On the Gaussian K-description problem under symmetric distortion constraints

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***Abstract*—We consider multiple description (MD) coding for the Gaussian source under the symmetric mean squared error distortion constraints. With focus on the three description problem, we provide inner and outer bounds for the rate region, between which the gap can be bounded by some small constants.**

At the heart of this result is a novel lower bound for the sum rate, which is derived through generalization of the well-known bounding technique by Ozarow. In contrast to the original method, we expand the probability space by more than one (instead of only one) random variable, and further impose a particular Markov structure on them. The outer bound is then established by applying this technique to several bounding planes of the rate region. For the inner bound, we consider a simple scheme of combining successive refinement coding and lossless multilevel diversity coding (MLD). Both the inner and outer bounds can be written as the intersection of ten half spaces with matching normal directions, and thus can be easily compared. The small gap between them, where the boundary of the MD rate region clearly resides, suggests the surprising competitiveness of this simple achievability scheme. The geometric structure of the MLD rate region provides important guidelines as to the normal directions of the outer bound hyperplanes, which demonstrates an intimate connection between MD and MLD coding. These results can be generalized and improved in various ways which are also discussed.

I. Introduction

In the multiple description (MD) problem, a source is encoded into several descriptions such that any subset of them can be used to reconstruct the source with certain specified quality. The problem is well motivated by source transmission over network with path failure or packet network with loss, since there exists uncertainty as to which transmissions (or packets) are received successfully by the receiver.

Even in the seemingly simple situation with two descriptions, the only solved case is the quadratic Gaussian problem [1], for which the achievable region in [2] is tight. Zhang and Berger showed that this achievable region is however not tight in general [3], and a complete characterization of the rate-distortion (R-D) region has not been found. In [4][5], an achievable rate was provided for the symmetric case with many (> 2) descriptions. Wang and Viswanath [6], [7] generalized the scalar MD problem to vector Gaussian source with many descriptions, and tight sum rate lower bound was established for certain cases with only *two levels* of distortion constraints.

In this work, we focus on the three description scalar Gaussian problem under symmetric quadratic distortion constraints, and provide both inner and outer bounds to the rate region. At the heart of the result is a novel lower bound for the sum rate, which is the first lower bound with more than two levels of distortion constraints to appear in the literature, to the best of our knowledge. We derive this lower bound by generalizing Ozarow's technique in treating the Gaussian two-description problem. More specifically, we expand the probability space of the problem by more than one auxiliary random variable, and impose certain Markov structure on these random variables. Ozarow's technique has been applied to various problems besides the MD problem [1], [6], [7], for example, the results on multi-terminal source coding by Wagner and Anantharam [8], and the joint source channel coding problem with bandwidth expansion by Reznic *et al.* [9]. However, in all these previous works the probability space was expanded by only one additional auxiliary random variable, which appears insufficient for the general K-description problem in consideration.

A simple inner bound is considered in this paper by combining successive refinement (SR) coding [10] and lossless multilevel diversity (MLD) coding [11], [12], which was in fact considered previously by Zamir and Yeung [13] in a slightly different context; we shall refer to this scheme as the SR-MLD scheme. This scheme provides an achievable region with the same geometric structure as the rate region of the MLD coding problem, which can be written as the intersection of ten half spaces. With this inner bound in mind, we derive ten lower bounds for the bounding planes of the MD rate region. The resulting outer bound is shown to differ from the inner bound by some small universal constants independent of the specific distortion constraints.

This result can be further improved in several ways. It can be generalized to any number of descriptions under symmetric distortion constraints, as well as asymmetric distortion constraints. We shall discuss certain aspects of these improvements and generalizations.

II. Notation and problem definition

Let $\{X(i)\}_{i=1,2,...}$ be a memoryless and stationary Gaussian source with zero-mean and unit-variance. The vector $X(1), X(2), ..., X(n)$ will be denoted as X^n. The mean squared error (MSE) distortion $d(x^n, y^n) = \frac{1}{n}\sum_{i=1}^{n}(x(i) - y(i))^2$ will be used. We adopt the notation in [12]. A length-n

block of the source samples is encoded into three descriptions. Let $\boldsymbol{v}$ be a vector in $\{0,1\}^3$, and denote the i-th component of $\boldsymbol{v}$ by v_i. Define

$$\Omega_3^\alpha = \{\boldsymbol{v} \in \{0,1\}^K : |\boldsymbol{v}| = \alpha\}, \quad \alpha = 1,2,3 \tag{1}$$

where $|\boldsymbol{v}|$ is the Hamming weight of $\boldsymbol{v}$, and define $\Omega_3 = \bigcup_{\alpha=1}^{3} \Omega_3^\alpha$. We have

$$\begin{aligned}\Omega_3 &= \Omega_3^1 \cup \Omega_3^2 \cup \Omega_3^3 \\ &= \{100, 010, 001\} \cup \{110, 101, 011\} \cup \{111\}. \end{aligned} \tag{2}$$

