Asynchronous Iterative Waterfilling for Gaussian Frequency-Selective Interference Channels: A Unified Framework

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***Abstract*— In this paper we give an overview of recent results on the rate maximization game in the Gaussian frequency-selective interference channel. We focus on the competitive maximization of information rates, subject to global power and spectral mask constraints.**

To achieve the so-called Nash equilibrium points of the game Yu, Ginis and Cioffi proposed the *sequential* Iterative Waterfilling Algorithm (IWFA), where, at each iteration, the users choose, one after the other, their power allocation to maximize their own information rate, treating the interference generated by the others as additive colored Gaussian noise. To overcome the potential slow convergence of the sequential update, specially when the number of users is large, the *simultaneous* IWFA was proposed by the authors, where, at each iteration, all the users update their power allocations simultaneously, rather than sequentially. Recently, the authors showed that both the sequential and the simultaneous IWFAs are just special cases of a more general unified framework, given by the *totally asynchronous* IWFA. In this more general algorithm, the users update their power spectral density in a completely distributed and asynchronous way. Furthermore, the asynchronous setup includes another form of lack of synchronism where the transmission by the different users contains time and frequency synchronization offsets. A unified set of convergence conditions were provided for the whole class of algorithms obtained from the asynchronous IWFA.

Interestingly, there is a key result used in the proof of convergence of the algorithms: an alternative interpretation of the waterfilling operator as a projector

I. Introduction

In this paper we focus on the frequency selective interference channel with Gaussian noise. The capacity region of the interference channel is still unknown, even for the simplest Gaussian two-user case [1]. Only some bounds are available (see, e.g., [2] for a summary of the known results about the Gaussian interference channel). A pragmatic approach that leads to an achievable region or inner bound of the capacity region is to restrict the system to operate as a set of independent units, i.e., not allowing multiuser encoding/decoding or the use of interference cancelation techniques. This achievable region is very relevant in practical systems with limitations on the decoder complexity and simplicity of the system. With this assumption, the multiuser interference is treated as noise and the transmission strategy for each user is simply its power allocation. The system design reduces then to finding the optimum Power Spectral Density (PSD) for the users according to some performance measure.

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The results existing in the current literature [3]-[17] have dealt with the maximization of the information rates of all the links, subject to individual transmit power and (possibly) spectral mask constraints. The latter constraint is motivated by current regulations that impose strict restrictions on the usage of certain frequency bands, in order to limit the amount of interference that each transmitter can generate. In [3]-[5] a centralized approach based on duality theory [18] was proposed to compute, under technical conditions, the largest achievable rate region of the system (i.e., the Pareto-optimal set of the achievable rates). Our interest is focused on distributed algorithms with no centralized control; therefore, we formulate the system design under the convenient framework of game theory. In particular, we formulate the rate maximization problem as a strategic non-cooperative game, where every link is a player that competes against the others by choosing the signaling that maximizes his own information rate. An equilibrium for the whole system is reached when every player is unilaterally optimum, i.e., when, given the current strategies of the others, any change in his own strategy would result in a rate loss. This equilibrium constitutes the celebrated notion of Nash Equilibrium (NE) in game theory [19].

All the NEs of the rate maximization game can be reached using Gaussian signaling and a proper PSD from each user [9], [13], [14]. To obtain the optimal PSD of the users, Yu, Ginis, and Cioffi proposed the *sequential* Iterative WaterFilling Algorithm (IWFA) [6] in the context of DSL systems, modeled as a Gaussian frequency-selective interference channel. The algorithm is an instance of the Gauss-Seidel scheme [20]: the users maximize their own information rates *sequentially* (one after the other), according to a fixed updating order. Each user performs the single-user waterfilling solution given the interference generated by the others as additive (colored) noise. The most appealing features of the sequential IWFA are its low-complexity and its distributed nature. In fact, to compute the waterfilling solution, each user only needs to measure the noise-plus-interference PSD, without requiring specific knowledge of the power allocations and the channel transfer functions of all other users.

The convergence of the sequential IWFA has been studied in a number of works [7]-[13], each time obtaining milder conditions that guarantee convergence. However, despite its appealing properties, the sequential IWFA may suffer from slow convergence if the number of users in the network is large because of the sequential updating strategy. In addition, the algorithm requires some form of central scheduling to determine the order in which users are updated.

To overcome the drawback of slow speed of convergence, the *simultaneous* IWFA was proposed in [9], [13], [14]. The simultaneous IWFA is an instance of the Jacobi scheme [20]: at each iteration, the users update their own strategies *simultaneously*, still according to the waterfilling solution, but using the interference generated by the others in the *previous* iteration. The simultaneous IWFA was shown to converge to the unique NE of the rate maximization game faster than the sequential IWFA and under weaker conditions on the multiuser interference than those given in [6], [7], [11], [12], [15] for the sequential IWFA. Furthermore, differently from [6], [7], [11], [15], the algorithm takes explicitly into account spectral masks constraints. However, the simultaneous IWFA still requires some form of synchronism as all the users need to be simultaneously updated. In a real network with many users, the form of synchronism requirements of the sequential and simultaneous IWFAs may not be feasible.

Recently, in [16], [17], the authors showed that both the sequential and the simultaneous IWFAs are just special cases of a more general unified framework, given by the *totally asynchronous* IWFA. In this more general algorithm, all users still update their power allocations according to the waterfilling solution, but the updates can be performed in a *totally asynchronous* way (in the sense of [20]). This means that some users may update their PSDs *more frequently* than others and they may even use an *outdated* measurement of the interference caused from the others. These features make the asynchronous IWFA appealing for all practical scenarios, either wired or wireless, where strong constraints on synchronization cannot be met. Furthermore, the asynchronous setup considered in [16], [17] includes also another form of lack of synchronism where the transmission by uncoordinated users contains time and frequency synchronization offsets, due to mismatch between the oscillators of different transmitters, propagation delays, Doppler effects, etc.. A unified set of convergence conditions were provided for the whole class of algorithms obtained from the asynchronous IWFA, as special cases. These convergence conditions enlarge those given in [6], [7], [11], [12], [15] for the sequential IWFA.

Interestingly, in [13], [16], [17], there is a common thread relating the algorithms and the derivation of their convergence conditions: the interpretation of the waterfilling operator as the Euclidean projector of a vector onto a convex set. In the single-user case, this provides an alternative perspective of the well-known waterfilling solution, that dates back to Shannon in 1949 [21]. Interestingly, in the multiuser case, this interpretation plays a key role in proving the convergence of the proposed algorithms.

The paper is organized as follows. Sec. II gives the system model and Sec. III formulates the optimization problem as a strategic non-cooperative game. Sec. IV provides the interpretation of the waterfilling operator as a projector. Sec. V contains the state of the art of distributed algorithms able to reach the NE of the game, along with their convergence properties. Sec. VI shows how to modify the original game theoretic formulation to take explicitly into account the effect of time and/or frequency synchronization errors in the system. Finally, Sec. VII draws the conclusions.

II. System Model

We consider a Gaussian frequency-selective interference channel composed by multiple links. Aiming at finding distributed algorithms, we focus on transmission techniques where no interference cancelation is performed and multiuser interference is treated as additive colored noise from each receiver. To deal easily with the frequency-selectivity of the channel, we adopt a multicarrier approach without loss of optimality (since it is a capacity-lossless structure for sufficiently large block length [22], [23]). Given the above system model, we make the following assumptions:

A.1 Each channel changes sufficiently slowly and thus can be considered fixed during the whole transmission, so that the information theoretic results are meaningful;

A.2 The channel from each source to its own destination is known to the intended receiver, but not to the other terminals, and each receiver is assumed to measure with no errors the PSD of the noise plus the interference due to the other links. Based on this information, each destination computes the optimal signaling for its own link and transmits it back to its transmitter through a low bit rate (error-free) feedback channel.[1]

A.3 *All* the users are block-synchronized with an uncertainty at most equal to the cyclic prefix length. We will relax these assumption in Sec. VI, where we will explicitly take into account the effect of time and/or frequency offsets.

