Diversity-Multiplexing Gain Tradeoff and DMT-Optimal Distributed Space-Time Codes for Certain Cooperative Communication Protocols: Overview and Recent Results

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***Abstract*— This paper considers protocols for fading, wireless relay networks that aim to achieve cooperative diversity through the use of distributed space-time codes. All point-to-point communication is modeled as taking place over quasi-static, Rayleigh-fading channels with each node in the network operating in half-duplex mode. Channel-state information is assumed to be present only at the receiver. The diversity-multiplexing gain tradeoff (DMT) is taken to be the measure of performance.**

Prior work in this area relating to the identification of the DMT of various protocols as well as to DMT-optimal code construction is first reviewed. This is followed by a description of our recent results in this area. These include determination of the DMT of the orthogonal amplify-and-forward and selection decode-and-forward protocols as well as some simply-described, DMT-optimal code constructions based on cyclic division algebras.

I. Introduction

Cooperative relay communication is a promising means of wireless communication in which cooperation is used to create a virtual transmit array between the source and the destination, thereby providing spatial diversity for combating the fading channel. Such cooperative communication is under consideration for example, by the IEEE 802.16j Relay Task Group and a standard is expected to be completed and approved by 2008, see [13], [12].

Consider a communication system as shown in Fig. 1, in which there are $n+1$ nodes that cooperate in the communication between source node S and destination node D. The remaining $(n-1)$ nodes thus act as relays.

A. Assumptions

In keeping with the literature, we assume:

- all nodes to have a single transmit and single receive antenna

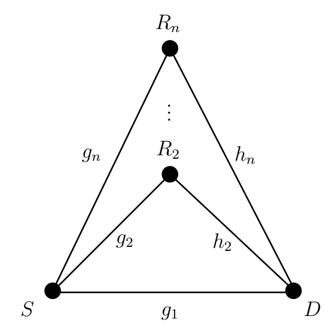

Fig. 1. Cooperative relaying in networks.

- the T channel uses over which communication relating to a single message vector takes place is short enough to invoke the quasi-static assumption, i.e., the channel fading coefficients $\{g_i, h_j\}$ are fixed for the duration of the communication, but vary randomly from one block of T channel uses to the next. Here, g_1 denotes the fading coefficient for the channel between S and D, and $\{g_i, h_i, 2 \leq i \leq n\}$ represent fading coefficients for the channels between S and the relay nodes $\{R_i\}$, and channels between the $\{R_i\}$ and D respectively. (Fig. 1).
- Rayleigh fading, i.e., all fade coefficients $\{g_i, h_i\}$ are i.i.d., circularly symmetric complex Gaussian $\mathbb{C}\mathcal{N}(0,1)$ random variables.
- half-duplex operation at each node, i.e., at any given instant, a node can either transmit or receive, but not do both.
- all additive-noise variables to be i.i.d., $\mathbb{C}\mathcal{N}(0,1)$ distributed.

B. Protocols

All protocols considered here involve two-phase communication. In the first or broadcast phase lasting p channel uses, the source S alone broadcasts to the relays and the destination

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D. In the second cooperation phase lasting q channel uses, the relays and possibly the source as well, communicate with the destination. A protocol is said to be *non-orthogonal* or *orthogonal* depending on whether or not the source continues to transmit in the cooperation phase. The protocol is said to be a *decode-and-forward* (DF) or an *amplify-and-forward* (AF) protocol depending upon whether the relays are required to decode the received message or not.

Within the class of DF protocols, one distinguishes between fixed decode-and-forward (FDF) and selection decode-and-forward (SDF) protocols. In a SDF protocol, a relay participates in the cooperation phase only if its measurements of the corresponding source-relay channel-fading coefficient g_i reveal the particular source-channel link to lie outside the outage region. In FDF, a relay always decodes. We will use OAF, OSDF to refer to orthogonal AF and SDF protocols while NAF and NSDF are the labels applied to non-orthogonal versions of the corresponding protocols. Other protocols such as compress-and-forward [11] and incremental AF [4] will not be discussed here.

C. Diversity-Multiplexing Gain Tradeoff

A point-to-point MIMO coding scheme $\{C(\rho)\}$ is a collection of space-time codes indexed by the SNR ρ. The scheme $\{C(\rho)\}$ is said to achieve spatial multiplexing gain r and corresponding diversity gain $d(r)$ if

$$\lim_{\rho\to\infty} \frac{R(\rho)}{\log\rho} \;=\; r, \quad \lim_{\rho\to\infty} \frac{\log P_e(\rho)}{\log\rho} \;=\; -d(r) \ ,$$

where $R(\rho)$ is the rate of the code $C(\rho)$ and $P_e(\rho)$ is the average codeword-error probability under maximum likelihood decoding. We use the symbol $\doteq$ to denote exponential equality, i.e.,

$$\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log\rho} = b \quad \Leftrightarrow \quad f(\rho) \doteq \rho^b$$

and $\dot{\geq}$, $\dot{\leq}$ are similarly defined. The DMT, introduced by Zheng and Tse in [1] for point-to-point channels, provides a means of evaluating and comparing the various proposed protocols.