Decoder $\boldsymbol{v}$, $\boldsymbol{v} \in \Omega_3$ has access to the $|\boldsymbol{v}|$ descriptions in the set $G_{\boldsymbol{v}} = \{i : v_i = 1\}$. The symmetric distortion constraints are given such that any decoder $\boldsymbol{v}$ can reconstruct the source to satisfy a certain distortion $D_{|\boldsymbol{v}|}$, i.e., the distortion constraint depends only on the number of descriptions the decoder has access to, but not the particular combination of descriptions. The notation I_k is used to denote the set $\{1, 2, ..., k\}$.

Formally, an $(n, M_1, M_2, M_3, (\Delta_{\boldsymbol{v}}, \boldsymbol{v} \in \Omega_3))$ code is defined by encoding functions S_i and decoding functions $T_{\boldsymbol{v}}$

$$\begin{aligned} S_i &: \mathbb{R}^n \to I_{M_i},\ i = 1,2,3, \\ T_{\boldsymbol{v}} &: \prod_{i \in G_{\boldsymbol{v}}} I_{M_i} \to \mathbb{R}^n,\ \boldsymbol{v} \in \Omega_3, \end{aligned}$$

and

$$\Delta_{\boldsymbol{v}} = \mathbb{E} d(X^n, T_{\boldsymbol{v}}(S_i(X^n), i \in G_{\boldsymbol{v}})), \quad \boldsymbol{v} \in \Omega_3, \tag{3}$$

where $\mathbb{E}$ is the expectation operator.

A rate triple (R_1, R_2, R_3) is (D_1, D_2, D_3)-achievable if for every $\epsilon > 0$, there exists for sufficiently large n an $(n, M_1, M_2, M_3, (\Delta_{\boldsymbol{v}}, \boldsymbol{v} \in \Omega_3))$ code such that

$$\frac{1}{n} \log M_i \le R_i + \epsilon,\ i = 1,2,3, \quad \Delta_{\boldsymbol{v}} \le D_{|\boldsymbol{v}|} + \epsilon,\ \boldsymbol{v} \in \Omega_3.$$

We are interested in the collection of all the (D_1, D_2, D_3)-achievable rate triples, which is denoted as $\mathcal{R}(\boldsymbol{D})$, where $\boldsymbol{D} = (D_1, D_2, D_3)$.

III. The SR-MLD coding scheme rate region

The symmetric MLD coding problem was considered in [11], which can roughly be described in the context we are considering as follows.

Three independent sources V_1, V_2 and V_3 are observed at the encoder, and encoded into three descriptions. The decoders T_{100}, T_{010} and T_{001}, which are called the first level decoders, should reconstruct V_1 losslessly in the Shannon sense, the decoders T_{110}, T_{101} and T_{011} should reconstruct (V_1, V_2), and decoder T_{111} should reconstruct (V_1, V_2, V_3).

The achievable rate region was characterized in [11] for this coding problem, which can be written as

$$R_i \ge H_1, \quad i = 1,2,3, \tag{4}$$

$$R_i + R_j \ge 2H_1 + H_2, \quad i \ne j, \quad i,j \in \{1,2,3\}, \tag{5}$$

$$\begin{aligned} &2R_i + R_j + R_k \ge 4H_1 + 2H_2 + H_3, \\ &\qquad\qquad i \ne j \ne k, \quad i,k,j \in \{1,2,3\}, \end{aligned} \tag{6}$$