We consider the following power constraints, required by different systems. For each transmitter q:

Co.1 Maximum overall transmit power:

$$\sum_{k=0}^{N-1} \bar{p}_q(k) \leq N P_q, \tag{1}$$

where $\bar{p}_q(k)$ denotes the power allocated by user q over carrier k, and P_q is power in units of energy per transmitted symbol.

Co.2 Spectral mask constraints:

$$\bar{p}_q(k) \leq \bar{p}_q^{\max}(k), \quad k = 0, \ldots, N-1, \tag{2}$$

where $\bar{p}_q^{\max}(k)$ represents the maximum power that is allowed to be allocated on the k-th frequency bin from the q-th user. Constraints like in (2) are imposed by current regulations and attempts to limit the amounts of interference generated by each transmitter at the other systems' receivers.

[1]In practice, both measurement and feedback are inevitably affected by errors. This scenario can be studied by extending our formulation to games with partial information [24], [25], but this goes beyond the scope of the present paper.

III. Problem Formulation as a Game

We formulate the *joint* maximization of mutual information on each link as a strategic non-cooperative game [24], [25], in which the players are the links and the payoff functions are the information rates on the links: Each player competes rationally[2] against the others by choosing the signaling (i.e. its strategy) that maximizes its own rate, given constraints on the transmit power and spectral masks. A NE of the game is reached when each user, given the strategy profile of the others, does not get any rate increase by changing its own strategy.

Under the signal model described in Sec. II, the achievable rate for each player q is computed as the maximum information rate on the q-th link, assuming *the other received signals as additive noise*. It is straightforward to see that a (pure or mixed strategy) NE is obtained if each user transmits using Gaussian signaling, with a proper PSD [9], [14], [13]. Hence, the maximum achievable rate for the q-th user is given by [22]

$$R_q = \frac{1}{N}\sum_{k=0}^{N-1}\log\left(1+\mathsf{sinr}_q(k)\right), \qquad (3)$$

with $\mathsf{sinr}_q(k)$ denoting the Signal-to-Interference plus Noise Ratio (SINR) on the k-th carrier for the q-th link:

$$\mathsf{sinr}_q(k) \triangleq \frac{\left|H_{qq}(k)\right|^2 p_q(k)}{\sigma^2_{w_q}(k)+\sum_{r\neq q}\left|H_{rq}(k)\right|^2 p_r(k)}, \qquad (4)$$

where $H_{rq}(k) = \bar{H}_{rq}(k)\sqrt{P_q/d_{rq}^{\gamma}}$, with $\bar{H}_{rq}(k)$ denoting the normalized frequency-response of the channel between source r and destination q; d_{rq} is the distance between source r and destination q, and γ is the path loss exponent; $p_q(k) = \bar{p}_q(k)/P_q$ is the normalized power allocated by the q-th user over the k-th subcarrier, subject to the spectral mask constraints $p_q(k) \leq p_q^{\max}(k)$ with $p_q^{\max}(k) = \bar{p}_q^{\max}(k)/P_q$, and the power constraint $(1/N)\sum_{k=0}^{N-1}p_q(k)\leq 1$.

Observe that in the case of practical coding schemes, where only finite order constellations can be used, we can use the gap approximation analysis [26], [27] and write the number of bits transmitted over the N substreams from the q-th source still as in (3) (for a given family of constellations and a given error probability), simply replacing $\left|H_{qq}(k)\right|^2$ in (4) with $\left|H_{qq}(k)\right|^2/\Gamma_q$, where $\Gamma_q \geq 1$ is the gap.

In summary, we have the following structure for the game:

$$\mathscr{G} = \left\{\Omega, \{\mathscr{P}_q\}_{q\in\Omega}, \{R_q\}_{q\in\Omega}\right\}, \qquad (5)$$

where $\Omega \triangleq \{1,2,\ldots,Q\}$ denotes the set of the active links, $\mathscr{P}_q$ is the set of admissible (normalized) power allocation strategies, across the N available carriers, for the q-th player, defined as[3]

$$\mathscr{P}_q \triangleq \left\{\mathbf{p}_q\in\mathbb{R}_+^N : \frac{1}{N}\sum_{k=0}^{N-1}p_q(k)=1,\ p_q(k)\leq p_q^{\max}(k),\ \forall k\right\}, \qquad (6)$$

and R_q is the payoff function of the q-th player, defined in (3).

The optimal strategy for the q-th player, given the power allocation of the others, is then the solution to the following maximization problem

$$\begin{array}{ll}\underset{\mathbf{p}_q}{\text{maximize}} & \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1}\log\left(1+\ \mathsf{sinr}_q(k)\right), \qquad \forall q\in\Omega \\ \text{subject to} & \mathbf{p}_q\in\mathscr{P}_q\end{array} \qquad (7)$$

where $\mathsf{sinr}_q(k)$ and $\mathscr{P}_q$ are given in (4) and (6), respectively. Note that, for each q, the maximum in (7) is taken over $\mathbf{p}_q$, for a *fixed* $\mathbf{p}_{-q} \triangleq (\mathbf{p}_1,\ldots,\mathbf{p}_{q-1},\mathbf{p}_{q+1},\ldots,\mathbf{p}_Q)$.

The solutions of (7) are the well-known Nash Equilibria, which are formally defined as follows [19], [24], [25].

Definition 1: A (pure) strategy profile $\mathbf{p}^\star = (\mathbf{p}_1^*,\ldots,\mathbf{p}_Q^*) \in \mathscr{P}_1\times\ldots\times\mathscr{P}_Q$ is a Nash Equilibrium of the game $\mathscr{G}$ in (5) if $R_q(\mathbf{p}_q^\star,\mathbf{p}_{-q}^\star)\geq R_q(\mathbf{p}_q,\mathbf{p}_{-q}^\star), \forall\mathbf{p}_q\in\mathscr{P}_q, \forall q\in\Omega$.

The definition of NE as given in Definition 1 can be generalized to contain mixed strategies [24], i.e. the possibility of choosing a randomization over a set of pure strategies (the randomizations of different players are independent). However, it is straightforward to see that one can indeed limit himself to adopt pure strategies w.l.o.g., since all the NEs of the game $\mathscr{G}$ in (5) are reached using pure strategies [9], [10], [13]. This follows directly from the strict concavity of each rate function $R_q(\mathbf{p}_q,\mathbf{p}_{-q})$ in $\mathbf{p}_q$ and the structure of the joint admissible strategy set of the players, i.e., $\mathscr{P}=\mathscr{P}_1\times,\cdots,\times\mathscr{P}_Q$.

According to (7), all the (pure) NEs of the game, if they exist, must satisfy the waterfilling solution *for each* user, i.e. the following system of *nonlinear* equations:

$$\mathbf{p}_q^\star = \mathsf{WF}_q\left(\mathbf{p}_1^\star,\ldots,\mathbf{p}_{q-1}^\star,\mathbf{p}_{q+1}^\star,\ldots,\mathbf{p}_Q^\star\right), \quad \forall q\in\Omega, \qquad (8)$$

with the waterfilling operator $\mathsf{WF}_q(\cdot)$ defined as

$$[\mathsf{WF}_q]_k \triangleq \left[\mu_q - \mathsf{insr}_q(k)\right]_0^{p_q^{\max}(k)},\ k=0,\ldots,N-1, \qquad (9)$$

where the symbol $[\cdot]_a^b$, with $b\geq a$ denotes the Euclidean projection on the interval $[a,b]$, and $\mathsf{insr}_q(k)$ is defined as

$$\mathsf{insr}_q(k) \triangleq \frac{\sigma^2_{w_q}(k)+\sum_{r\neq q}\left|H_{rq}(k)\right|^2 p_r(k)}{\left|H_{qq}(k)\right|^2}. \qquad (10)$$

The water-level μ_q in (9) is chosen to satisfy the power constraint $(1/N)\sum_{k=0}^{N-1}p_q^\star(k)=1$.

