Poisson convergence can yield very sharp transitions in geometric random graphs

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***Abstract*— We investigate how quickly phase transitions can occur in some geometric random graphs where n points are distributed uniformly and independently in the unit cube $[0,1]^d$ for some positive integer d. In the case of graph connectivity for the one-dimensional case, we show that the transition width behaves like $\mathbf{n^{-1}}$ (when the number $\mathbf{n}$ of users is large), a significant improvement over general asymptotic bounds given recently by Goel et al. for monotone graph properties. We outline how the approach used here could be applied to higher dimensional graphs and to other graph properties. The key ingredient is the availability of a Poisson paradigm complementing the "zero-one" law usually occurring for many graph properties.**

I. Introduction

Over the past few years, geometric random graphs have provided a useful abstraction for studying large wireless networks [6], [9], [10], [13]. Much attention has focused on the basic situation where n points are distributed uniformly and independently in the unit cube $[0,1]^d$ (in $\mathbb{R}^d$) for some positive integer d. Given a fixed threshold $\tau > 0$, two points are then said to be directly connected if their Euclidean distance is less than τ. This notion of connectivity gives rise to an undirected geometric random graph, denoted $\mathbb{G}_d(n;\tau)$.

In the context of wireless networks, with $d = 1, 2, 3$, these n points represent users equipped with a transmitter/receiver of transmission range τ. In first approximation, if we neglect details of channel behavior, it is reasonable to model two users as being able to communicate with each other if their Euclidean distance is less than τ. This approach has been taken by a number of authors, e.g., [3], [4], [5], [6], [9], [10], [13].

Randomizing user locations makes it possible for many properties of $\mathbb{G}_d(n;\tau)$ (including connectivity) to reveal a *typical behavior* when n becomes large. This manifests itself as follows: Consider a monotone increasing graph property A defined in the usual manner [12],[1] graph connectivity being such a property. For each $n = 2, 3, \ldots$, let $P_A(n;\tau)$ denote the probability that A occurs in $\mathbb{G}_d(n;\tau)$. The mapping $\tau \to P_A(n;\tau)$ is monotone increasing with $0 < P_A(n;\tau) < 1$ in some finite interval and $P_A(n;\tau) = 0$ or 1 outside it. As earlier simulation results already indicate for various properties of interest [3], [4], [5], [13], there often exists a *phase transition* from $P_A(n;\tau) \simeq 0$ to $P_A(n;\tau) \simeq 1$ as τ varies across some critical range. A natural question therefore consists in estimating how quickly this transition takes place.

To address this issue, for each $n = 2, 3, \ldots$, we define

$$\tau_{A,n}(a) = \inf\left(\tau > 0 : \; P_A(n;\tau) \geq a\right), \quad a \in (0,1)$$

and we set

$$\delta_{A,n}(a) = \tau_{A,n}(1-a) - \tau_{A,n}(a), \quad a \in (0, \frac{1}{2}).$$

The transition width $\delta_{A,n}(a)$ measures how quickly $P_A(n;\tau)$ climbs from level a to level $1-a$, thereby giving an indication of the sharpness of the phase transition. Given the rather complex dependence of $\delta_{A,n}(a)$ on n and a, it is desirable to find asymptotic bounds (if nothing else) on its behavior for large n.

Recently, Goel et al. [8] have derived such asymptotic bounds for any monotone graph property in $\mathbb{G}_d(n;\tau)$. For any such property A, their results imply that $\delta_{A,n}(a) = o(1)$, a fact captured by the terminology that the monotone property A has a *sharp* threshold. However, these general results leave open the question as to whether these asymptotic bounds can be further sharpened for *specific* monotone graph properties.

Here, we tackle this issue for the probability of graph connectivity. For ease of telling the story, we restrict the discussion to one-dimensional geometric random graph models ($d = 1$); such models have been investigated in the references [3], [4], [5], [7] which contain some of the needed results. Our main result takes the form of *exact* asymptotic expansions (in n) for the thresholds [Theorem 1]. This leads to transition widths of order n^{-1} with known preconstants, so that these graph properties are *very sharp* indeed! Such information can be leveraged in network design when network connectivity is an important concern.

