAM}} \right\},$$

achieves for every $0 \leq r \leq \min\{n_t, n_r\}$, the D-MG tradeoff of the i.i.d. Rayleigh-fading channel for any number n_r of receive antennas.

□

Remark 1: The theorem above presents a single unified construction that covers the square minimum-delay case as well as the general situation when $T > n_t$. The code matrices arising from the construction correspond in the $n_t = T$

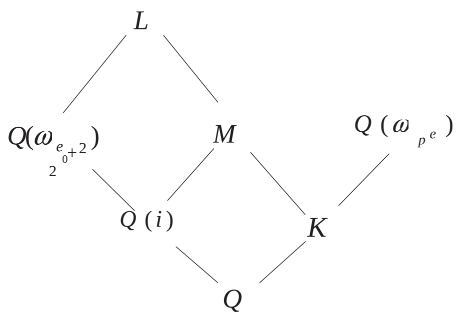

Fig. 2. *Constructing the Number Field Underlying the D-MG Optimal ST Code Construction.*

case to the matrix representation of elements of a cyclic division algebra. For the rectangular case, other methods of construction are also possible (see [22]). D-MG optimality of the codes constructed in [22] for the case of an arbitrary fading channel was shown by Tavildar and Vishwanath in [10] (see also [6]).

Note that this means of code construction calls for a cyclic extension of $\mathbb{Q}(\imath)/\mathbb{Q}$ of degree T containing a prime ideal β that remains inert in the extension. A method of constructing such a cyclic extension is presented below, valid for all T. An analogous D-MG optimal construction is possible for the case when $\mathbb{Q}(\imath)$ is replaced by $\mathbb{Q}(\exp(\imath\frac{2\pi}{3}))$.

III. Constructing the Desired Number Field

Let $T = 2^{e_0}n_1$, n_1 odd. For every integer $m \geq 3$, let $\omega_m = \exp(\imath\frac{2\pi}{m})$.

Theorem 2 (Number Field Construction [22]): Let p^e be the smallest rational prime-power for which $n_1 \mid p^e(p-1)$. Let G denote the Galois group $\text{Gal}(\mathbb{Q}(\omega_{p^e})/\mathbb{Q})$ of the cyclic extension $\mathbb{Q}(\omega_{p^e})/\mathbb{Q}$. Let H denote the unique cyclic subgroup of G of index n_1. Let $\mathbb{K}$ be the subfield of $\mathbb{Q}(\omega_{p^e})$ fixed element-wise by H. Let $\mathbb{M}$ be the compositum of $\mathbb{Q}(\imath)$ and $\mathbb{K}$ and $\mathbb{L}$ be the compositum of $\mathbb{M}$ and $\mathbb{Q}(\omega_{2^{e_0+2}})$ (Fig. 2). Then $\mathbb{L}/\mathbb{Q}(\imath)$ is a cyclic extension of $\mathbb{Q}(\imath)$ of degree T. Let q be a prime satisfying

$$q = \begin{cases} \alpha & \pmod{p^e} \\ 5 & \pmod{2^{e_0+2}} \end{cases}$$

where α is an element of $\mathbb{Z}^*_{p^e}$ having maximum possible order $= p^{e-1}(p-1)$. Then the prime ideal $q\mathbb{Z}[\imath]$ splits into the product of two prime ideals β_1, β_2 in $\mathbb{L}/\mathbb{Q}(\imath)$. Each prime ideal β_i remains inert in $\mathbb{L}/\mathbb{Q}(\imath)$.

□

IV. Perfect Codes

In [16], Oggier et. al. define a square $(n \times n)$ STBC to be a perfect code if

- the code is a full rate linear dispersion code using M^2 information symbols either QAM or HEX.
- the minimum determinant of the infinite code is non zero (so that in particular the rank criterion is satisfied).
- the $2M^2$dimensional real lattice generated by the vectorized codewords, is either Z^{2M^2} or $A_2^{M^2}$
- the code induces uniform average transmitted energy per antenna in all T time slots., i.e., all the coded symbols in the code matrix have the same average energy

It can be shown that perfect codes are D-MG optimal. The authors of [16] show the existence of perfect CDA-based space-time codes for dimensions $n = 2, 3, 4, 6$ and conclude that perfect CDA-based ST codes do not exist for any other value of n. This was proven under the additional assumption that γ is an algebraic integer.