Observe that in the absence of spectral mask constraints (i.e. when $p_q^{\max}(k)=+\infty$, $\forall q$, $\forall k$), the NEs of the game $\mathscr{G}$ in (5) are given by the classical simultaneous waterfilling

[2] The rationality assumption means that each user will never chose a strictly dominated strategy. A strategy profile $\mathbf{x}_q$ is strictly dominated by $\mathbf{z}_q$ if $\Phi_q(\mathbf{x}_q,\mathbf{y}_{-q}) < \Phi_q(\mathbf{z}_q,\mathbf{y}_{-q})$, for a given admissible $\mathbf{y}_{-q} \triangleq \mathbf{y}_1,\ldots,\mathbf{y}_{q-1},\mathbf{y}_{q+1},\ldots,\mathbf{y}_Q$, where Φ_q denotes the payoff function of player q.

[3] In order to avoid the trivial solution $p^\star(k) = p^{\max}(k)$ for all k, $\sum_{k=0}^{N-1}p^{\max}(k) > N$ is assumed. Furthermore, in the feasible strategy set of each player, we can replace, w.l.o.g., the original *inequality* power constraint in (1) with equality, since, at the optimum, this constraint must be satisfied with equality.

solutions [6], [7], where $\mathsf{WF}_q(\cdot)$ in (8) is still obtained from (9) simply setting $p_q^{\max}(k) = +\infty$, $\forall q$, $\forall k$. In this special case, the game $\mathscr{G}$ in (5) is usually referred to in the literature as the Gaussian Interference Game [6], [7], and alternative (sufficient) conditions for the existence and uniqueness of a NE are given in [6], [7], [9], [11], [12].

In the presence of spectral mask constraints, the derivations in [6], [7], [11] cannot be applied and thus a solution for the system of nonlinear equations (8) can not be guaranteed for any set of channels and spectral masks. However, the following Proposition, whose proof comes directly from standard results of game theory [24], [25], provides a positive answer on the existence of a NE for the game $\mathscr{G}$ in (5).

Proposition 1 ([9], [13]): The game $\mathscr{G}$ in (5) always admits at least one NE in pure-strategies, for any set of channel realizations, power and spectral mask constraints.

Once proved that a NE always exists, the problem of how to reach such an equilibrium arises. We address this issue in the forthcoming sections. By direct product of our derivations, we also provide sufficient conditions for the uniqueness of the equilibrium. We refer the reader to [13] for more general conditions for the uniqueness of the NE.

IV. Waterfilling Solution as a Projection

In this section we provide an interpretation of the waterfilling operator as a proper Euclidean projector. This interpretation is the key result to prove the convergence properties of the distributed algorithms described in the subsequent sections [13], [16], [17].

A. A New Look at the Single-user Waterfilling Solution

Consider a parallel additive colored Gaussian noise channel composed of N subchannels with coefficients $\{H(k)\}$, subject to some spectral mask constraints $\{p^{\max}(k)\}$ and to a global average transmit power constraint across the subchannels. It is well-known that the capacity-achieving solution for this channel is obtained using independent Gaussian signaling across the subchannels with the following waterfilling power allocation [17]

$$p^{\star}(k) = \left[\mu - \frac{\sigma_k^2}{|H(k)|^2}\right]_0^{p^{\max}(k)}, \quad k = 0, \ldots, N-1, \quad (11)$$

where σ_k^2 denotes the variance of the noise on the k-th subchannel, $p^{\star}(k)$ is the optimal power allocation over the k-th subchannel. The water-level μ in (11) is chosen in order to satisfy the power constraint $(1/N)\sum_{k=0}^{N-1} p^{\star}(k) = 1$.

In [13] we showed that, interestingly, the solution in (11) can be interpreted as the Euclidean projection of the vector $-\mathsf{\mathbf{insr}}$, defined as

$$\mathsf{\mathbf{insr}} \triangleq \left[\sigma_0^2/|H(0)|^2, \ldots, \sigma_{N-1}^2/|H(N-1)|^2\right]^T \quad (12)$$

onto the simplex

$$\mathcal{S} \triangleq \left\{ \mathbf{x} \in \mathbb{R}^N : \frac{1}{N}\sum_{k=0}^{N-1} x_k = 1,\ 0 \le x_k \le p^{\max}(k), \quad \forall k, \right\}. \quad (13)$$

Lemma 1 ([13]): The Euclidean projection of the N-dimensional real nonpositive vector $-\mathbf{x}_0 \triangleq -[x_{0,0}, \ldots, x_{0,N-1}]^T$ onto the simplex $\mathcal{S}$ defined in (13), denoted by $[-\mathbf{x}_0]_{\mathcal{S}}$, is by definition the solution to the following convex optimization problem:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \|\mathbf{x} - (-\mathbf{x_0})\|_2^2 \\ \text{subject to} & 0 \le x_k \le p^{\max}(k), \quad k = 0, 1, \ldots, N-1, \\ & \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} x_k = 1. \end{array} \quad (14)$$

whose optimal solution is

$$x_k^{\star} = [\mu - x_{0,k}]_0^{p^{\max}(k)}, \quad k = 0, 1, \ldots, N-1, \quad (15)$$

where $\mu > 0$ is chosen to satisfy the constraint $(1/N)\sum_{k=0}^{N-1} x_k^{\star} = 1$.

Corollary 1: The waterfilling solution $\mathbf{p}^{\star} = [p^{\star}(0), \ldots, p^{\star}(N-1)]^T$ in (11) can be expressed as the projection of $-\mathsf{\mathbf{insr}}$ given in (12) onto the simplex $\mathcal{S}$ in (13):

$$\mathbf{p}^{\star} = [-\mathsf{\mathbf{insr}}]_{\mathcal{S}}. \quad (16)$$

Corollary 2: The waterfilling solution in the form

$$p^{\star}(k) = \left[\frac{\mu}{w_k} - \frac{\sigma_k^2}{|H(k)|^2}\right]_0^{p^{\max}(k)}, \quad k = 0, \ldots, N-1, \quad (17)$$

where $\mathbf{w} = [w_0, \ldots, w_{N-1}]^T$ is any positive vector, can be expressed as the projection with respect to the weighted Euclidean norm[4] with weights $w_0, \ldots, w_{N-1}$, of $-\mathsf{\mathbf{insr}}$ given in (12) onto the simplex $\mathcal{S}$ in (13):

$$\mathbf{p}^{\star} = [-\mathsf{\mathbf{insr}}]_{\mathcal{S}}^{\mathbf{w}}. \quad (18)$$

The graphical interpretation of the waterfilling solution as a Euclidean projector, for the single-user two-carriers case, is given in Fig. 1: for any $\mathbf{insr} \equiv (\mathrm{insr}_1, \mathrm{insr}_2)$ corresponding to a point in the interior of the gray region (e.g., point A), the waterfilling solution allocates power over both the channels. If, instead, the vector $\mathbf{insr}$ is outside the gray region (e.g., point B), all the power is allocated only over one channel, the one with the highest normalized gain.