The one-dimensional case ($d = 1$) may be construed as perhaps too limited or not too relevant to practice. However, we stress that the main contribution of the paper lies in identifying an *approach of wide applicability* to establish sharp asymptotics on the transition width: The key ingredient is the availability of a *Poisson paradigm* complementing the "zero-one" law usually occurring for many graph properties.

The paper is organized as follows: The model and preliminaries are given in Section II. The main results concerning the

[1] The case of monotone decreasing graph properties can be discussed *mutatis mutandis*.

behavior of thresholds for graph connectivity are presented in Section III. In Section IV, we explain how the appropriate "zero-one" laws and companion Poisson convergence lead to the correct asymptotics for the threshold width. This is followed by a formal proof in Section V. In Section VI we briefly contrast our results against the results of Goel et al.; we also provide a rough roadmap for establishing similar results in higher dimensions ($d \geq 2$) and for other graph properties.

II. Model and preliminaries

The one-dimensional model has been considered by a number of authors [3], [4], [5], [7]. To define it, let $\{X_i,\ i = 1, 2, \ldots\}$ denote a sequence of i.i.d. rvs distributed uniformly in the interval $[0, 1]$.

For each $n = 2, 3, \ldots$, we think of $X_1, \ldots, X_n$ as the locations of n nodes (or users), labelled $1, \ldots, n$, in the interval $[0, 1]$. Given a fixed distance $\tau > 0$, two nodes are said to be directly connected if their distance is at most τ, i.e., nodes i and j are connected if $|X_i - X_j| \leq \tau$, in which case an undirected edge is said to exist between these two users. This notion of connectivity gives rise to the undirected geometric random graph $\mathbb{G}_1(n; \tau)$, thereafter denoted $\mathbb{G}(n; \tau)$.

Let the rvs $X_{n,1}, \ldots, X_{n,n}$ denote the locations of these n users arranged in increasing order, i.e., $X_{n,1} \leq \ldots \leq X_{n,n}$ with the convention $X_{n,0} = 0$ and $X_{n,n+1} = 1$. The rvs $X_{n,1}, \ldots, X_{n,n}$ are the *order statistics* associated with the n i.i.d. rvs $X_1, \ldots, X_n$. Also define

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \ldots, n + 1.$$

Obviously $L_{n,1} + \ldots + L_{n,n+1} = 1$. It is well known [2, Eq. (6.4.3), p. 135] that for any fixed subset $I \subseteq \{1, \ldots, n\}$, we have

$$\mathbf{P}\left[L_{n,k} > t_k,\ k \in I\right] = \left(1 - \sum_{k \in I} t_k\right)_+^n, \quad t_k \in [0, 1],\ k \in I$$

with the notation $x_+^n = x^n$ if $x \geq 0$ and $x_+^n = 0$ if $x \leq 0$.

Fix $\tau > 0$ and $n = 2, 3, \ldots$. The geometric random graph $\mathbb{G}(n; \tau)$ is said to be *(path) connected* if every pair of users can be linked by at least one path over the edges of the graph, and we write

$$P(n; \tau) := \mathbf{P}\left[\mathbb{G}(n; \tau) \text{ is connected}\right].$$

Obviously, the graph $\mathbb{G}(n; \tau)$ is connected if and only if $L_{n,k} \leq \tau$ for all $k = 2, \ldots, n$, so that

$$P(n; \tau) = \mathbf{P}\left[L_{n,k} \leq \tau,\ k = 2, \ldots, n\right]. \tag{1}$$

The closed form expression

$$P(n; \tau) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (1 - k\tau)_+^n \tag{2}$$

has been rediscovered by several authors, e.g., Godehardt and Jaworski [7, Cor. 1, p. 146], and Desai and Manjunath [3] (as Eqn (8) with $z = 1$ and $r = \tau$).