The combination of physical wireless channel depicted in Fig. 1 and cooperative-communication protocol results in an equivalent point-to-point MIMO channel which we shall refer to as the *induced channel*. The DMT of a protocol will then refer to the tradeoff between the multiplexing gain r and the negative SNR exponent $d_{\text{out}}(r)$ of the probability of outage of the induced channel. In this paper, a DMT optimal code for a protocol will refer to a family of codes, indexed by ρ, whose tradeoff between r and $d(r)$ coincides with the DMT of the protocol.

II. Prior Work

The idea of obtaining spatial diversity through the cooperation of mobile users was first proposed in [2], [3] while the setting adopted in this talk is drawn from [4]. The protocols of SDF and incremental AF are introduced in [4] where the focus is on the case of a single relay.

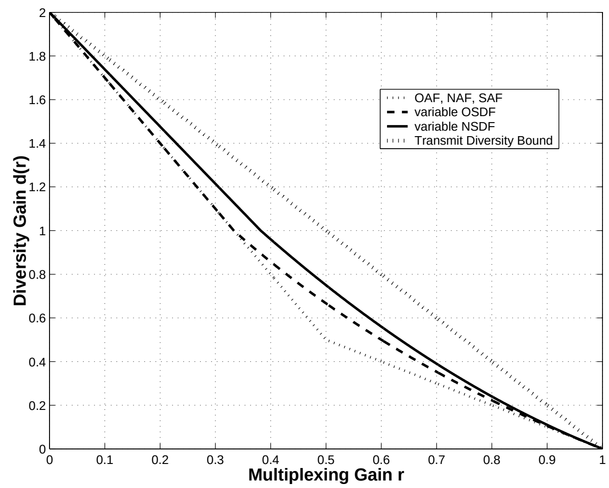

Fig. 2. *Optimal DMT for single relay cooperative communication protocols.*

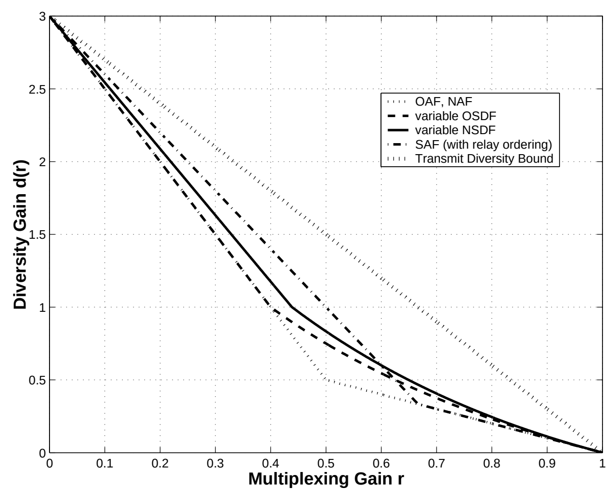

Fig. 3. Diversity-multiplexing gain tradeoff of some cooperative communication protocols in the case of 2 relay antennas.

It can be shown that the DMT of a cooperative relay network with $(n-1)$ relay nodes can at most equal the DMT

$$d(r) = n(1-r) \ , \tag{1}$$

and the right-hand side consequently, serves as an upper bound to the DMT of any cooperative relay network protocol involving $(n-1)$ relay nodes. This is first pointed out in [4] for the case of a single relay antenna where the authors refer to this bound as the *transmit-diversity bound*. The DMT of the OAF, FDF, SDF and incremental AF protocols is determined here for the single-relay case when the two phases of all the protocols are of equal duration.

In [5], the authors expand their attention to the network as a whole. They consider a wireless network with n cooperating terminals $\mathcal{M} = \{1, 2, \ldots, n\}$, with each terminal $s \in \mathcal{M}$ having an interest in communicating to a corresponding destination terminal $d(s) \notin \mathcal{M}$. The communication between

TABLE I
DMT OF VARIOUS AMPLIFY-AND-FORWARD PROTOCOLS.