B. Simultaneous Multiuser Waterfilling

In the multiuser scenario described in the game $\mathscr{G}$ defined in (5), the optimal power allocation of each user also depends on the power allocation of the other users through the received interference, according to the simultaneous multiuser waterfilling solution in (8). As in the single-user case, invoking Lemma 1, we obtain the following.

Corollary 3: The waterfilling operator $\mathsf{WF}_q(\mathbf{p}_{-q})$ in (9) can be expressed as the projection of $-\mathsf{\mathbf{insr}}_q(\mathbf{p}_{-q})$ onto the simplex $\mathscr{P}_q$ defined in (6):

$$\mathsf{WF}_q(\mathbf{p}_{-q}) = [-\mathsf{\mathbf{insr}}_q(\mathbf{p}_{-q})]_{\mathscr{P}_q}, \quad (19)$$

with $\mathsf{\mathbf{insr}}_q(\mathbf{p}_{-q})$ defined in (10).

[4]The weighted Euclidean norm $\|\mathbf{x}\|_{2,\mathbf{w}}$ is defined as $\|\mathbf{x}\|_{2,\mathbf{w}} \triangleq (\sum_i w_i |x_i|^2)^{1/2}$ [20].

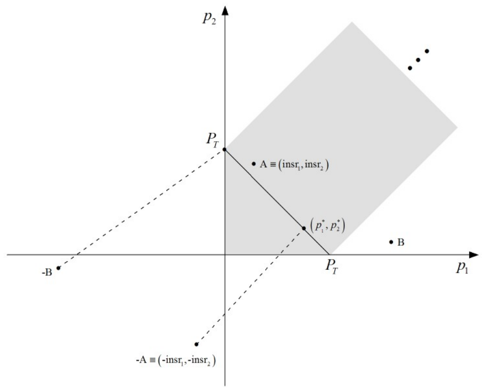

Fig. 1. Graphical interpretation of waterfilling solution (11) as a projection onto the two-dimensional simplex.

Comparing (8) with (19), it is straightforward to see that all the NEs of the game $\mathscr{G}$ in (5) can be alternatively obtained as the fixed-points of the mapping defined in (19), whose existence is guaranteed by Proposition 1:

$$\mathbf{p}_q^{\star} = \left[-\textsf{\textbf{insr}}_q(\mathbf{p}_{-q}^{\star})\right]_{\mathscr{P}_q}, \quad \forall q \in \Omega. \tag{20}$$

In [13], [16], [17], the authors provided the key properties of the mapping in (19), that are instrumental to obtain sufficient conditions for the convergence of the distributed iterative algorithms, described in the next section.

V. Distributed Algorithms

In this section we review the state of the art of the distributed algorithms able to reach the NEs of the game $\mathscr{G}$ in (5), along with their convergence properties.

A. Sequential Iterative Waterfilling Revisited

The sequential IWFA is an instance of the Gauss-Seidel scheme by which each user is sequentially updated [20] based on the waterfilling mapping (9). In fact, in sequential IWFA each player, sequentially and according to a fixed order, maximizes its own rate (3), performing the single-user waterfilling solution in (9), given the others as interference. The sequential IWFA can be written in compact form as in Algorithm 1 [13], [14].

Algorithm 1: Sequential Iterative Waterfilling

Set $\mathbf{p}_q^{(0)}$ = any feasible power allocation;
`for` $n = 0$: Number_of_ iterations,

$$\mathbf{p}_q^{(n+1)} = \begin{cases} \textsf{WF}_q\left(\mathbf{p}_{-q}^{(n)}\right), & \text{if } (n+1) \bmod Q = q, \\ \mathbf{p}_q^{(n)}, & \text{otherwise}, \end{cases} \quad \forall q \in \Omega; \tag{21}$$

`end`

Observe that Algorithm 1 generalizes the well-known sequential iterative waterfilling algorithm proposed by Yu et al. in [6] to the case where the spectral mask constraints are explicitly taken into account. In fact, the algorithm in [6] can be obtained as a special case of Algorithm 1, by removing the spectral mask constraints in each set $\mathscr{P}_q$ in (6), (i.e. setting $p_q^{\max}(k) = +\infty, \forall k, q$), so that the waterfilling operator in (9) becomes the classical waterfilling solution [22]:

$$\textsf{WF}_q\left(\mathbf{p}_{-q}\right) = \left(\mu_q \mathbf{1}_N - \textsf{\textbf{insr}}_q\right)^+ , \tag{22}$$

where $(x)^+ = \max(0, x)$ and $\textsf{\textbf{insr}}_q \triangleq [\textsf{insr}_q(0), \ldots, \textsf{insr}_q(N-1)]^T$, with $\textsf{insr}_q(k)$ given in (10).

The following sufficient conditions for the convergence of the sequential IWFA of [6] to the NE were derived in [6] (for $Q = 2$) and in [7], [15] (for $Q > 2$):

$$\max_{k=0,\ldots,N-1} \left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} < \frac{1}{Q-1}, \quad \forall\, r, q \neq r \in \Omega. \tag{C1}$$

Conditions (C1) are also sufficient for the (existence [6] and) uniqueness of the equilibrium [6], [7].

However, in the presence of spectral mask constraints, the results of [6], [7], [11], [15] cannot be used anymore to guarantee the convergence of sequential IWFA as given in Algorithm 1.

Recently, new sufficient conditions for the convergence of sequential IWFA, larger than those given in [6], [7], [11] were independently provided in [12] and [13], [14]. Specifically, in [12], the sequential IWFA as given in Algorithm 1 was proved to converge to the unique NE of the game $\mathscr{G}$ in (5) if

$$\rho\left(\boldsymbol{\Upsilon}\right) < 1, \tag{C2}$$

where $\rho\left(\boldsymbol{\Upsilon}\right)$ denotes the spectral radius of the matrix $\boldsymbol{\Upsilon} \triangleq \left(\mathbf{I} - \overline{\mathbf{H}}_{\text{low}}\right)^{-1} \overline{\mathbf{H}}^{\text{upp}}$, with $\overline{\mathbf{H}}_{\text{low}}$ and $\overline{\mathbf{H}}^{\text{upp}}$ denoting the strictly lower and strictly upper triangular part of the matrix $\overline{\mathbf{H}}$, respectively, and $\overline{\mathbf{H}}$ is defined, in our notation, as

$$[\overline{\mathbf{H}}]_{qr} \triangleq \begin{cases} \max\limits_{k=0,\cdots,N-1} \left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q}, & \text{if } q \neq r \\ 1, & \text{otherwise}. \end{cases} \tag{23}$$

In [13], [14], the authors obtained the following conditions for the convergence of sequential IWFA.

Theorem 1 ([13], [14]): The sequential IWFA, described in Algorithm 1, converges *geometrically* to the unique NE of the game $\mathscr{G}$ in (5), if one of the two following set of conditions is satisfied

$$\frac{1}{w_q} \sum_{r=1, r\neq q} \max_{k \in \mathcal{D}_r \cap \mathcal{D}_q} \left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} w_r < 1, \ \forall q \in \Omega, \tag{C3}$$

$$\frac{1}{w_r} \sum_{q=1, q\neq r} \max_{k \in \mathcal{D}_r \cap \mathcal{D}_q} \left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} w_q < 1, \ \forall r \in \Omega, \tag{C4}$$

where $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T$ is any positive vector, and $\mathcal{D}_q$ is defined as $\mathcal{D}_q \triangleq \{k \in \{0, \ldots, N-1\} : \exists\ \mathbf{p}_{-q} \in \mathscr{P}_{-q}$ such that $[\textsf{WF}_q\left(\mathbf{p}_{-q}\right)]_k \neq 0\}$ with $\textsf{WF}_q\left(\cdot\right)$ given in (9).