III. Main results

For each $n = 2, 3, \ldots$, the mapping $\tau \rightarrow P(n; \tau)$ can be shown to be continuous and strictly monotone increasing. Given fixed a in $(0, 1)$, this property guarantees the existence and uniqueness of solutions to the equation

$$P(n; \tau) = a, \quad \tau \in (0, 1). \tag{3}$$

Let $\tau_n(a)$ denote this unique solution, and whenever a lies in the interval $(0, \frac{1}{2})$, we set

$$\delta_n(a) := \tau_n(1 - a) - \tau_n(a).$$

The main result concerning the behavior of $\tau_n(a)$ for large n is given first.

Theorem 1: For every a in the interval $(0, 1)$, it holds that

$$\tau_n(a) = \frac{\log n}{n} - \frac{1}{n} \log\left(\log\left(\frac{1}{a}\right)\right) + o\left(n^{-1}\right). \tag{4}$$

Theorem 1 is established in Section V. The desired result on the width of the transition interval flows as an easy corollary.

Corollary 1: For every a in the interval $(0, \frac{1}{2})$, we have

$$\delta_n(a) = \frac{C(a)}{n} + o\left(n^{-1}\right) \tag{5}$$

with constant $C(a)$ given by

$$C(a) = \log\left(\frac{\log a}{\log(1 - a)}\right). \tag{6}$$

It is a simple matter to check that $a \rightarrow C(a)$ is decreasing on the interval $(0, \frac{1}{2})$ with $\lim_{a \downarrow 0} C(a) = \infty$ and $\lim_{a \uparrow \frac{1}{2}} C(a) = 0$. These qualitative features are in line with one's intuition.

IV. How to guess the result

We now present a plausibility argument which allows us to guess the validity of Theorem 1, and which eventually paves the way to its proof: Our point of departure is the "zero-one" law available for the property of graph connectivity under the asymptotic regime created by having n become large and the threshold parameter scaled appropriately with n. We shall find it useful to say that a threshold function $\tau : \mathbb{N} \rightarrow [0, 1]$ is *admissible* if $\lim_{n\rightarrow\infty} \tau_n = 0$. There is no loss of generality in writing such an admissible threshold function in the form

$$\tau_n = \frac{1}{n}\left(\log n + \alpha_n\right), \quad n = 2, 3, \ldots. \tag{7}$$

where $\alpha : \mathbb{N} \rightarrow \mathbb{R}$ satisfies $\alpha_n = o(n)$.

Theorem 2: For any admissible threshold function $\tau : \mathbb{N} \rightarrow [0, 1]$ written in the form (7), it holds that

$$\lim_{n\rightarrow\infty} P(n; \tau_n) = \begin{cases} 0 & \text{if } \lim_{n\rightarrow\infty} \alpha_n = -\infty \\ & \\ 1 & \text{if } \lim_{n\rightarrow\infty} \alpha_n = +\infty. \end{cases} \tag{8}$$

Theorem 2 follows from Theorem 1 in [1, p. 352], but can also be derived by direct arguments based on the method of first and second moments [11].

The convergence (8) identifies the critical scaling

$$\tau^\star_{\mathrm{con},n} = \frac{\log n}{n}, \quad n = 2, 3, \ldots$$

as the threshold function which defines a *boundary* in the space of scalings. Intuition suggests that mild fluctuations about this boundary, say of order n^{-1}, are likely to hold the key to the form of $\tau_n(a)$ for large n. To explore this idea further, for each x in $\mathbb{R}$, define the $[0,1]$-valued sequence $\{\sigma_n(x),\ n = 1, 2, \ldots\}$ by

$$\sigma_n(x) = \min\left(1, \left(\frac{\log n + x}{n}\right)_+\right), \quad n = 1, 2, \ldots \tag{9}$$

so that

$$\sigma_n(x) = \frac{\log n + x}{n} \tag{10}$$

for n large enough. The next result complements the "zero-one" law (8), and in fact implies it; it is given as part of Theorem 12 in [7, p. 157].