$$R_1 + R_2 + R_3 \ge 3H_1 + \frac{3}{2} H_2 + H_3. \tag{7}$$

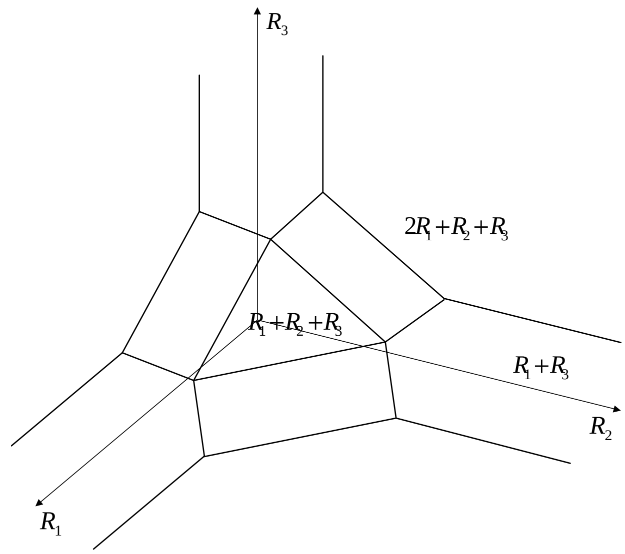

Fig. 1. A typical $\hat{\mathcal{R}}(\boldsymbol{D})$.

where $H_i \triangleq H(V_i)$ for $i = 1,2,3$. The coding scheme has two effective steps for each source V_i: 1) the source vector V_i^n is encoded (independently of other sources) into a stream B_i of rate approximately H_i, 2) this stream is mapped by three hash functions (similar to binning) into hash indices, one for each description, and the hash indices are written into the descriptions. It is easily seen that there exists hash functions such that with a set of descriptions, as long as the sum rate of the hash functions for each B_i is larger than H_i, these descriptions can be used to jointly decode B_i and subsequently recover V_i^n losslessly in the usual Shannon sense.

Consider now constructing the bitstream B_i using the i-th layer of a successive refinement code for the Gaussian source, to satisfy distortion constraint D_i, for $i = 1,2,3$. Since the Gaussian source is successively refinable [10], the following rate of B_i is achievable

$$\hat{H}_i = \frac{1}{2} \log \frac{D_{i-1}}{D_i}, \quad i = 1,2,3. \tag{8}$$

where $D_0 \triangleq 1$. Clearly the result [11], i.e., (4)-(7), implies that the following rate region, denoted as $\hat{\mathcal{R}}(\boldsymbol{D})$, is achievable for the MD problem.

$$R_i \ge \frac{1}{2} \log \frac{1}{D_1}, \quad i = 1,2,3, \tag{9}$$

$$R_i + R_j \ge \frac{1}{2} \log \frac{1}{D_1 D_2}, \quad i \ne j, \quad i,j \in \{1,2,3\}, \tag{10}$$

$$\begin{aligned} &2R_i + R_j + R_k \ge \frac{1}{2} \log \frac{1}{D_1^2 D_2 D_3}, \\ &\qquad\qquad i \ne j \ne k, \quad i,k,j \in \{1,2,3\}, \end{aligned} \tag{11}$$

$$R_1 + R_2 + R_3 \ge \frac{1}{4} \log \frac{1}{D_1^3 D_2 D_3^2}. \tag{12}$$

Formally we have the following theorem.

***Theorem** 1:* $\hat{\mathcal{R}}(\boldsymbol{D}) \subseteq \mathcal{R}(\boldsymbol{D})$.

This theorem is also implied by the result in [13], though the problem considered there is defined to be more restrictive: the reconstructions by decoders $T_{\boldsymbol{v}}$, $|\boldsymbol{v}|=k$, are exactly the same for any fixed value k. In Fig. 1, we show a typical rate region of the lossless MLD coding problem, which clearly bears the same geometric structure as $\hat{\mathcal{R}}(\boldsymbol{D})$. This region is a polytope with ten faces, and our plan is to derive an outer bound which is also a polytope with ten faces of matching normal directions.

IV. A SUM RATE LOWER BOUND

We first provide a lemma that will be useful for proving the outer bound.

***Lemma** 1:* Let S_i, $i\in\{1,2,3\}$ be a set of encoding functions such that there exist decoding functions to satisfy the distortion constraints $\boldsymbol{D}$. Let $Y_b=X+N_b$ and $Y_a=X+N_a+N_b$, where N_a and N_b are mutually independent Gaussian random variables independent of the Gaussian source X, with variance σ_a^2 and σ_b^2, respectively. Then by defining $\sigma_b^2=d_b$ and $\sigma_a^2+\sigma_b^2=d_a$, we have

$$I(S_i,i\in G_{\boldsymbol{v}};Y_a^n)\geq\frac{n}{2}\log\frac{1+d_a}{D_{|\boldsymbol{v}|}+d_a},\tag{13}$$

and

$$I(S_i,i\in G_{\boldsymbol{v}};Y_b^n)-I(S_i,i\in G_{\boldsymbol{v}};Y_a^n)\geq\frac{n}{2}\log\frac{(1+d_b)(D_{|\boldsymbol{v}|}+d_a)}{(1+d_a)(D_{|\boldsymbol{v}|}+d_b)}.\tag{14}$$

Note that clearly Lemma 1 holds true whenever $0\leq d_b\leq d_a$. Next we proceed to prove this lemma.