Remark 1. The set $\mathcal{D}_q$ defined in Theorem 1 represents the set $\{0, \ldots, N-1\}$ (possibly) deprived of the carrier indices that user q would never use as the best response set to any strategies used by the other users, for the given set of transmit power and propagation channels. Observe that one can always choose $\mathcal{D}_q = \{0, \ldots, N-1\}$. However, less stringent conditions are obtained by removing unnecessary subcarriers, which are never used. A simple algorithm to estimate the set $\mathcal{D}_q$ was given in [13].

The optimal vector $\mathbf{w}$ in (C3)-(C4) can be obtained as a solution of a geometric programming [13].

Remark 2. If finite order constellations are used, Theorem 1 is still valid using the gap-approximation method [26], [27] as pointed out in Sec. III. It is sufficient to replace each $|H_{qq}(k)|^2$ in above conditions with $|H_{qq}(k)|^2/\Gamma_q$.

Remark 3. As expected, the convergence of the sequential IWFA and the uniqueness of NE are ensured if the links are sufficiently far apart from each other. In fact, from (C1), (C3)-(C4) one infers that there exists a minimum distance beyond which the convergence of the algorithm (and the uniqueness of NE) is guaranteed, corresponding to the maximum level of interference that may be tolerated by each receiver. But, the most interesting result coming from (C3)-(C4) is that, contrary to what one could infer from (C1) and (C2), the convergence of the sequential IWFA is robust against the worst normalized channels $|H_{rq}(k)|^2/\,|H_{qq}(k)|^2$; in fact, the subchannels corresponding to the highest ratios $|H_{rq}(k)|^2/|H_{qq}(k)|^2$ (and, in particular, the subchannels where $|H_{qq}(k)|^2$ is vanishing) do not necessarily affect the convergence of the algorithm, as their carrier indices may not belong to the set $\mathcal{D}_q$.

This property strongly enlarges the conditions for the convergence, as shown in Fig. 2, where we compare the range of validity of our convergence conditions (C3)-(C4) with (C1) and (C2).

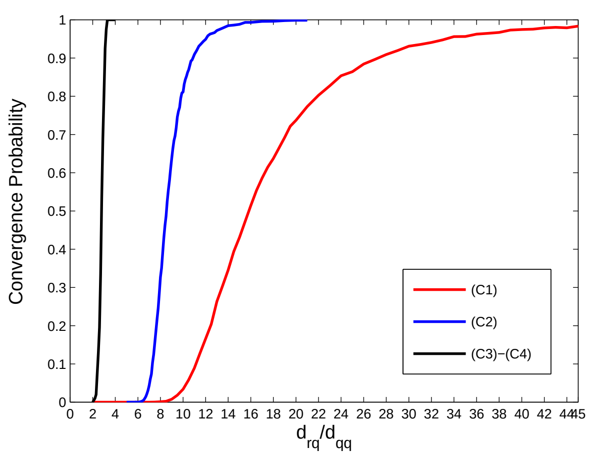

Fig. 2. Probability of (C1), (C2) and (C3)-(C4) versus d_{rq}/d_{qq}; $\gamma = 2.5$, $d_{rq} = d_{qr}$, $d_{rr} = d_{qq} = 1$, $P_q = P_r$, $P_q/\sigma_w^2 = 15$ dB, $P_r/(\sigma_w^2 d_{rq}^{\gamma}) = 5$ dB, $\forall r, q \in \Omega$, $Q = 15$.

In the figure, we plot the probability that conditions (C1), (C2) and (C3)-(C4) are satisfied versus the ratio d_{rq}/d_{qq} (which measures how far apart the links are from each other), in a system with $Q = 15$ active users. For the sake of simplicity, we assumed $d_{rq} = d_{qr}$, $P_q = P_r$ $\forall r, q$, and $\mathbf{w} = \mathbf{1}$. We tested our conditions considering the set $\mathcal{D}_q$, obtained using the algorithm given in [13].

We can see, from Fig. 2, that the probability of guaranteeing convergence increases as the links become more and more separated of each other (i.e. the ratio d_{rq}/d_{qq} increases). More interestingly, the probability that (C3)-(C4) are satisfied, differently from (C1) and (C2) exhibits a neat threshold behavior since it transits very rapidly from the non-convergence guarantee to the almost certain convergence, as the inter-user distance ratio d_{rq}/d_{qq} increases by a small percentage. This shows that the convergence conditions depend, fundamentally, on the inter-link distance, rather than on the channel realizations. Finally, it is worthwhile noticing that our conditions have a broader validity than (C1) and (C2). As an example, for a system with probability of guaranteeing convergence of 0.99, conditions (C3)-(C4) only require $d_{rq}/d_{qq} \simeq 4.3$, whereas conditions (C1) and (C2) require $d_{rq}/d_{qq} > 45$ and $d_{rq}/d_{qq} \simeq 20$, respectively. Furthermore, one can see that this difference increases as the number Q of links increases.

Remark 4. It is useful in practice to have a simple criterion to stop the iterative algorithm, when the desired accuracy (defined in terms of some distance's measure from the NE) is reached. The following result provides as error estimate (to be exact, an upper bound of the error) obtained by the sequential IWFA in each iteration.

Proposition 2 ([17]): Under condition (C3) of Theorem 1, the sequence of power vectors $\{\mathbf{p}^{(n)}\}$ generated by the sequential IWFA converges to the unique NE $\mathbf{p}^{\star}$ of the game $\mathscr{G}$ in (5), satisfying the following error estimate:

$$\|\mathbf{p}^{(n)} - \mathbf{p}^{\star}\|_{2,\mathrm{b}}^{\mathbf{w}} \leq \frac{\varepsilon_{\mathbf{w}}}{1-\varepsilon_{\mathbf{w}}}\,\|\mathbf{p}^{(n)} - \mathbf{p}^{(n-1)}\|_{2,\mathrm{b}}^{\mathbf{w}},\;\; n = 1, 2, \ldots, \tag{24}$$

where $\|\cdot\|_{2,\mathrm{b}}^{\mathbf{w}}$ denotes the weighted block maximum norm[5], and

$$\varepsilon_{\mathbf{w}} \triangleq \max_{q\in\Omega} \frac{1}{w_q} \sum_{r\neq q} \max_{k\in\mathcal{D}_r\cap\mathcal{D}_q} \left\{ \frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2} \right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} w_r < 1, \tag{25}$$

with $\mathbf{w} = [w_1, \ldots, w_Q]^T > \mathbf{0}$ given by (C3), and $\mathcal{D}_q$ defined as in Theorem 1.

Remark 5. The sequential IWFA can be implemented in a distributed way, since each user, to maximize its own rate, needs only to measure the PSD of the thermal noise plus the received multiuser interference (see (10)). However, the inner loop in IWFA represents a bottleneck that slows down the whole algorithm when the number of users increases. We manage this issue in the next section.

B. Simultaneous Iterative Waterfilling

The simultaneous IWFA proposed in [9], [13], [14] is an instance of the Jacobi scheme [20]: the users update their own PSD *simultaneously* at each iteration, performing the single user waterfilling solution (9), given the interference generated

[5] Given $\mathbf{x}$, partitioned as $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_Q]$, with each $\mathbf{x}_q \in \mathbb{R}^N$, and a positive vector $\mathbf{w} = [w_1, \ldots, w_Q]^T$, the weighted block maximum norm, denoted by $\|\mathbf{x}\|_{2,\mathrm{b}}^{\mathbf{w}}$, is defined as $\|\mathbf{x}\|_{2,\mathrm{b}}^{\mathbf{w}} = \max_q \frac{\|\mathbf{x}_q\|_2}{w_q}$, where $\|\cdot\|_2$ is the Euclidean norm.

by the other users in the *previous* iteration. The sequential IWFA is described in Algorithm 2 [9], [13], [14].