Protocol	Authors	No of relays	Duration of two phases	$d_{\text{out}}(r)$	Delay of optimal code	Remark
OAF	LTW [4]	1	$p = q$ (chosen)	$2(1-2r)$	–	
	EVAK [15]	$n-1$	$p = n$, $q = n-1$ (optimal)	$(1-r)^{+} + (n-1)(1-2r)^{+}$	$2n-1$ [15]	1
NAF	AGS [6]	$n-1$	$p = q$ (optimal)	$(1-r)^{+} + (n-1)(1-2r)^{+}$	$4(n-1)$ [9]	2
SAF	YB [10]	2	$\frac{p}{q} = \frac{1}{2}$ (chosen)	$(1-r)^{+} + \left(1-\frac{3}{2}r\right)^{+}$ 3-slot	9 [10]	3
		$n-1$	$\frac{p}{q} = \frac{1}{n-1}$ (chosen)	$\leq (1-r)^{+} + \left(1-\frac{M}{M-1}r\right)^{+}$ M-slot	–	

TABLE II
DMT OF VARIOUS DECODE-AND-FORWARD PROTOCOLS WITH $n-1$ RELAYS.

Protocol	Authors	Duration of two phases	$d_{\text{out}}(r)$	Delay of optimal code	Remarks
DDF	AGS [6]	$\frac{p}{q}$ varies with$\{g_i\}$	$\begin{array}{ll} n(1-r), & 0 \leq r \leq \frac{1}{n} \\ 1+\frac{(n-1)(1-2r)}{1-r}, & \frac{1}{n} \leq r \leq \frac{1}{2} \\ \frac{1-r}{r}, & \frac{1}{2} \leq r \leq 1 \end{array}$	–	
Variable NSDF	EVAK [15]	$p = \kappa q$ κ varies with r (optimal)	$\begin{array}{ll} n\left(1-\frac{(n-1)(\kappa_n+1)}{n}r\right), & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{(n-r)(1-r)}{(n-2)r+1}, & \frac{1}{\kappa_n+1} \leq r \leq 1 \end{array}$, where $\kappa_n = \frac{1+\sqrt{1+4(n-1)^2}}{2(n-1)}$	For fixed (p,q) $(p+q)(p+nq)$ [15]	4
Variable OSDF	EVAK [15]	$p = \kappa q$ κ varies with r (optimal)	$\begin{array}{ll} n\left(1-\frac{(n-1)(\kappa_n+1)}{n}r\right), & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{n(1-r)}{(n-1)r+1}, & \frac{1}{\kappa_n+1} \leq r \leq 1 \end{array}$, where $\kappa_n = \frac{n}{n-1}$	For fixed (p,q) $(p+q)(p+(n-1)q)$ [15]	

source s and destination $d(s)$, with the remaining terminals in $\mathcal{M} \setminus \{s\}$ acting as potential relays, is envisaged to take place over a collection of orthogonal channels (for eg., orthogonal in frequency), one channel per source terminal in $\mathcal{M}$. The focus in [5] is on OSDF protocols. A terminal in $\mathcal{M} \setminus s$ will decode the source message only if the corresponding source-terminal channel is not in outage and will then proceed to relay the signal to the destination. Two variations of the OSDF protocol are considered, labelled as repetition and space-time coded OSDF protocols respectively. In the repetition-code-based OSDF protocol, each relay node is assigned a distinct time slot in which it repeats the decoded message to the destination. In the space-time coded version, all relays are permitted to transmit simultaneously and are permitted to re-encode the message using independent codebooks. The DMT of the repetition-based OSDF protocol is determined as are bounds on the DMT of the space-time coded version of the protocol.

In [6], Azarian *et al.* analyze the class of NAF protocols, introduced earlier by Nabar *et al.* [16]. The authors show that these NAF protocols have improved performance in comparison with either the OAF protocol presented in [4] or the class of OSDF protocols considered in [5]. The authors also introduce the dynamic decode-and-forward (DDF) protocol where the time for which the relays listen to the source is dependent on the magnitude of the source-relay channel gain coefficient g_i. They show that the DMT of the DDF protocol achieves the transmit diversity bound for $r \leq 0.5$ in the case of a single relay. The authors also consider protocols for the cooperative broadcast and cooperative multiple-access channel and determine the corresponding DMT.

Jing and Hassibi [7] consider an OAF protocol in which relay nodes are permitted to apply a linear transformation to the received signal. The authors restrict attention to the case where both source and relays transmit for equal time durations and where the linear transformations applied by the relays are unitary. Performance is measured in terms of pairwise error probability. Subsequent work on this protocol, including the construction of distributed space-time codes with lesser decoding complexity, can be found in [8].