The construction below due to Elia et. al. [23] recognizes that this assumption is not essential to the existence of CDA-based D-MG optimal ST codes and provides $(n \times n)$ perfect ST codes for every value of n.

Theorem 3 (Perfect Code Construction [23]): Let $n_t = T = n$. Let $n = 2^{e_0}n_1$ for odd n_1. Let p, q be primes satisfying

$$\begin{aligned} p &= 1 \pmod{n_1} \\ q &= \begin{cases} 5 & \pmod{2^{e_0+2}} \\ \alpha & \pmod{p} \end{cases} \end{aligned}$$

where $\alpha \in \mathbb{Z}^*_p$ has maximal order $(p-1)$. Let G denote the Galois group $\text{Gal}(\mathbb{Q}(\omega_p)/\mathbb{Q})$ of the cyclic extension $\mathbb{Q}(\omega_p)/\mathbb{Q}$. Let H denote the unique cyclic subgroup of G of index n_1. Let $\mathbb{K}$ be the subfield of $\mathbb{Q}(\omega_p)$ fixed element-wise by H. Let $\mathbb{M}$ be the compositum of $\mathbb{Q}(\imath)$ and $\mathbb{K}$ and $\mathbb{L}$ be the compositum of $\mathbb{M}$ and $\mathbb{Q}(\omega_{2^{e_0+2}})$. Then $\mathbb{L}/\mathbb{Q}(\imath)$ is a cyclic extension of $\mathbb{Q}(\imath)$ of degree n.

In the ring $\mathbb{Z}[\imath]$, q can be factored in the form $q = \pi\pi^*$ for a suitable Gaussian prime π. Set $\gamma = \frac{1+2\imath}{1-2\imath}$ if $n_1 = 1$ and $\gamma = \frac{\pi_1}{\pi_1^*}$ otherwise.

Let σ denote a generator of $\text{Gal}(\mathbb{L}/\mathbb{Q}(\imath))$. Let $\{\beta_i \mid i = 1, 2, \cdots, T\}$ be an integral basis for $\mathbb{L}/\mathbb{Q}$, such that the rows of the matrix $[\sigma^i(\beta_j)]$ are pairwise orthogonal[1]. Then the CDA based space-time code

$$\mathcal{X} = \left\{ \theta \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{n-1}) & \dots & \gamma\sigma^{n-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \dots & \gamma\sigma^{n-1}(\ell_2) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n-1} & \sigma(\ell_{n-2}) & \dots & \sigma^{n-1}(\ell_0) \end{bmatrix} \mid \ell_i \in \mathcal{L} \right\},$$

where

- $M^2 = \text{SNR}^{\frac{r}{n}}$, $\theta^2 = \text{SNR}^{1-\frac{r}{n}}$,
- the signal constellation $\mathcal{L}$ is as defined earlier,

is a perfect code for any number n_r of receive antennas.

□

V. ARQ Channel

Under the ARQ signalling protocol introduced by El Gamal, Caire and Damen [8], in place of transmitting a matrix X of fixed time duration T, each message symbol from the source is associated with a unique block $\theta[X_1\ X_2 \cdots X_L]$ of L $(n_t \times T)$ matrices θX_i in such a way that it is possible to uniquely decode the message symbol given $\theta[X_1 X_2 \cdots X_l]$ for any $1 \leq l \leq L$, in particular, given just θX_1. The scalar θ is chosen to ensure the energy constraint

$$\mathbb{E}\left[\theta^2 \|X_l\|_F^2\right] \leq T\ \text{SNR},\ 1 \leq l \leq L. \quad (4)$$

[1]Such a basis can always be found, see [23])

a) Bank of Codes: Let $\mathcal{X}_{\text{ARQ}}$ denote the ARQ ST code, i.e., the collection of matrices $\theta[X_1 \ X_2 \cdots X_L]$ corresponding to all possible message symbols. We will use $\mathcal{X}_{\text{ARQ},l}$ to denote the ST code truncated to l rounds and call it the lth round ST code. The specific codes $\mathcal{X}_{\text{ARQ},1}$ and $\mathcal{X}_{\text{ARQ},L}$ will be termed the single-round and full-length ST codes respectively.