Proof of Lemma 1: Define $Z_a=N_a+N_b$ and $Z_b=N_b$. To proof the first statement, we consider the following chain of inequalities

$$\begin{aligned}
&I(S_i,i\in G_{\boldsymbol{v}};Y_a^n)\\
&=nh(Y_a)-h(Y_a^n|S_i,i\in G_{\boldsymbol{v}})\\
&=nh(Y_a)-h(X^n+Z_a^n|S_i,i\in G_{\boldsymbol{v}})\\
&=nh(Y_a)-h(X^n+Z_a^n-\hat{X}_{\boldsymbol{v}}^n|S_i,i\in G_{\boldsymbol{v}})\\
&\overset{(a)}{\geq} nh(Y_a)-h(X^n+Z_a^n-\hat{X}_{\boldsymbol{v}}^n)\\
&\overset{(b)}{\geq} nh(Y_a)-\sum_{i=1}^n h[X(i)+Z_a(i)-\hat{X}_{\boldsymbol{v}}(i)]\\
&\overset{(c)}{\geq} nh(Y_a)\\
&\quad-\sum_{i=1}^n\frac{1}{2}\log\left\{(2\pi e)\mathbb{E}[(X(i)+Z_a(i)-\hat{X}_{\boldsymbol{v}}(i))^2]\right\}\\
&=nh(Y_a)-\sum_{i=1}^n\frac{1}{2}\log\left[(2\pi e)(\mathbb{E}d(X(i),\hat{X}_{\boldsymbol{v}}(i))+d_a)\right],
\end{aligned}$$

where $\hat{X}_{\boldsymbol{v}}^n$ is the reconstruction with descriptions $S_i,i\in G_{\boldsymbol{v}}$, and its i-th position is denoted as $\hat{X}_{\boldsymbol{v}}(i)$. The inequality (a) is because conditioning reduces entropy, (b) is because of the chain rule for differential entropy and the fact conditioning reduces entropy, and (c) is because Gaussian distribution maximizes the differential entropy for a given second moment. Since $\log(\cdot)$ is a concave function, we have

$$\begin{aligned}
&\sum_{i=1}^n\frac{1}{2}\log\left[(2\pi e)(\mathbb{E}d(X(i),\hat{X}_{\boldsymbol{v}}(i))+d_a)\right]\\
&\qquad\leq\frac{n}{2}\log\left(\mathbb{E}d(X^n,\hat{X}_{\boldsymbol{v}}^n)+d_a\right).
\end{aligned}$$

And it follows

$$\begin{aligned}
&I(S_i,i\in G_{\boldsymbol{v}};Y_a^n)\\
&\geq nh(Y_a)-\frac{n}{2}\log\left(\mathbb{E}d(X^n,\hat{X}_{\boldsymbol{v}}^n)+d_a\right)\\
&\geq nh(Y_a)-\frac{n}{2}\log\left(D_{|\boldsymbol{v}|}+d_a\right)\\
&=\frac{n}{2}\log\frac{1+d_a}{D_{|\boldsymbol{v}|}+d_a},
\end{aligned}$$

which is the first claim.

To prove the second claim, we write the following

$$\begin{aligned}
&I(S_i,i\in G_{\boldsymbol{v}};Y_b^n)-I(S_i,i\in G_{\boldsymbol{v}};Y_a^n)\\
&=nh(Y_b)-nh(Y_a)\\
&\qquad+h(Y_a^n|S_i,i\in G_{\boldsymbol{v}})-h(Y_b^n|S_i,i\in G_{\boldsymbol{v}}).
\end{aligned}$$