DMT-optimal codes for the NAF protocol are constructed in [9]. In [10], the same authors consider a new class of NAF protocols called slotted amplify-and-forward (SAF) protocols, and show that these improve upon the performance of the NAF protocol of [6] for the case of two relays. The authors also provide an upper bound on the DMT of the SAF protocol with any number of slots, which tends towards the transmit diversity bound as the number of slots increases. Under the assumption of relay isolation and relay ordering, a SAF scheme proposed in [10] achieves the SAF-protocol upper bound. A general means of constructing DMT-optimal codes can be found here, although the codes so constructed are not necessarily of minimum delay. A key distinction between the SAF and NAF protocols is that whereas the NAF protocol permits relays to only forward signals received by them during the period when no relay is transmitting, the SAF protocol does not place such a restriction.

III. Our Recent Results

Our recent results [15] relate to cooperative relay communication under the OAF, OSDF and NSDF protocols. Our protocols differ from those considered by other researchers in that we permit the duration p, q of the broadcast and cooperation phases respectively, to be chosen arbitrarily.

For all three protocols, we construct codes that are optimal with respect to the corresponding DMT. Our codes are simply constructed, sphere decodable and, in certain cases, incur minimum-possible delay. In the process, we also determine the DMT of the OAF, NSDF and OSDF protocols. Code construction draws from elementary number theory as well as from the theory of cyclic division algebras, see [14], [17]–[19].

Included in our results, is the perhaps surprising finding that the DMT of the OAF protocol is identical to the DMT of the NAF protocol. This point appears to have been overlooked in the earlier literature, possibly because the earlier literature on OAF assumed equal durations for the broadcast and cooperation phases, i.e., $p = q$, whereas it is shown here that the best OAF performance results when the ratio $\frac{p}{q}$ is chosen to equal $\frac{n}{(n-1)}$ where $(n-1)$ is the number of relays.

Two variants of the NSDF protocol are considered in [15], which we term as fixed and variable NSDF. In the variable-NSDF protocol, the parameter p is allowed to vary with the multiplexing gain r of communication. Among the class of static amplify-and-forward and decode-and-forward protocols having a closed-form expression, the variable-NSDF protocol is shown to have the best-known DMT for any number of relays apart from the two-relay case where the SAF protocol has the better performance for lower ranges of r. Our results also establish that the fixed-NSDF protocol has a better DMT than the NAF protocol for any number of relays.

A. Orthogonal Amplify and Forward Protocol

In this protocol, the source S transmits a signal to the relays $\{R_j\}$ and to the destination D for p channel uses. Over the next q channel uses, the relays transmit a linear transformation of the received signal, while the source remains silent.

1) MISO Channel Model: The channel model for the OAF protocol is given by

$$\begin{aligned}
\left[\begin{array}{cc} \underline{y}_1^t & \underline{y}_2^t \end{array}\right] &= \left[\begin{array}{cccc} g_1 & g_2h_2 & \cdots & g_nh_n \end{array}\right] \left[\begin{array}{c|c} \underline{x}^t & \underline{0}^t \\ \underline{0}^t & \underline{x}^t A_2^t \\ \vdots & \vdots \\ \underline{0}^t & \underline{x}^t A_n^t \end{array}\right] \\
&\quad + \left[\begin{array}{cc} \underline{w}_1^t & \sum_{j=2}^{n} h_j \underline{v}_j^t A_j^t + \underline{w}_2^t \end{array}\right] \\
&= HX + \underline{n}^t
\end{aligned}$$

where

- $\underline{x}$ is the signal transmitted by the source
- $[\underline{y}_1^t \ \underline{y}_2^t]$ is the signal received by the destination
- $\{A_j\}$ are $(q \times p)$ matrices that represent the linear transformation taking place at the relay nodes.
- $H = [g_1 \ g_2h_2 \ \cdots \ g_nh_n]$ is the MISO channel seen by the OAF protocol
- and the vectors $\{\underline{v}_j\}_{j=2}^n$ and $\{\underline{w}_1, \underline{w}_2\}$, represent the additive noise seen by the receivers located at the relay nodes and destination respectively.

The code matrix for the OAF protocol is given by

$$X = \left[\begin{array}{c|c} \underline{x}^t & \underline{0}^t \\ \underline{0}^t & \underline{x}^t A_2^t \\ \vdots & \vdots \\ \underline{0}^t & \underline{x}^t A_n^t \end{array}\right].$$

We set the Frobenius norm of the relay matrices $\{A_j\}$ to be $\|A_j\|_F^2 = \alpha_j^2$ in order to constrain the average energy of the signal transmitted by the relay R_j, where the average is computed over statistics of fading coefficients $\{g_i\}$, message-bearing vector $\underline{x}$ and the noise vectors $\{\underline{v}_j\}$.