Algorithm 2: Simultaneous Iterative Waterfilling

Set $\mathbf{p}_q^{(0)} =$ any feasible power allocation;
`for` $n = 0 : \text{Number\_of\_ iterations}$,

$$\mathbf{p}_q^{(n+1)} = \mathsf{WF}_q\left(\mathbf{p}_{-q}^{(n)}\right), \qquad \forall q \in \Omega; \tag{26}$$

`end`

Sufficient conditions for the convergence of the sequential IWFA are given in the following.

Theorem 2 ([13], [14]): SIWFA, given in Algorithm 2, converges *geometrically* to the unique NE of the game $\mathcal{G}$ in (5) if

$$\rho\left(\mathbf{H}^T(k)\mathbf{H}(k)\right) < 1, \qquad \forall k = 0, \ldots, N-1, \tag{C4}$$

where $\rho\left(\mathbf{H}^T(k)\mathbf{H}(k)\right)$ denotes the spectral radius of the matrix $\mathbf{H}^T(k)\mathbf{H}(k)$ [20], and $\mathbf{H}(k)$ is defined as

$$[\mathbf{H}(k)]_{qr} \triangleq \begin{cases} \dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\dfrac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}\dfrac{P_r}{P_q}, & \text{if } \; k \in \mathcal{D}_q \cap \mathcal{D}_r, q \neq r \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

To give additional insight into the physical interpretation of sufficient conditions for the convergence of SIWFA, we provide the following.

Corollary 4 ([13], [14]): SIWFA, given in Algorithm 2, converges *geometrically* to the unique NE of the game $\mathcal{G}$ in (5) if conditions of Theorem 1 are satisfied.

Remark 1. The simultaneous IWFA keeps the most appealing features of the IWFA, namely its low-complexity and distributed nature. In addition, thanks to the Jacobi-based update, the simultaneous IWFA is expected to be faster than the sequential IWFA, especially if the number of active users in the network is large.

To measure the rate of convergence of the these two algorithms, in [17] the authors provided an upper bound of the the asymptotic convergence exponent for the worst-case convergence rate, defined as

$$d = -\sup_{\mathbf{p}^{(0)} \neq \mathbf{p}^{\star}} \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{\left\|\mathbf{p}^{(n)} - \mathbf{p}^{\star}\right\|}{\left\|\mathbf{p}^{(0)} - \mathbf{p}^{\star}\right\|}\right), \tag{28}$$

where $\mathbf{p}^{\star}$ and $\mathbf{p}^{(n)}$ denote the NE of the game $\mathscr{G}$ in (5) and the power allocation vector obtained by the algorithm at the n-th iteration, respectively. The factor d_{asym} as defined in (28) gives the (asymptotic) number of iterations for the error to decrease by the factor $1/e$.

Since the waterfilling operator is not a monotone mapping, only (upper) bounds for the asymptotic convergence exponent can be obtained [28], as given in the following.

Proposition 3 ([17]): Let $d_{\text{seq}}^{\text{upp}}$ and $d_{\text{sim}}^{\text{upp}}$ be the upper bound of d in (28) obtained using sequential IWFA in Algorithm 1 and simultaneous IWFA in Algorithm 2, respectively. Under condition (C3) of Corollary Theorem 1, we have

$$d_{\text{sim}}^{\text{upp}} = -\log\left(\max_q \frac{1}{w_q}\sum_{r\neq q}\max_{k\in\mathcal{D}_r\cap\mathcal{D}_q}\left\{\frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\}\frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}}\frac{P_r}{P_q}w_r\right), \tag{29}$$

$$d_{\text{seq}}^{\text{upp}} = d_{\text{seq}}^{\text{upp}}/Q. \tag{30}$$

Expression (29) shows that the convergence speed of the algorithms depends, as expected, on the level of interference: the convergence speed increases as the interference level decreases.

Since $d_{\text{sim}}^{\text{upp}}$ and $d_{\text{seq}}^{\text{upp}}$ are only bounds of the asymptotic convergence exponent, a comparison between the sequential IWFA and the simultaneous IWFA by $d_{\text{sim}}^{\text{upp}}$ and $d_{\text{seq}}^{\text{upp}}$ might not be fair. These bound becomes meaningful if $d_{\text{sim}}^{\text{upp}}$ and $d_{\text{seq}}^{\text{upp}}$ approximate with equality d_{sim} and d_{seq}, respectively, for some initial conditions (cf. [28]).

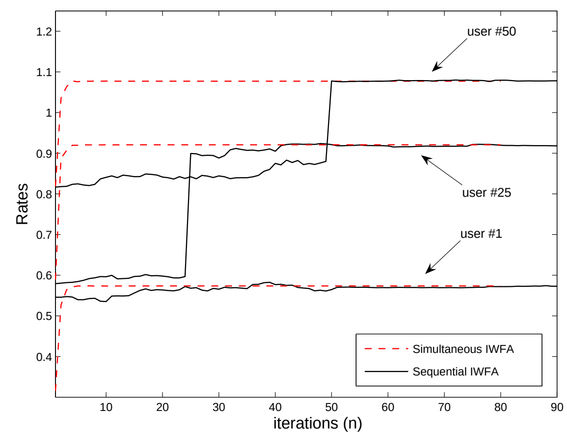

Fig. 3. Rates of the users versus iterations: sequential IWFA (solid line curves), simultaneous IWFA (dashed line curves), $Q = 50$, d_{rq}/d_{qr}, $d_{rr} = d_{qq} = 1$, $\gamma = 2.5$, $P_q = P_r$, $P_q/\sigma_w^2 = 10$ dB, $P_r/(\sigma_w^2 d_{rq}^{\gamma}) = 5$ dB, $\forall r, q \in \Omega$.

In Fig. 3 we compare the performance of the sequential and simultaneous IWFA, in terms of convergence speed. We consider a network composed of 50 links and we show the rate evolution of three of the links corresponding to the sequential IWFA and simultaneous IWFA as a function of the iteration index n as defined in Algorithms 1 and 2. To make the figure not excessively overcrowded, we report only the curves of 3 out of 50 links. As expected, the sequential IWFA is slower than the simultaneous IWFA, especially if the number of active links Q is large, since each user is forced to wait for all the other users scheduled before updating its power allocation.

Remark 2. As for the sequential IWFA, also for the simultaneous IWFA one can obtain an upper bound of the error estimates generated by the algorithm, similarly to the results given in Proposition 2 [17].

C. Asynchronous Iterative Waterfilling

We show now that the sequential and simultaneous IWFAs described in the previous sections are just special cases of a more general unified framework, based on the asynchronous IWFA [16], [17]. The asynchronous IWFA is an instance of the totally asynchronous scheme of [20]: all the users maximize their own rate in a *totally asynchronous* way via the single user waterfilling solution. According to this asynchronous procedure, some users are allowed to update their strategy more frequently than the others, and they might perform these updates using *outdated* information on the interference caused from the others. We show in the following that, whatever the asynchronous mechanism is, such a procedure converges to a stable NE of the game $\mathscr{G}$ in (5), under mild conditions on the multiuser interference.