For the latter two terms, we have

$$\begin{aligned}
&h(Y_a^n|S_i,i\in G_{\boldsymbol{v}})-h(Y_b^n|S_i,i\in G_{\boldsymbol{v}})\\
&\overset{(a)}{=} h(Y_a^n|S_i,i\in G_{\boldsymbol{v}})-h(Y_b^n|N_a^n,\{S_i,i\in G_{\boldsymbol{v}}\})\\
&\overset{(b)}{=} h(Y_a^n|S_i,i\in G_{\boldsymbol{v}})-h(Y_a^n|N_a^n,\{S_i,i\in G_{\boldsymbol{v}}\})\\
&=I(Y_a^n;N_a^n|S_i,i\in G_{\boldsymbol{v}}),
\end{aligned}$$

where (a) is because N_a^n is independent of Y_b^n and $\{S_i,i\in G_{\boldsymbol{v}}\}$; (b) is by the definition of Y_a. Continuing along this line, we have

$$\begin{aligned}
&I(Y_a^n;N_a^n|S_i,i\in G_{\boldsymbol{v}})\\
&\overset{(a)}{=} h(N_a^n)-h(N_a^n|X^n+N_a^n+N_b^n,\{S_i,i\in G_{\boldsymbol{v}}\})\\
&=h(N_a^n)-h(N_a^n|X^n+N_b^n+N_a^n,\hat{X}_{\boldsymbol{v}}^n,\{S_i,,i\in G_{\boldsymbol{v}}\})\\
&\overset{(b)}{\geq} h(N_a^n)-h(N_a^n|X^n-\hat{X}_{\boldsymbol{v}}^n+N_a^n+N_b^n)\\
&\overset{(c)}{\geq}\sum_{i=1}^n\left(h(N_a(i))-h(N_a(i)|X(i)-\hat{X}_{\boldsymbol{v}}(i)+N_a(i)+N_b(i)\right)\\
&=\sum_{i=1}^n I(N_a(i);X(i)-\hat{X}_{\boldsymbol{v}}(i)+N_b(i)+N_a(i))\\
&\overset{(d)}{\geq}\sum_{i=1}^n\frac{1}{2}\log\frac{\mathbb{E}d(X(i),\hat{X}_{\boldsymbol{v}}(i))+d_a}{\mathbb{E}d(X(i),\hat{X}_{\boldsymbol{v}}(i))+d_b}\\
&\overset{(e)}{\geq}\frac{n}{2}\log\frac{D_{|\boldsymbol{v}|}+d_a}{D_{|\boldsymbol{v}|}+d_b},
\end{aligned}$$

where (a) is because N_a is independent of $S_i,i\in G_{\boldsymbol{v}}$; (b) is because conditioning reduces entropy; (c) is by applying the chain rule, and the facts that N_a^n is an i.i.d. sequence and conditioning reduces entropy; (d) is by applying the mutual information game result (see page 263, [14]) that Gaussian noise is the worst additive noise under a variance constraint, and taking $N_a(i)$ as channel input; finally (e) is due to the convexity and monotonicity of $\log \frac{x+d_a}{x+d_b}$ in $x \in (0,\infty)$ when $d_a \geq d_b \geq 0$. This completes the proof for the second claim. ∎

A similar line of argument was used in [6] to derive an lower bound for a system with two levels of distortion constraints. However, Lemma 1 generalizes that result since there exists only one auxiliary random variable in the setting of [6], and we have two auxiliary random variables Y_a and Y_b.

Next we prove the following theorem, which in fact provides a set of lower bounds to the sum rate.

***Theorem** 2:*

$$R_1 + R_2 + R_3 \geq \frac{3}{2}\log\frac{1+d_1}{D_1+d_1} + \frac{3}{4}\log\frac{(1+d_2)(D_2+d_1)}{(1+d_1)(D_2+d_2)} + \frac{1}{2}\log\frac{(D_3+d_2)}{(1+d_2)D_3}, \quad (15)$$

for any $d_1 \geq d_2 \geq 0$.

Proof: Define $Y_2 = X + N_2$ and $Y_1 = X + N_1 + N_2$, where N_1 and N_2 are mutually independent Gaussian random variables independent of the Gaussian source X, with variance σ_1^2 and σ_2^2, respectively. Define $\sigma_2^2 = d_2$, $\sigma_1^2 + \sigma_2^2 = d_1$. We start with the following chain of inequalities

$$\begin{aligned}
&n(R_1+R_2+R_3)\\
&\stackrel{(a)}{\geq} H(S_1)+H(S_2)+H(S_3)-H(S_1S_2S_3|X^n)\\
&\stackrel{(b)}{=} H(S_1)+H(S_2)+H(S_3)-H(S_1S_2S_3|X^n)\\
&\quad -\frac{1}{2}\left[H(S_1S_2)+H(S_2S_3)+H(S_1S_3)\right]\\
&\quad +\frac{1}{2}\left[H(S_1S_2)+H(S_2S_3)+H(S_1S_3)\right]\\
&\quad -H(S_1S_2S_3)+H(S_1S_2S_3) \triangleq \check{H}_3, \qquad (16)
\end{aligned}$$

where (a) is because S_i, $i=1,2,3$, are deterministic functions of X^n; (b) is by adding and subtracting the same term.