2) Induced Channel Model: We can rewrite the signal model for the protocol, in matrix form as

$$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} g_1 I_p \\ \sum_{j=2}^{n} g_j h_j A_j \end{bmatrix} \underline{x} + \begin{bmatrix} \underline{w}_1 \\ \sum_{j=2}^{n} h_j A_j \underline{v}_j + \underline{w}_2 \end{bmatrix}.$$

This is the induced channel model, referred before, for the OAF protocol and is useful in computing an upper bound to the DMT of the protocol.

3) General OAF Upper Bound: We proceed to give an upper bound on the DMT of the class of OAF protocols.

Theorem 1: Consider the collection of OAF protocols described above (different protocols can be obtained by varying p, q and $\{A_j\}$ for a given n). Then regardless of the choice of the transformation matrices $\{A_j\}$, the DMT of any protocol satisfies the upper bounds given below.

If $\frac{p}{m} \geq \frac{n}{2n-1}$, where $m = p + q$,

$$d(r) \quad \leq \quad \begin{cases} n\left(1 - \frac{(n-1)mr}{nq}\right), & 0 \leq r \leq \frac{q}{m} \\ \frac{p}{p-q}\left(1 - \frac{mr}{p}\right), & \frac{q}{m} \leq r \leq \frac{1}{2} \\ (1-r), & \frac{1}{2} \leq r \leq 1 \end{cases} . \tag{2}$$

If $\frac{p}{m} \leq \frac{n}{2n-1}$, then

$$d(r) \quad \leq \quad \begin{cases} n(1 - \frac{mr}{p}), & 0 \leq r \leq \frac{(n-1)}{n\frac{m}{p}-1} \\ (1-r), & \frac{(n-1)}{n\frac{m}{p}-1} < r \leq 1 \end{cases} . \tag{3}$$

In deriving these bounds for the protocols, cooperative relaying is avoided whenever it is advantageous to do so.

Also, the highest value of the upper bound on the DMT occurs for the choice $\frac{p}{m} = \frac{n}{2n-1}$. In this case, we get

$$d(r) \quad \leq \quad \begin{cases} n\left(1 - \frac{(2n-1)r}{n}\right), & 0 \leq r \leq \frac{1}{2} \\ (1-r), & \frac{1}{2} < r \leq 1 \end{cases} . \tag{4}$$

4) Specific OAF protocol: Consider a specific OAF protocol with parameters $p = n$ and $q = n - 1$. Choose the $(n-1) \times n$ matrices $\{A_j\}$ as follows:

$$A_j(k,l) = \begin{cases} \alpha_j & k = j-1, l = j \\ 0 & \text{elsewhere} \end{cases} , \tag{5}$$

i.e., the $(j-1, j)^{\text{th}}$ entry of A_j is equal to α_j and remaining entries are 0. The DMT of this specific protocol meets the upper bound in (4) (see [15]).

5) DMT Optimal Codes for the OAF Protocol: In this subsection, we provide an explicit construction of a DMT optimal code, based on CDAs, for the OAF protocol for any number of relays. If the number of relays is $n-1$, we choose $p = n$ and $q = (n-1)$ since this choice of parameters has the best DMT (see Theorem 1).

For M even, let $\mathcal{A}_{\text{QAM}}$ denote the M^2-QAM constellation given by

$$\mathcal{A}_{\text{QAM}} = \{a + \imath b \mid |a|, |b| \leq M-1, \ a, b \text{ odd}\}.$$

Consider a CDA having center $\mathbb{F} = \mathbb{Q}(\imath)$ and maximum subfield $\mathbb{L}$ that is a degree-n cyclic Galois extension $\mathbb{L}/\mathbb{F}$ of $\mathbb{F}$. Let σ be the generator of the cyclic Galois group $\text{Gal}(\mathbb{L}/\mathbb{F})$. Let $\mathcal{O}_{\mathbb{F}}$ and $\mathcal{O}_{\mathbb{L}}$ denote the ring of algebraic integers in $\mathbb{F}$ and $\mathbb{L}$ respectively. It is known that $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}[\imath]$. Let $\{\beta_1, \ldots, \beta_n\}$ be an integral basis for $\mathcal{O}_{\mathbb{L}}/\mathcal{O}_{\mathbb{F}}$. Let $\text{D}(\mathbb{L}/\mathbb{F}, \sigma, \gamma)$ denote the associated CDA. The interested reader can see [14] for an introduction to CDAs.