In order to provide a formal description of the asynchronous IWFA, we need some preliminary definitions, as we introduce next. We assume, without loss of generality, that the set of times at which one or more users update their strategies is the discrete set $T = \mathbb{N}_+ = \{0, 1, 2, \ldots\}$. Let $\mathbf{p}_q^{(n)}$ denote the power allocation of user q at the n-th iteration, and let $T_q \subseteq T$ denote the set of times n at which $\mathbf{p}_q^{(n)}$ is updated (thus, at time $n \notin T_q$, $\mathbf{p}_q^{(n)}$ is left unchanged). Let $\tau_r^q(n)$ denote the most recent time at which the interference from user r is perceived by user q at the n-th iteration (observe that $\tau_r^q(n)$ satisfies $0 \le \tau_r^q(n) \le n$). Hence, if user q updates its power allocation at the n-th iteration, then it waterfills, according to (9), the interference level caused by

$$\mathbf{p}_{-q}^{(\boldsymbol{\tau}^q(n))} \triangleq \left(\mathbf{p}_1^{(\tau_1^q(n))}, \ldots, \mathbf{p}_{q-1}^{(\tau_{q-1}^q(n))}, \mathbf{p}_{q+1}^{(\tau_{q+1}^q(n))}, \ldots, \mathbf{p}_Q^{(\tau_Q^q(n))}\right). \tag{31}$$

The overall system is said to be totally asynchronous if the following weak assumptions are satisfied for each q [20]: A1) $0 \le \tau_r^q(n) \le n$; A2) $\lim_{k\to\infty} \tau_r^q(n_k) = +\infty$; A3) $|T_q| = \infty$, where $\{n_k\}$ is a sequence of elements in T_q that tends to infinity. Assumption A1)-A3) are standard in asynchronous convergence theory [20], and they are fulfilled in any practical implementation.

Using the above notation, the asynchronous IWFA is described in Algorithm 3 [14], [17].

Algorithm 3: Asynchronous Iterative Waterfilling

Set $\mathbf{p}_q^{(0)} =$ any feasible power allocation;
`for` $n = 0$: Number_of_ iterations,

$$\mathbf{p}_q^{(n+1)} = \begin{cases} \mathsf{WF}_q\left(\mathbf{p}_{-q}^{(\boldsymbol{\tau}^q(n))}\right), & \text{if } n \in T_q, \\ \mathbf{p}_q^{(n)}, & \text{otherwise;} \end{cases} \quad \forall q \in \Omega \tag{32}$$

`end`

Remark 1. Since the asynchronous IWFA is based on the waterfilling solution (8), it can be implemented in a distributed way, where each user, to maximize its own rate, only needs to locally measure the PSD of the interference-plus-noise (see (4)) and waterfill over this level. More interestingly, according to the asynchronous scheme, the users may update their strategies using a potentially outdated version of the PSD of the interference and, furthermore, some users are allowed to update their power allocation more often than others, without affecting the convergence of the algorithm. These features strongly relax the constraints required on the synchronization of the updates of the users for the sequential IWFA [6], [7] and simultaneous IWFA [13], [14].

Any particular choice of the sets $\{T_q\}$ and the values of the variables $\{\tau_r^q(n)\}$ will provide a different scheduling of the users in the updates of the PSDs. The interesting result is that this choice does not affect the convergence of the algorithm (provided that A1)-A3) are satisfied), as proved in the following.

Theorem 3 ([14], [17]): The asynchronous IWFA, described in Algorithm 3, converges to the unique NE of the game $\mathscr{G}$ in (5), if the following condition is satisfied

$$\rho\left(\mathbf{H}^{\max}\right) < 1, \tag{C5}$$

where $\rho\left(\mathbf{H}^{\max}\right)$ denotes the spectral radius of the matrix $\mathbf{H}^{\max}$ [20], defined as

$$[\mathbf{H}^{\max}]_{qr} = \begin{cases} \max\limits_{k\in\mathcal{D}_r\cap\mathcal{D}_q} \left\{\dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2}\right\} \dfrac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \dfrac{P_r}{P_q}, & \text{if } r \neq q, \\ 0, & \text{otherwise,} \end{cases} \tag{33}$$

and $\mathcal{D}_q$ is defined as in Theorem 1.

Sufficient conditions for (C5) are given in the following.

Corollary 5 ([14], [17]): The asynchronous IWFA, described in Algorithm 3, converges to the unique NE of the game $\mathscr{G}$ in (5), if conditions of Theorem 1 are satisfied.

Remark 2. The asynchronous IWFA introduced in the previous section represents a general framework to solve the rate maximization game $\mathscr{G}$ in (5), as it contains as special cases a plethora of algorithms, each one obtained by a possible choice of the scheduling of the users in the updating procedure (i.e. the parameters $\{\tau_r^q(n)\}$ and $\{T_q\}$). The important result here is that all the algorithms resulting as special cases of the asynchronous IWFA are guaranteed to reach the unique NE of the game, under the same set of convergence conditions (Theorem 3). For example the sequential and simultaneous IWFAs introduced in the previous sections are special cases of the asynchronous IWFA described in Algorithm 3, using the following parameters $T_q = \{kQ+q,\, k \in \mathbb{N}_+\}$, $\tau_r^q(n) = n$ and $T_q = \mathbb{N}_+$, $\tau_r^q(n) = n$, $\forall r, q$, respectively.

By direct product of this generalized framework, one infer that the convergence for these two algorithms is robust to situations where some users may fail to follow the sequential or simultaneous scheduling of updates. What is affected in this case is only the convergence time. Moreover, Theorem 3 provides alternative conditions for the convergence of both the sequential and simultaneous IWFAs.

VI. Asynchronous Iterative Waterfilling in the Presence of Time and Frequency Offsets

In large scale distributed systems, where no cooperation among different users is allowed, the assumption of perfect synchronization in time and/or frequency among the transmissions of all the links, as made in A.3 of Sec. II, may not be satisfied, because of large propagation delays, timing errors, and/or transmit-receive oscillators' mismatch. Whenever this happens, multiuser Inter-Carrier Interference (ICI) arises, since the signal transmitted by each source over one carrier interferes with the other links not only at the same carrier, but also at neighboring frequencies.

As every link results in a different (unknown) time/frequency shift from the others, the loss of the orthogonality among the carriers cannot be recovered by trying to compensate time/frequency offsets with a proper tuning of each local oscillator, as in single-user systems. The correction made with respect to one user would in fact misalign other already aligned users. Moreover, as our interest is in totally distributed algorithms, we do not consider multiuser ICI cancelation, and ICI is treated as additive noise at the receivers, which leads to carrier-coupling in the (information) rate of each link.

In the presence of ICI, the game theoretic approach proposed in Sec. III is not adequate anymore, since it ignores the presence of carrier-coupling in the expression of the rates, and thus the resulting NEs can lead to poor performance. The main scope of this section is then to reformulate the competitive optimization proposed in Section III, taking explicitly into account the presence of multiuser ICI, due to time and/or frequency offsets, and show how to modify the asynchronous IWFA proposed in Algorithm 3 so that it still converges to the unique NE of the new game [17], [29].

A. Game Theoretic Formulation in the Presence of ICI

Given the multiuser ICI as additive noise, an unified expression for the SINR $\mathsf{sinr}_q(k)$ on the k-th carrier for the q-th link in both the cases of time and frequency offsets is obtained in [17], [29]:

$$\mathsf{sinr}_q(k) = \frac{|H_{qq}(k)|^2\, p_q(k)}{\sigma^2_{w_q}(k) + \sum_{r\neq q}\sum_{k'}\eta_{rq}(k-k')\,|H_{rq}(k')|^2\, p_r(k')}, \tag{34}$$

where $\eta_{rq}(k)$ is the ICI function defined as

$$\eta_{rq}(k) \triangleq \begin{cases} \dfrac{2}{N^2}\dfrac{\sin^2\left(\frac{\pi}{N}k\nu_{rq}\right)}{\sin^2\left(\frac{\pi}{N}k\right)}, & \text{if } k\neq 0 \\ \dfrac{\nu_{rq}^2 + (N-\nu_{rq})^2}{N^2}, & \text{otherwise,} \end{cases} \tag{35}$$

in the case of time synchronization errors, and as

$$\eta_{rq}(k) \triangleq \frac{1}{N^2}\frac{\sin^2\left(\pi\left(k - N\Delta f_{rq}\right)\right)}{\sin^2\left(\frac{\pi}{N}\left(k-N\Delta f_{rq}\right)\right)}. \tag{36}$$

in the case of frequency synchronization offsets [30]. In (35), ν_{rq} denotes the (unknown) time offset at receiver q between the block transmitted from user q and the block transmitted from user r; whereas in (36) Δf_{rq} is the carrier frequency offset between transmitter r and receiver q.