The next step is essential for establishing the lower bound, which differs signigicantly from the technique of [1] and [6] in that we now utilize the two auxiliary random variables Y_1 and Y_2. Consider the following quantity

$$\begin{aligned}
\acute{H}_3 = &\Big\{H(S_1|Y_1^n)+H(S_2|Y_1^n)+H(S_3|Y_1^n)\\
&-\frac{1}{2}\left[H(S_1S_2|Y_1^n)+H(S_2S_3|Y_1^n)+H(S_1S_3|Y_1^n)\right]\Big\}\\
&+\Big\{\frac{1}{2}\left[H(S_1S_2|Y_2^n)+H(S_2S_3|Y_2^n)+H(S_1S_3|Y_2^n)\right]\\
&\qquad -H(S_1S_2S_3|Y_2^n)\Big\}. \qquad (17)
\end{aligned}$$

It is seen that $\acute{H}_3 \geq 0$, because each brace in (17) is nonnegative by applying the conditional version of Han's inequality [14]. Thus it follows

$$\begin{aligned}
&n(R_1+R_2+R_3) \geq \check{H}_3 - \acute{H}_3\\
&= I(S_1;Y_1^n)+I(S_2;Y_1^n)+I(S_3;Y_1^n)\\
&+\frac{1}{2}[I(S_1S_2;Y_2^n)-I(S_1S_2;Y_1^n)]\\
&+\frac{1}{2}[I(S_2S_3;Y_2^n)-I(S_2S_3;Y_1^n)]\\
&+\frac{1}{2}[I(S_1S_3;Y_2^n)-I(S_1S_3;Y_1^n)]\\
&+[I(S_1S_2S_3;X^n)-I(S_1S_2S_3;Y_2^n)]. \qquad (18)
\end{aligned}$$

Clearly we can now apply the first statement in Lemma 1 to the first three terms in (18), and the second statement in Lemma 1 to the first three brackets in (18). For the last bracket, let $\sigma_b^2 = 0$ and $\sigma_a^2 = \sigma_2^2$ in Lemma 1, then again the second statement can be applied. Summarizing these bounds gives the claimed result. ∎

This lower bound on the sum rate is quite powerful, since we can choose arbitrarily valid d_1 and d_2 and it remains a lower bound. One could maximize this lower bound, however without a matching inner bound, solving this rather involved optimization problem offers little insight. Instead, we shall choose some specific values in the next section for d_1 and d_2, which indeed provides insightful results.

V. An outer bound to the rate region

Let $\underline{\mathcal{R}}(\boldsymbol{D}, d_1, d_2)$ be the set of rate triples (R_1, R_2, R_3) satisfying (15) and the following conditions

$$R_i \geq \frac{1}{2}\log\frac{1}{D_1}, \quad i=1,2,3, \qquad (19)$$

$$R_i + R_j \geq \frac{1}{2}\log\frac{(1+d_1)(D_2+d_1)}{(D_1+d_1)^2 D_2},\ i\neq j,\ i,j\in\{1,2,3\}, \qquad (20)$$

$$\begin{aligned}
&2R_i + R_j + R_k\\
&\quad \geq \frac{1}{2}\log\frac{(1+d_1)^2(1+d_2)(D_2+d_1)^2(D_3+d_2)}{(D_1+d_1)^4(D_2+d_2)^2 D_3},\\
&\qquad\qquad i\neq j\neq k, \quad i,k,j\in\{1,2,3\} \qquad (21)
\end{aligned}$$

We have the following theorem.

***Theorem** 3:* $\underline{\mathcal{R}}(\boldsymbol{D}, d_1, d_2) \supseteq \mathcal{R}(\boldsymbol{D})$ for any $d_1 \geq d_2 \geq 0$.