Let

$$\ell_i \in \mathcal{A}_{\text{QAM}}(\beta_1, \ldots, \beta_n) \tag{6}$$

where

$$\mathcal{A}_{\text{QAM}}(\beta_1, \ldots, \beta_n) = \left\{ \sum_i a_i \beta_i \mid a_i \in \mathcal{A}_{\text{QAM}} \right\}.$$

Now to specify a code we should identify the parameters $(p, q, \{A_j\})$. We set $p = n, \ q = n - 1$ and select $\{A_j\}$ as specified in (5). Without loss of generality (insofar as DMT is concerned), for simplicity we set

$$\alpha_j = 1, \quad 2 \leq j \leq n.$$

Let the signal transmitted by the source in the first $p = n$ channel uses be given by

$$\underline{x} = \theta \begin{bmatrix} \ell_0 & \sigma(\ell_0) & \ldots & \sigma^{n-1}(\ell_0) \end{bmatrix}^t$$

where $\ell_0 \in \mathcal{A}_{\text{QAM}}(\beta_1, \ldots, \beta_n)$ and θ is a normalizing parameter. The relay $R_j, \ 2 \leq j \leq n$, then transmits the signal $A_j \underline{x}$ in the next $q = n-1$ channel uses. Hence, the code matrices will be as shown in (7). This code is DMT optimal for the OAF protocol.

Remark 1: In this subsection, we have used CDA based codes to construct a DMT optimal code for the OAF protocol. However, ℓ_0 need not be drawn from a maximal subfield $\mathbb{L}$ of a division algebra. It is enough if $\mathbb{L}$ is an algebraic extension (not necessarily cyclic) of $\mathbb{Q}[\imath]$ of degree n and ℓ_0 is as defined in (6). In this case, $\{\sigma(\ell_0), \ldots, \sigma^{n-1}(\ell_0)\}$ would be replaced by the appropriate conjugates of ℓ_0.

Example 1: Let the number of relays be 2 so that $n = 3$. We choose $p = 3$, $q = 2$ and

$$A_2 = \begin{bmatrix} 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}.$$

and the code matrices are given by

$$X = \theta \left[\begin{array}{ccc|cc} \ell_0 & \sigma(\ell_0) & \sigma^2(\ell_0) & 0 & 0 \\ 0 & 0 & 0 & \sigma(\ell_0) & 0 \\ 0 & 0 & 0 & 0 & \sigma^2(\ell_0) \end{array} \right],$$

where $\ell_0 \in \mathcal{A}_{\text{QAM}}(\beta_1, \beta_2, \beta_3)$.

For these parameters, the DMT of the OAF protocol is

$$d(r) = \begin{cases} 3 - 5r & , \quad 0 \leq r \leq \frac{1}{2} \\ 1 - r & , \quad \frac{1}{2} < r \leq 1 \end{cases} .$$

$$X = \theta \left[\begin{array}{cccc|ccc} \ell_0 & \sigma(\ell_0) & \cdots & \sigma^{n-1}(\ell_0) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \sigma(\ell_0) & & \\ \vdots & \ddots & & \vdots & & \ddots & \\ 0 & & \ddots & 0 & & & \sigma^{n-1}(\ell_0) \end{array} \right], \quad \text{where } \ell_0 \in \mathcal{A}_{\text{QAM}}(\beta_1, \cdots, \beta_n). \tag{7}$$

B. Non-Orthogonal Selection Decode and Forward Protocol

In the non-orthogonal selection decode-and-forward (NSDF) protocol, the source transmits a signal to the destination and the relays for p channel uses in the first phase. All the relays, which are not in outage[3], will decode the source message and participate in the second phase. In the second phase, the relays will separately encode and transmit a vector of length q. The source continues to transmit to the destination in the second phase.

We only consider the case when $p \geq q$. To compute the best possible DMT, we allow p and q to vary with the multiplexing gain r and choose the value of $\kappa = \frac{p}{q}$ which maximizes the DMT for a given r. This version of the protocol will be called the variable-NSDF protocol. We also compute the DMT of the fixed-NSDF protocol, wherein the ratio $\kappa = \frac{p}{q}$ is fixed for all r, and construct CDA based codes which achieve the DMT of the variable and fixed NSDF protocols.

1) DMT of NSDF Protocol:

Theorem 2: The DMT of the variable-NSDF protocol is given by

$$d(r) = \begin{cases} n\left(1 - \frac{(n-1)(\kappa_n+1)}{n} r\right), & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{(n-r)(1-r)}{(n-2)r+1}, & \frac{1}{\kappa_n+1} \leq r \leq 1 \end{cases}, \tag{8}$$

where $\kappa_n = \frac{1+\sqrt{1+4(n-1)^2}}{2(n-1)}$.