Taking ICI into account, the structure of the rate maximization game in both cases of time and frequency offsets becomes

$$\tilde{\mathscr{G}} = \left\{\Omega, \{\mathscr{P}_q\}_{q\in\Omega}, \{\widetilde{R}_q\}_{q\in\Omega}\right\}, \tag{37}$$

where Ω and $\mathscr{P}_q$ are defined as in the original game $\mathscr{G}$ in (7), $\widetilde{R}_q$ is defined in (3), with $\mathsf{sinr}_q(k)$ given in (34), and the ICI function $\eta_{rq}(k)$ is defined in (35) for time offsets, and in (36) for frequency offsets[6].

All the NEs of the modified game $\tilde{\mathscr{G}}$ in (37) are reached using, for each transmitter, Gaussian signaling and a power allocation satisfying the simultaneous waterfilling solution as in (8), where the waterfilling operator $\mathsf{WF}_q(\cdot)$ is replaced with the following

$$\left[\mathsf{WF}_q(\mathbf{p}_{-q})\right]_k \triangleq \left[\mu_q - \mathsf{insr}_{q,k}(\mathbf{p}_{-q})\right]_0^{p_q^{\max}(k)}, \; k=0,\ldots,N-1, \tag{38}$$

with

$$\mathsf{insr}_{q,k}(\mathbf{p}_{-q}) \triangleq \frac{\sigma^2_{w_q}(k) + \sum_{r\neq q}\sum_{k'}\eta_{rq}(k-k')\,|H_{rq}(k')|^2\, p_r(k')}{|H_{qq}(k)|^2}. \tag{39}$$

The water-level μ_q in (38) is chosen to satisfy the power constraint $(1/N)\sum_{k=0}^{N-1} p_q^{\star}(k) = 1$.

The existence of at least one NE for the game $\tilde{\mathscr{G}}$ in (37) is guaranteed by the following.

Proposition 4 ([17]): The game $\tilde{\mathscr{G}}$ in (37) always admits at least one NE in pure-strategies, for any set of time/frequency offsets, channel realizations, power and spectral mask constraints.

The (unique) NE of the game $\tilde{\mathscr{G}}$ can be reached in a *totally asynchronous* way (in the sense described in Sec. V-C), still using the asynchronous IWFA given in Algorithm 3, provided that the waterfilling operator $\mathsf{WF}_q(\cdot)$ in (32) be replaced by the modified expression given in (38).

Remark 1. Also in the presence of ICI, the asynchronous IWFA can be implemented in a totally distributed way since it just needs a local measure of the multiuser ICI and interference. More importantly, it does not require knowledge of the *unknown* time/frequency offsets by each link.

Remark 2. The convergence of the asynchronous IWFA in the presence of ICI is guaranteed under the following sufficient conditions.

Theorem 4 ([17]): The asynchronous IWFA in the presence of ICI converges to the unique NE of the game $\tilde{\mathscr{G}}$ in (37), if the following condition is satisfied

$$\rho(\widetilde{\mathbf{H}}^{\max}) < 1, \tag{C6}$$

where $\rho(\widetilde{\mathbf{H}}^{\max})$ denotes the spectral radius of the matrix $\widetilde{\mathbf{H}}$ [20], defined as

$$[\widetilde{\mathbf{H}}^{\max}]_{qr} = \begin{cases} \|\widetilde{\mathbf{\Upsilon}}_{rq}\widetilde{\mathbf{H}}_{rq}\|_2, & \text{if } r\neq q, \\ 0, & \text{otherwise,} \end{cases} \tag{40}$$

with

$$[\widetilde{\mathbf{\Upsilon}}_{rq}]_{kk'} = \begin{cases} \eta_{rq}(k-k'), & \text{if } k\in\widetilde{\mathcal{D}}_q \text{ and } k'\in\widetilde{\mathcal{D}}_r, \\ 0, & \text{otherwise,} \end{cases} \tag{41}$$

[6]In the presence of both time and frequency offsets, an additional term of ICI has to be considered in writing the SINR. Since this term is similar to the one already written in (34), for the sake of notation we will not consider it in the following.

and $\widetilde{\mathbf{H}}_{rq}$ is a diagonal matrix, whose diagonal entries are:

$$[\widetilde{\mathbf{H}}_{rq}]_{kk} = \begin{cases} \dfrac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2} \dfrac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \dfrac{P_r}{P_q}, & \text{if } k \in \widetilde{\mathcal{D}}_r, \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

The set $\widetilde{\mathcal{D}}_q$ is defined as in Theorem 1, with $\mathsf{WF}_q(\cdot)$ given in (38).

Corollary 6 ([17]): A sufficient condition for (C6) is given by one of the following

$$\frac{1}{w_q} \sum_{r=1, r\neq q} \left\| \widetilde{\boldsymbol{\Upsilon}}_{rq} \right\|_2 \max_{k \in \widetilde{\mathcal{D}}_r} \left\{ \frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2} \right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} w_r < 1, \ \forall q \in \Omega, \tag{C7}$$

$$\frac{1}{w_r} \sum_{q=1, q\neq r} \left\| \widetilde{\boldsymbol{\Upsilon}}_{rq} \right\|_2 \max_{k \in \widetilde{\mathcal{D}}_r} \left\{ \frac{|\bar{H}_{rq}(k)|^2}{|\bar{H}_{qq}(k)|^2} \right\} \frac{d_{qq}^{\gamma}}{d_{rq}^{\gamma}} \frac{P_r}{P_q} w_q < 1, \ \forall r \in \Omega, \tag{C8}$$

where $\widetilde{\boldsymbol{\Upsilon}}_{rq}$ is defined in (41) and $\mathbf{w} \triangleq [w_1, \ldots, w_Q]^T$ is any positive vector.

Remark 3. As expected, in the presence of ICI, the convergence of the asynchronous IWFA is affected by both the MUI and the coupling due to the ICI. Observe that in the absence of ICI condition (C6) coincides with (C5).

VII. Conclusion

In this paper, we have provided an overview of the current results on the competitive rate maximization problem in Gaussian frequency-selective interference channels. We have shown that the totally asynchronous IWFA proposed in [14], [17] represents a unified framework that collects, as special cases, all the previous results in the literature. In fact, we showed that the well-known sequential IWFA and the recently proposed simultaneous IWFA, are an instance of the asynchronous IWFA. The main advantage of the asynchronous IWFA is that no rigid scheduling in the updates of the users is required, since the users are allowed to update their own strategies in a totally asynchronous way. This indeed relaxes the synchronization requirements among the users needed in the sequential and simultaneous IWFAs. We then have provided a unified set of sufficient conditions ensuring the convergence of all the algorithms that can be obtained from the asynchronous IWFA as special cases. In addition, we have provided results on the convergence speed and on the error estimation at each iteration. Finally, we have considered the extension of the asynchronous IWFA to the case where there are time and/or frequency offsets among the links

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