Theorem 3: For each x in $\mathbb{R}$, it holds that

$$\lim_{n\to\infty} P(n; \sigma_n(x)) = p(x) \tag{11}$$

with

$$p(x) = e^{-e^{-x}}. \tag{12}$$

To see in what sense the convergence (11) underpins Theorem 1, consider the following heuristic arguments: For each x in $\mathbb{R}$, the convergence (11) yields the approximation

$$P(n; \sigma_n(x)) \simeq p(x)$$

for large enough n. The mapping $p : \mathbb{R} \to \mathbb{R}_+ : x \to p(x)$ is strictly monotone and continuous with $\lim_{x\to-\infty} p(x) = 0$ and $\lim_{x\to\infty} p(x) = 1$. Therefore, for each a in the interval $(0, 1)$, there exists a unique scalar, denoted x_a, such that $p(x_a) = a$. In fact,

$$x_a = -\log\left(-\log a\right). \tag{13}$$

Given a in the interval $(0, 1)$, we find that

$$P(n; \sigma_n(x_a)) \simeq a$$

for large n. This suggests (but not quite yet proves) that $\sigma_n(x_a)$ and $\tau_n(a)$ behave in tandem asymptotically, thereby laying the grounds for the validity of (4) – Just insert (13) into (10) and (12). These ideas form the basis for the proof of Theorem 1 found in Section V.

To gain some perspective on (11)–(12), we introduce the notion of *breakpoint* user. For each $i = 1, \ldots, n$, user i is said to be a breakpoint user in the random graph $\mathbb{G}(n; \tau)$ whenever (i) it is not the leftmost user in $[0, 1]$ and (ii) there is no user in the random interval $[X_i - \tau, X_i]$. The number $C_n(\tau)$ of breakpoint nodes in $\mathbb{G}(n; \tau)$ is given by

$$C_n(\tau) = \sum_{k=2}^{n} \chi_{n,k}(\tau)$$

with $\{0, 1\}$-valued rvs $\chi_{n,1}(\tau), \ldots, \chi_{n,n+1}(\tau)$ defined as the indicator functions

$$\chi_{n,k}(\tau) := \mathbf{1}\left[L_{n,k} > \tau\right], \quad k = 1, \ldots, n+1.$$

As a result,

$$P(n; \tau) = \mathbf{P}\left[C_n(\tau) = 0\right] \tag{14}$$

and Theorem 3 is now a mere byproduct of a stronger result [Theorem 4] on *Poisson convergence* [7, Thm. 12, p. 157].

Theorem 4: For each x in $\mathbb{R}$, it holds that $C_n(\sigma_n(x)) \Longrightarrow_n \Pi(e^{-x})$ where $\Pi(\mu)$ denotes a Poisson rv with parameter μ and $\Longrightarrow_n$ denotes convergence in distribution with n going to infinity.

V. A proof of Theorem 1

Fix x in $\mathbb{R}$. We restate (11) by noting that for each $\varepsilon > 0$, there exists a finite integer $n^\star(\varepsilon, x)$ such that

$$p(x) - \varepsilon < P(n; \sigma_n(x)) < p(x) + \varepsilon, \quad n \ge n^\star(\varepsilon, x). \tag{15}$$