Proof: It is clear (19) is by the conventional rate distortion theorem. To get (20), we write

$$\begin{aligned}
&n(R_i+R_j) \geq H(S_i)+H(S_j)-H(S_iS_j|X^n) \qquad (22)\\
&\stackrel{(a)}{\geq} H(S_i)+H(S_j)-H(S_iS_j)+H(S_iS_j)-H(S_iS_j|X^n)\\
&\qquad -[H(S_i|Y_1^n)+H(S_j|Y_1^n)-H(S_iS_j|Y_1^n)] \qquad (23)\\
&= I(S_i;Y_1^n)+I(S_j;Y_1^n)+[I(S_iS_j;X^n)-I(S_iS_j;Y_1^n)], \qquad (24)
\end{aligned}$$

where (a) is adding and subtracting the same term, and the fact S_i and S_j is deterministic given X^n and $I(S_i;S_j|Y_1^n) \geq 0$. We can now apply Lemma 1 to (24), and reach the claimed bound. For the only remaining bound, we write

$$\begin{aligned}
&n(2R_i+R_j+R_k) \geq 2H(S_i)+H(S_j)+H(S_k)\\
&\geq 2H(S_i)+H(S_j)+H(S_k)-H(S_iS_j)-H(S_iS_k)\\
&\quad+H(S_iS_j)+H(S_iS_k)-H(S_iS_jS_k)+H(S_iS_jS_k)\\
&\quad-[H(S_i|Y_1^n)+H(S_j|Y_1^n)-H(S_iS_j|Y_1^n)]\\
&\quad-[H(S_i|Y_1^n)+H(S_k|Y_1^n)-H(S_iS_k|Y_1^n)]\\
&\quad-[H(S_iS_j|Y_2^n)+H(S_iS_k|Y_2^n)-H(S_iS_jS_k|Y_2^n)]\\
&\quad-H(S_1S_2S_3|X^n),
\end{aligned} \tag{25}$$

where the brackets are nonnegative due to the nonnegativeness of $I(S_i;S_j|Y_1^n)$, $I(S_i;S_k|Y_1^n)$ and $I(S_iS_j;S_iS_k|Y_2^n)$, respectively. Through some algebra, we arrive at

$$\begin{aligned}
&n(2R_i+R_j+R_k)\\
&\geq 2I(S_i;Y_1^n)+I(S_j;Y_1^n)+2I(S_k;Y_1^n)\\
&\quad+[I(S_iS_j;Y_2^n)-I(S_iS_j;Y_1^n)]\\
&\quad+[I(S_iS_k;Y_2^n)-I(S_iS_k;Y_1^n)]\\
&\quad+[I(S_iS_jS_k;X^n)-I(S_iS_jS_k;Y_2^n)],
\end{aligned} \tag{26}$$

to which Lemma 1 can again be applied, and this completes the proof. ■

Next we consider specializing the above outer bound by a particular choice of d_1 and d_2. Without loss of generality we may assume $D_1 \geq D_2 \geq D_3$. Thus $d_1 = D_1$ and $d_2 = D_2$ are a valid choice, and subsequently we have

$$R_i+R_j \geq \frac{1}{2}\log\frac{(1+D_1)(D_2+D_1)}{(2D_1)^2D_2} \overset{(a)}{\geq} \frac{1}{2}\log\frac{1}{4D_1D_2}, \tag{27}$$

where (a) is by the fact $1+D_1 \geq D_1$ and $D_2+D_1 \geq D_1$. For the other bound we have

$$\begin{aligned}
&2R_i+R_j+R_k\\
&\quad\geq \frac{1}{2}\log\frac{(1+D_1)^2(1+D_2)(D_2+D_1)^2(D_3+D_2)}{(2D_1)^4(2D_2)^2D_3}\\
&\quad\geq \frac{1}{2}\log\frac{1}{64D_1^2D_2D_3}
\end{aligned} \tag{28}$$

Finally, for the sum rate we arrive at the following expression using a similar argument

$$R_1+R_2+R_3 \geq \frac{1}{4}\log\frac{1}{2^9D_1^3D_2D_3^2}. \tag{29}$$

We are interested in the Euclidean distance between the inner bound and the tightest outer bound along a certain direction $\boldsymbol{u}=(u_1,u_2,u_3)$. More precisely, we define

$$\begin{aligned}
\delta_{\boldsymbol{u}} = &\min_{(R_1,R_2,R_3)\in\hat{\mathcal{R}}(\mathbf{D})} \frac{u_1R_1+u_2R_2+u_3R_3}{\|\boldsymbol{u}\|}\\
&-\max_{d_1,d_2:d_1\geq d_2\geq 0}\left[\min_{(R_1,R_2,R_3)\in\underline{\mathcal{R}}(\mathbf{D},d_1,d_2)} \frac{u_1R_1+u_2R_2+u_3R_3}{\|\boldsymbol{u}\|}\right],
\end{aligned} \tag{30}$$

where $\|\boldsymbol{u}\|$ is Euclidean norm of $\boldsymbol{u}$.