In deriving the above DMT, we have allowed p and q to vary with the multiplexing gain r. The source and the relays choose a code corresponding to each p and q. For the case $p \geq q$, we select the value of (p, q) which maximizes the DMT for a given r. Suppose $p = \kappa q$, then the optimal value of κ is given by

$$\kappa = \begin{cases} \kappa_n, & 0 \leq r \leq \frac{1}{\kappa_n+1} \\ \frac{1+(n-2)r}{(n-1)(1-r)}, & \frac{1}{\kappa_n+1} < r \leq 1 \end{cases}.$$

For a fixed choice $\kappa = \frac{p}{q}$, the DMT of the fixed-NSDF protocol is given by:

if $1 \leq \kappa \leq \kappa_n$,

$$d(r) = (n-1)\left(1 - \frac{mr}{p}\right)^+ + (1-r), \quad 0 \leq r \leq 1, \tag{9}$$

else if $\kappa \geq \kappa_n$,

$$d(r) = \begin{cases} n\left(1 - \frac{m(n-1)}{nq} r\right), & 0 \leq r \leq \frac{q}{m} \\ \frac{m}{p}(1-r), & \frac{q}{m} \leq r \leq \frac{np-m}{(n-2)m+p} \\ n\left(1 - \frac{(n-1)m+p}{np} r\right), & \frac{np-m}{(n-2)m+p} \leq r \leq \frac{p}{m} \\ 1-r & \frac{p}{m} \leq r \leq 1 \end{cases}. \tag{10}$$

Surprisingly, for the case of one relay the optimal ratio κ_2 turns out to be the Golden Number, $\kappa_2 = \frac{1+\sqrt{5}}{2}$!

2) DMT Optimal Codes for NSDF Protocol: In this subsection, we construct a DMT optimal code for the NSDF protocol. We shall use cyclic division algebras (CDA) to construct ST codes and derive a code for the NSDF protocol from the set of matrices comprising the ST code. For a fixed $\kappa = \frac{p}{q}$, where p and q are relatively prime, we outline the construction of a ST code when there are $(n-1)$ relays in the cooperative network[4].

Let $t = p + nq$. Consider a CDA having center $\mathbb{F} = \mathbb{Q}(\imath)$ and maximum subfield $\mathbb{L}$ that is a degree-t cyclic Galois extension $\mathbb{L}/\mathbb{F}$ of $\mathbb{F}$. Let $\mathrm{D}(\mathbb{L}/\mathbb{F}, \sigma, \gamma)$ denote the associated CDA. Consider the space time code $\mathcal{X}$ comprising of matrices corresponding to the left-regular representation of all the elements in the CDA D. Let $\mathcal{Z}$ denote the normalized code

$$\mathcal{Z} = \{\theta X \mid X \in \mathcal{X}\}$$

where θ is chosen to ensure that

$$\|\theta X\|_F^2 \leq t\rho, \quad \text{for all} \quad X \in \mathcal{X}.$$

The transmitted code matrix, denoted by Z, will be of the form

$$\theta \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_{t-1}) & \gamma\sigma^2(\ell_{t-2}) & \ldots & \gamma\sigma^{t-1}(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \gamma\sigma^2(\ell_{t-1}) & \ldots & \gamma\sigma^{t-1}(\ell_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{t-1} & \sigma(\ell_{t-2}) & \sigma^2(\ell_{t-3}) & \ldots & \sigma^{t-1}(\ell_0) \end{bmatrix},$$

where $\ell_i \in \mathcal{A}_{\text{QAM}}(\beta_1, \cdots, \beta_t)$.

In the first phase the source transmits the first p rows of Z and in the second phase the source transmits the next q rows of Z. The j^{th} relay R_j, $2 \leq j \leq n$, if not in outage, will decode the p rows transmitted by the source and will transmit q rows of Z numbering from $p + (j-1)q + 1$ to $p + jq$.

This code is DMT optimal for the fixed-NSDF protocol. Total delay of the DMT optimal code will be $(p+q)(p+nq)$ time slots. By constructing a DMT optimal code for each value

[3] We say that a relay is not in outage if the corresponding source-relay channel is not in outage.

[4] We are considering only rational values of κ here although, while computing the DMT, κ was allowed to take on irrational values too.

of the ratio $\kappa = \frac{p}{q}$, we can construct DMT optimal codes for the variable-NSDF protocol.

Example 2: Let the number of relays be n. We set $p = q = 1$. The DMT for the NSDF protocol is then given by,

$$d(r) = (1-r)^+ + (1-2r)^+ \ , \quad 0 \le r \le 1.$$

The transmitted code matrices are simply-described and are given by

$$X = \begin{bmatrix} \ell_0 & \gamma\sigma(\ell_n) & \dots & \gamma\sigma^n(\ell_1) \\ \ell_1 & \sigma(\ell_0) & \dots & \gamma\sigma^n(\ell_2) \\ \ell_2 & \sigma(\ell_1) & \dots & \gamma\sigma^n(\ell_3) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_n & \sigma(\ell_{n-1}) & \dots & \sigma^n(\ell_0) \end{bmatrix} \begin{matrix} \text{Source} \\ \text{Source} \\ R_2 \\ \vdots \\ R_n \end{matrix} ,$$

i.e., the source is given the first two rows of the code matrix and every relay node is given one of the remaining rows of the code matrix.