Now fix a in the interval $(0, 1)$, and pick ε sufficiently small such that $0 < 2\varepsilon < a$ and $a + 2\varepsilon < 1$. Repeatedly applying (15) with $x = x_{a+\varepsilon}$ and $x = x_{a-\varepsilon}$, we get

$$p(x_{a+\varepsilon}) - \varepsilon < P(n; \sigma_n(x_{a+\varepsilon})) < p(x_{a+\varepsilon}) + \varepsilon \tag{16}$$

whenever $n \ge n^\star(\varepsilon, x_{a+\varepsilon})$, and

$$p(x_{a-\varepsilon}) - \varepsilon < P(n; \sigma_n(x_{a-\varepsilon})) < p(x_{a-\varepsilon}) + \varepsilon \tag{17}$$

whenever $n \ge n^\star(\varepsilon, x_{a-\varepsilon})$. In the remainder of this proof, all inequalities are now understood to hold for $n \ge n^\star(a; \varepsilon)$ where we have set

$$n^\star(a; \varepsilon) = \max\left(n^\star(x_a), n^\star(\varepsilon, x_{a+\varepsilon}), n^\star(\varepsilon, x_{a-\varepsilon})\right)$$

where $n^\star(x)$ denotes the finite integer beyond which the representation (10) holds.

Since $p(x_{a\pm\varepsilon}) = a \pm \varepsilon$, the two chains of inequalities at (16) and (17) can be rewritten as

$$a < P(n; \sigma_n(x_{a+\varepsilon})) < a + 2\varepsilon$$

and

$$a - 2\varepsilon < P(n; \sigma_n(x_{a-\varepsilon})) < a.$$

Thus,

$$P(n; \tau_n(a)) < P(n; \sigma_n(x_{a+\varepsilon})) < P(n; \tau_n(a + 2\varepsilon))$$

and

$$P(n; \tau_n(a - 2\varepsilon)) < P(n; \sigma_n(x_{a-\varepsilon})) < P(n; \tau_n(a)),$$

and the strict monotonicity of $\tau \to P(n; \tau)$ yields

$$\tau_n(a) < \sigma_n(x_{a+\varepsilon}) < \tau_n(a + 2\varepsilon)$$

and

$$\tau_n(a - 2\varepsilon) < \sigma_n(x_{a-\varepsilon}) < \tau_n(a).$$

Combining these last two inequalities, we conclude that

$$\sigma_n(x_{a-\varepsilon}) < \tau_n(a) < \sigma_n(x_{a+\varepsilon}). \tag{18}$$

Upon writing

$$\xi_n(a) = \tau_n(a) - \sigma_n(x_a), \quad n = 2, 3, \ldots \tag{19}$$

we obtain from (18) that

$$\sigma_n(x_{a-\varepsilon}) - \sigma_n(x_a) < \xi_n(a) < \sigma_n(x_{a+\varepsilon}) - \sigma_n(x_a)$$

with

$$\sigma_n(x_{a\pm\varepsilon}) - \sigma_n(x_a) = \frac{x_{a\pm\varepsilon} - x_a}{n}. \tag{20}$$

As a result, $x_{a-\varepsilon} - x_a \leq \liminf_{n\to\infty} (n\xi_n(a))$ and $\limsup_{n\to\infty} (n\xi_n(a)) \leq x_{a+\varepsilon} - x_a$. Given that ε can be taken to be arbitrary small, it follows that

$$\liminf_{n\to\infty} (n\xi_n(a)) = \limsup_{n\to\infty} (n\xi_n(a)) = 0$$

since

$$\lim_{\varepsilon\downarrow 0} (x_{a-\varepsilon} - x_a) = \lim_{\varepsilon\downarrow 0} (x_{a+\varepsilon} - x_a) = 0.$$

Thus, $\lim_{n\to\infty} (n\xi_n(a)) = 0$, whence $\xi_n(a) = o\left(\frac{1}{n}\right)$. Reporting into (19) leads to

$$\tau_n(a) = \sigma_n(x_a) + o(n^{-1}), \quad n = 2, 3, \ldots$$

and the desired result readily follows from (9) and (13).