By the proceeding calculation and Theorem 1, we have

$$\delta_{\boldsymbol{u}} = 0, \quad \boldsymbol{u}\in\{(1,0,0),(0,1,0),(0,0,1)\} \tag{31}$$

$$\delta_{\boldsymbol{u}} \leq \frac{1}{\sqrt{2}} \approx 0.7071, \quad \boldsymbol{u}\in\{(1,1,0),(0,1,1),(1,0,1)\} \tag{32}$$

$$\delta_{\boldsymbol{u}} \leq \frac{\sqrt{3}}{\sqrt{2}} \approx 1.2247, \quad \boldsymbol{u}\in\{(2,1,1),(1,2,1),(1,1,2)\} \tag{33}$$

$$\delta_{1,1,1} \leq \frac{3\sqrt{3}}{4} \approx 1.2990. \tag{34}$$

Thus the simple SR-MLD scheme is in fact not very far away from optimality, since it is within a small constant of the outer bound. This illustrates the surprising competitiveness of this simple scheme.

VI. Improvements and Generalizations

We have improved and generalized the results given here in several ways.

- By using another inner bound based on a generalization of the multilayer coding scheme for the symmetric MD problem in [5], we can significantly reduce the distance between the inner and outer bounds for the three description cases, i.e., reduce the upper bounding constants on quantities defined similarly as $\delta_{\boldsymbol{u}}$ but for the improved inner bound. This improved result is illustrated in Fig. 2.
- The sum rate result can be generalized to any number of descriptions. In fact, we obtain a result that bound the (symmetric rate) individual rate distortion function within one bit for the general Gaussian K description problem [15].
- In [16] we formulate the asymmetric lossless MLD coding problem, and provide a complete solution for the three description case. Combining this result with the bounding technique here, we have obtained an approximate characterization for the asymmetric Gaussian MD problem.
- For the general K description problem, approximate characterization of the rate region is obtained by combining the lower bounding technique and the α-resolution result given in [12].
- Similar results can be derived for general sources under the mean squared error distortion measure.

These generalizations and improvements will be presented at other venues, and are not given here in detail. Instead, we have chosen to illustrate the underlying principles for the simplest non-trivial case in this paper.

The main contribution of our work can be understood as two-fold. The first is a general lower bounding technique which requires expanding the probability space by more than one auxiliary random variable. It appears that the random variables with the given Markov structure suite well with the multiple description problem. However, it is conceivable that for other information theoretical problems other structures may be more suitable. The second contribution is an illustration of

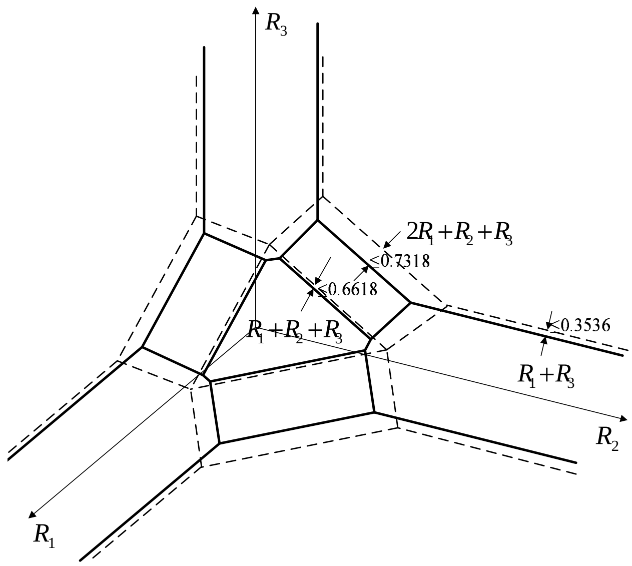

Fig. 2. The outer bound is drawn in bold lines, and the improved inner bound in dashed line. The Euclidean distances between corresponding planes are also given.

the philosophical guideline that there usually exists intimate relation between a lossy (Gaussian) source coding problem and its lossless counterpart. Moreover, a system based on a separation of lossy quantization and the lossless source coding component usually provides competitive performance, though it may not be optimal. Indeed, the SR-MLD scheme is exactly such a system. It will be interesting to investigate whether similar phenomenon occur for other information theoretic network source coding problems.

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