C. Orthogonal Selection Decode and Forward Protocol

The OSDF protocol is the same as the NSDF protocol, except that the source remains silent in the second phase.

Theorem 3: The DMT of the variable-OSDF protocol is given by

$$d(r) = \begin{cases} n\left(1 - \frac{2n-1}{n} r\right) , & 0 \le r \le \frac{n-1}{2n-1} \\ \frac{n(1-r)}{(n-1)r+1} , & \frac{n-1}{2n-1} \le r \le 1 \end{cases} .$$

We allow p and q to vary with the multiplexing gain r. We choose the value of (p, q) which maximizes the diversity, for a given r, for the case $p \ge q$. Suppose $p = \kappa q$, then the optimal value of κ is given by

$$\kappa = \begin{cases} \frac{n}{n-1} , & 0 \le r \le \frac{n-1}{2n-1} \\ \frac{1+(n-1)r}{(n-1)(1-r)} , & \frac{n-1}{2n-1} < r \le 1 \end{cases} .$$

For a fixed choice $\kappa = \frac{p}{q}$, the DMT of the fixed-OSDF protocol is given by:

if $1 \le \kappa \le \frac{n}{n-1}$,

$$d(r) = \begin{cases} n\left(1 - \frac{mr}{p}\right) , & 0 \le r \le \frac{(n-1)p}{nm-p} \\ (1-r) , & \frac{(n-1)p}{nm-p} \le r \le 1 \end{cases}$$

else if $\kappa \ge \frac{n}{n-1}$,

$$d(r) = \begin{cases} n\left(1 - \frac{m(n-1)}{nq} r\right) , & 0 \le r \le \frac{q}{m} \\ \frac{m}{p}(1-r) , & \frac{q}{m} \le r \le \frac{np-m}{m(n-1)} \\ n\left(1 - \frac{mr}{p}\right) , & \frac{np-m}{m(n-1)} \le r \le \frac{(n-1)p}{nm-p} \\ (1-r) , & \frac{(n-1)p}{nm-p} \le r \le 1 \end{cases} .$$

An approximately universal CDA code of dimension $(p + (n-1)q) \times (p + (n-1)q)$, where $(n-1)$ is the number of relays, will be DMT optimal for the OSDF protocol. The transmission of various rows of the code matrices by the source and the relays will be on similar lines to that mentioned in Section III-B.2 for the NSDF protocol.

IV. Plots and Tables

The DMT of the various protocols for the case of a single relay (corresponding to $n = 2$) and two relays (corresponding to $n = 3$) are presented in Fig. 2 and Fig. 3 respectively.

In Tables I and II, we present the DMTs of the various protocols. We also provide information relating to DMT-optimal code construction whenever such an explicit code construction exists. Also shown in the tables are the delays incurred by the DMT-optimal distributed space-time codes as well as corresponding values of the parameters p and q assumed in the code construction. The following remarks are made with respect to Tables I and II:

Remark 2: Among the class of OAF protocols, the best DMT is achieved when $\frac{p}{q} = \frac{n}{n-1}$. Since n and $n-1$ are relatively prime, the code construction in [15] has minimum delay possible $= (2n-1)$.

When each node in the system has only one transmit and one receive antenna, the DMT optimal code for the OAF protocol has lesser delay than the code proposed in [9] for the NAF protocol.

Remark 3: The construction of the DMT-optimal code for the NAF protocol proposed in [6] can be found in [9].

Remark 4: The expression for DMT for the case of two relay antennas in [10] is derived under the assumption of relay ordering. The upper bound for the general, $(n-1)$-relay case given in the table is achieved by a code provided in [10] under the assumption of relay isolation, i.e., under the assumption that the signal transmitted by one relay cannot be heard by a second relay.

Remark 5: In [15], we determine the DMT of fixed and variable versions of OSDF and NSDF and in addition, construct optimal, simply-described, codes for these protocols. The word "optimal" in the table indicates that for each r, we have optimally chosen p, q. The DMT in the fixed case has slightly worse performance in comparison with that of the corresponding variable-SDF protocol. However, the fixed SDF protocols have the advantage of not requiring adjustment of the parameters p, q if r is varied.

When $p = q = 1$, the DMT of the fixed-NSDF protocol coincides with that of the NAF protocol. However, the DMT optimal code for the fixed-NSDF protocol has a delay $2(n+1)$ which is shorter than the delay for the DMT optimal codes for the NAF protocol constructed in [9] for $n \ge 3$. The codes in [9] have delay $4(n-1)$.

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