VI. Discussion

A. Theorem 1 vs. Goel et al. [8]

For $d = 1$, the model considered by Goel et al. [8] coincides with the one-dimensional situation discussed here. They show [8, Thm. 1.1] that for every monotone graph property A, the corresponding transition width for property A satisfies

$$\delta_{A,n}(a) = O\left(\sqrt{\frac{-\log a}{n}}\right). \tag{21}$$

The results obtained here for graph connectivity markedly improve on (21) in that *exact* asymptotics were provided and the rate of decay (namely, n^{-1}) is found to be a lot faster than the rough asymptotic bound given by (21). These authors also show [8, Thm. 1.2] that there *exists* some monotone graph property, say B, such that

$$\delta_{B,n}(a) = \Omega\left(\sqrt{\frac{-\log a}{n}}\right). \tag{22}$$

Obviously, graph connectivity cannot be such a property!

B. Isolated nodes

Similar arguments can be made for graph properties other than graph connectivity. Here is another example: Fix $\tau > 0$ and $n = 2, 3, \ldots$. For each $i = 1, \ldots, n$, node i is said to be *isolated* in the random graph $\mathbb{G}(n;\tau)$ whenever $|U_i - U_j| > \tau$ for all $j \neq i$, $j = 1, \ldots, n$. In terms of the order statistics introduced earlier, we see that the user at location $X_{n,k}$ is isolated (i) if $L_{n,2} > \tau$ for $k = 1$; (ii) if $L_{n,k} > \tau$ and $L_{n,k+1} > \tau$ whenever $k = 2, \ldots, n-1$; and (iii) if $L_{n,n} > \tau$ for $k = n$. As a result, the total number $I_n(\tau)$ of isolated nodes in $\mathbb{G}(n;\tau)$ can be represented as

$$I_n(\tau) = \chi_{n,2}(\tau) + \sum_{k=2}^{n-1} \chi_{n,k}(\tau)\chi_{n,k+1}(\tau) + \chi_{n,n}(\tau)$$

and the probability $P_{\mathrm{iso}}(n;\tau)$ that no node is isolated in $\mathbb{G}(n;\tau)$ is given by

$$P_{\mathrm{iso}}(n;\tau) := \mathbf{P}\left[I_n(\tau) = 0\right]. \tag{23}$$

Here, it is convenient to represent an admissible threshold function $\tau : \mathbb{N} \to [0,1]$ in the form

$$\tau_n = \frac{1}{2n}\left(\log n + \alpha_n\right), \quad n = 2, 3, \ldots. \tag{24}$$

where $\alpha : \mathbb{N} \to \mathbb{R}$ satisfies $\alpha_n = o(n)$. The analog of Theorem 2 now takes the form

$$\lim_{n\to\infty} P_{\mathrm{iso}}(n;\tau(n)) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \alpha_n = -\infty \\ 1 & \text{if } \lim_{n\to\infty} \alpha_n = +\infty \end{cases} \tag{25}$$

This follows from Theorem 2 in [1, p. 353]. This time the critical scaling is given by

$$\tau^\star_{\mathrm{iso},n} = \frac{\log n}{2n} = \frac{1}{2}\tau^\star_{\mathrm{con},n}, \quad n = 2, 3, \ldots$$

and the complement to the "zero-one" law (25) takes the form

$$\lim_{n\to\infty} P_{\mathrm{iso}}(n; \frac{1}{2}\sigma_n(x)) = p(x), \quad x \in \mathbb{R} \tag{26}$$

with $p(x)$ given by (12). Again (26) flows from a Poisson convergence result, namely $I_n(\frac{1}{2}\sigma_n(x)) \Longrightarrow_n \Pi(e^{-x})$. A proof of this result is omitted due to space limitations.

C. A roadmap via Poisson convergence

The discussion of Section IV provides a roadmap to deriving corresponding results in higher dimensional graphs ($d \geq 2$) and for other graph properties: For a given graph property A, we first need to identify the critical threshold associated with the "zero-one" law it satisfies. The effect of "small" perturbations (of the property-specific appropriate order) from the critical threshold can then be explored with the help of the Poisson convergence paradigm.