With each code $\mathcal{X}_{\text{ARQ},l}$, we associate a decoder $\mathcal{D}_l$. While each of the decoders $\mathcal{D}_l, \ 1 \ \leq l \leq (L-1)$ is permitted to decline to decode, the decoder $\mathcal{D}_L$ in contrast, is a maximum-likelihood (ML) decoder and will always decode.

b) Signaling Protocol: The presence of a noiseless feedback channel capable of conveying one bit (ACK or NACK) of information per ARQ round is assumed. Also assumed, is the presence of an infinite buffer at the transmitter end. At the receiver end, the receiver applies the decoder $\mathcal{D}_1$ to the corresponding $(n_r \times T)$ received matrix Y_1. If decoder $\mathcal{D}_1$ is able to decode the underlying message symbol from Y_1, then an ACK is passed on to the transmitter, otherwise a NACK is sent. Upon receipt of an ACK, the transmitter moves on to transmit the next message symbol. Upon receipt of a NACK however, the transmitter proceeds to transmit θX_2. The receiver this time, attempts to decode the message symbol applying decoder $\mathcal{D}_2$ to the concatenated signal $[Y_1 Y_2]$. This process is continued until the transmitter receives an ACK. Since the decoder $\mathcal{D}_L$ is a ML decoder, the ARQ process will never run beyond L rounds.

To simplify notation, we will abbreviate and write

$$\begin{aligned} \mathbf{X}_l &= [X_1 X_2 \cdots X_l], \\ \mathbf{Y}_l &= [Y_1 Y_2 \cdots Y_l], \\ \mathbf{W}_l &= [W_1 W_2 \cdots W_l], \end{aligned}$$

for $1 \leq l \leq L$. These are related by

$$\mathbf{Y}_l = \theta H \mathbf{X}_l + \mathbf{W}_l, \qquad 1 \leq l \leq L \tag{5}$$

under the long-term static channel model introduced in [7]. The matrices H associated with transmission of different message symbols will be assumed to be statistically independent.

A. Rate and Reliability

Let the rate $R = r\log(\text{SNR})$ and and the error probability $P_e(r) \doteq \text{SNR}^{-d(r)}$ be as before. Then under this setting, El Gamal et al. [7] showed that for channels with Rayleigh fading, the maximum possible value $d^*_{\text{Rayleigh}}(r)$ of the diversity gain at each value of spatial-multiplexing gain r is given by

$$d^*_{\text{Rayleigh}}(r) = (n_t - \frac{r}{L})(n_r - \frac{r}{L}),$$

for integral values of $0 \leq r \leq \min\{n_t, n_r\}$ and through straight-line interpolation in between (Fig. 3).

This was termed the Diversity-Multiplexing-Delay (DMD) tradeoff in [8] in which it is also shown that an adaptation of the LAST codes [7] for the ARQ Rayleigh fading channel known as IR (incremental redundancy) LAST codes, are optimal with respect to the corresponding DMD tradeoff.

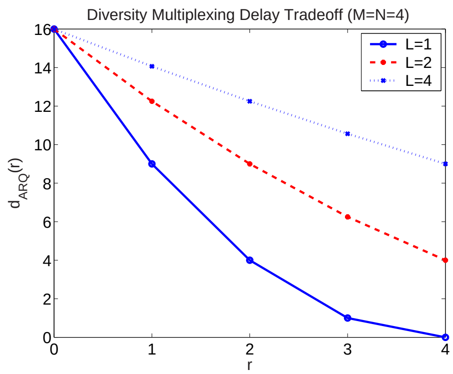

Fig. 3. *DMD Tradeoff for different L. Here $M = n_t = N = n_r = 4$.*

B. Our Recent Results

a) Construction of DMD-optimal Codes when $L|n_t$: Set $T = n_t/L$. The construction of DMD-optimal ARQ codes in this case is derived from the square $(LT \times LT)$ space-time code $\mathcal{X}_{\text{ARQ},LT}$ obtained via the construction in Theorem 1 with $n = LT$. Every codeword $X \in \mathcal{X}_{\text{ARQ},LT}$ is thus of the form

$$X = \theta \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{LT-1}) & \cdots & \gamma\sigma^{LT-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \cdots & \gamma\sigma^{LT-1}(\ell_2) \\ \ell_2 & \sigma(\ell_1) & \cdots & \gamma\sigma^{LT-1}(\ell_3) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{LT-1} & \sigma(\ell_{LT-2}) & \cdots & \sigma^{LT-1}(\ell_0) \end{bmatrix}.$$