Poisson convergence is a common occurrence in the context of random graphs. It has its roots in the fact that many graph properties can be captured through counting sums of many indicator functions which become vanishingly small and increasingly decorrelated with n large under the appropriate (perturbed) scaling. This property has been well studied in the case of Bernoulli graphs for a number of important graph properties [12, Chap. 3], e.g., node isolation or the existence of *at least one copy of a given graph* G.

Until recently Poisson convergence has received little attention in the context of geometric random graphs. However, its validity for the properties of connectivity and node isolation in the one-dimensional case is already apparent from (14) and (23), respectively. The picture is far less complete in higher dimensions even for the property of graph connectivity.

For $d \geq 2$ critical thresholds have been identified for graph connectivity by a number of authors for the unit cube model [1], [15] or for the unit disk model [9], [10], [17].

In particular, for $d = 2$, with points distributed uniformly over a *disk* of unit area (rather than over a square), the critical threshold is known [9], [10], [17] to be given by

$$\pi \left(\tau^{\star}_{\mathrm{con},n}\right)^2 = \frac{\log n}{n}, \quad n = 2, 3, \ldots \tag{27}$$

Venkatesh [17] has also shown that the number of isolated users indeed converges to a Poisson rv $\Pi(e^{-x})$ when this critical scaling (27) is perturbed to

$$\sigma_n(x) = \min\left(1, \sqrt{\left(\frac{\log n + x}{\pi n}\right)_+}\right), \quad n = 1, 2, \ldots$$

for each x in $\mathbb{R}$. This is known [1] to imply the convergence

$$\lim_{n\to\infty} P(n; \sigma_n(x)) = e^{-e^{-x}}. \tag{28}$$

This result also follows from developments by Penrose [14].

D. Retooling Theorem 1

For graph connectivity in one dimension, the proof of Theorem 1 leads to the asymptotic equivalence

$$\tau_n(a) = \sigma_n(x_a) + o(n^{-1}), \quad a \in (0, 1).$$

However, for other graph properties or in higher dimensions, this conclusion may need to be modified accordingly.

For instance, for graph connectivity with $d = 2$, the convergence (28) requires that (20) be modified to read

$$\sigma_n(x_{a\pm\varepsilon}) - \sigma_n(x_a) = \sqrt{\frac{\log n + x_{a\pm\varepsilon}}{\pi n}} - \sqrt{\frac{\log n + x_a}{\pi n}}$$

for sufficiently large n. It is then a simple matter to check that

$$\lim_{n\to\infty} 2\sqrt{\pi n \log n}\,(\sigma_n(x_{a\pm\varepsilon}) - \sigma_n(x_a)) = x_{a\pm\varepsilon} - x_a.$$

By the same arguments as in the proof of Theorem 1, we conclude that

$$\lim_{n\to\infty} \sqrt{n \log n}\,(\tau_n(a) - \sigma_n(x_a)) = 0$$

so that

$$\tau_n(a) = \sqrt{\frac{\log n + x_a}{\pi n}} + o\left(\sqrt{\frac{1}{n \log n}}\right). \tag{29}$$

Easy calculations readily give

$$\tau_n(a) = \sqrt{\frac{\log n}{\pi n}} + \frac{x_a}{2}\sqrt{\frac{1}{\pi n \log n}}\,(1 + o(1))$$

with a similar expression for $\tau_n(1-a)$, whence

$$\delta_n(a) = \frac{C(a)}{2}\sqrt{\frac{1}{\pi n \log n}}\,(1 + o(1)). \tag{30}$$

For $d = 2$, Goel et al. show that (21) and (22) need to be replaced as follows [8, Thms. 1.1 and 1.2]: For every monotone graph property A, we have

$$\delta_{A,n}(a) = O\left(\frac{(\log n)^{3/4}}{\sqrt{n}}\right) \tag{31}$$

but there exists some graph property B such that

$$\delta_{B,n}(a) = \Omega\left(\sqrt{\frac{-\log a}{n}}\right). \tag{32}$$

Again the asymptotics (30) show that (31) is quite conservative, and that graph connectivity does not yield the worst case at (32).

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