Each round of the ARQ transmission corresponds to transmitting T successive columns from the matrix θX, i.e., during the l^{th} round, we transmit $\theta X_l = \theta[\underline{c}_{(l-1)T+1} \cdots \underline{c}_{lT}]$ for $l = 1, 2, \ldots, L$, where $\underline{c}_i$ denotes the i^{th} column of X.

Theorem 4 (ARQ Construction when $L|n_t$): The space-time code $\mathcal{X}_{\text{ARQ}}$ constructed above for the case $L|n_t$ achieves the MIMO-ARQ diversity-multiplexing-delay tradeoff of the long-term-static ARQ channel for $n_r \geq n_t$ under the power constraint (4), i.e.,

$$d_{\mathcal{X}_{\text{ARQ}}}(r, L) = d^*\left(\frac{r}{L}\right), \quad 0 \leq r < n_t$$

for block length $T = \frac{n_t}{L}$.

□

b) Construction for the Case $n_t|L$: Let $L = n_t k$, for k an integer. Consider the CDA-based D-MG optimal $(n_t \times L)$ rectangular space-time code $\mathcal{X}_{n_t \times L}$ constructed using Theorem 1 with L in place of T. Each codeword $\theta X \in \mathcal{X}_{\text{ARQ},L}$ is then of the form

$$\theta \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{L-1}) & \ldots & \gamma\sigma^{L-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \ldots & \gamma\sigma^{L-1}(\ell_2) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{L-1} & \sigma(\ell_{L-2}) & \ldots & \sigma^{L-1}(\ell_0) \end{bmatrix}_{n_t}.$$

We identify a set of $k-1$ columns of the matrix X, which along with the first column will contain all the independent variables $\{\ell_i\}_{i=0}^{L-1}$. Consider a column vector $\underline{c}$ obtained by taking a linear combination of these k columns.

Since the transmitted message symbol can be recovered from the elements $\{\ell_i\}_{i=0}^{L-1}$, column vector $\underline{c}$ has the same information content as does the entire ST code matrix X. Next, replace the first column of X with column $\underline{c}$. The above procedure of combining $k-1$ columns into the first column is an elementary column operation that does not change either the determinant of the $(n_t \times L)$ ST matrix X. Using this, it can be shown that the modified $(n_t \times L)$ ST code is also optimal with respect to the D-MG tradeoff. We now define the $n_t \times L$ ARQ space-time scheme $\mathcal{X}_{\text{ARQ}}$ as one in which the L ARQ rounds involve transmission of the L columns of the modified $(n_t \times L)$ ST code matrix in turn, beginning with the first. (Thus in effect, we have set the ARQ parameter $T = 1$.)

Theorem 5 (ARQ Construction when $n_t|L$): The ARQ ST code $\mathcal{X}_{\text{ARQ}}$ constructed above for the case when $n_t|L$ achieves the MIMO-ARQ diversity-multiplexing-delay tradeoff of the long term static ARQ channel for $n_r \geq n_t$ under the power constraint (4), i.e.,

$$d_{\mathcal{X}_{\text{ARQ}}}(r, L) = d^*\left(\frac{r}{L}\right), \quad 0 \leq r < n_t$$

for block length $T = 1$.

C. Regular Fading Channels

Optimality of the ARQ ST codes constructed above can be shown to hold not only for the Rayleigh fading channel, but also for the larger class of channels characterized by a channel matrix H such that the density function $p_{\lambda_{\min}}(\lambda)$ of the smallest eigenvalue $\lambda_{\min}$ of $H^\dagger H$ is finite and well-behaved near zero (details may be found in [24]).

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