Combinations of Context-Free Shifts and Shifts of Finite Type

Hiroshi Kamabe
Department of Information Science
Gifu University
1-1, Yanagido, Gifu, 501-1193, JAPAN
Email: kamabe@ieee.org

***Abstract*— A Dyck shift and a Motzkin shift are mathematical models for constraints on genetic sequences. In terms of the theory of symbolic dynamics, neither of the Dyck shift nor the Motzkin shift is sofic. In terms of the mathematical language theory, they are non-regular and context free languages. Therefore we can not use the Perron-Frobenius theory to calculate capacities of these constraints. O. Milenkovic has shown that the DSV (Delèst-Schẗzenberger-Viennot) theory for grammars gives us a method of calculating capacities of constraints modeled with context-free grammars. On the other hand, W. Krieger shown that the capacity of the Dyck shift with brackets of n kinds is $\log(n+1)$. Recently, K. Inoue has shown that the capacity of the Motzkin shift with brackets of n kinds and neutral symbols of m kinds is $\log(n+m+1)$. We give alternative proofs for these results by using the DSV theory. We also show that the DSV method allow us to calculate the capacity of a constraint given as a combinations of context free languages and shifts of finite type.**
keywords: DSV method, Dyck shift, Motzkin shift, capacity, input constraints, context-free grammar

I. Introduction

In digital storage devices, information sequences are encoded by an error correcting code and then by a recording code(there are coding schemes in which the recording code is applied first). We use the recording code because encoded sequences should satisfy some constraints which ensure that the sequences would be correctly retrieved or transmitted. Such constraints are called input constraints for the devices.

When we would use genetic sequences for processing, recording or transmitting information, we may impose input constraints on the sequences so that processing, recording or transmitting would be done without difficulties. Sets of Dyck paths and Motzkin paths can be regarded as mathematical models of such input constraints for genetic sequences.

Usually Dyck paths are defined with one pair of a left and a right brackets. By extending the number of pairs of brackets, we can define a special class of symbolic dynamics. It was shown by W. Krieger that the class of dynamical systems is very different from the class of sofic shifts [1]. The results were extended by K. Inoue[2] to dynamical systems modeled as Motzkin paths.

The topological entropy of a subshift is completely determined by an adjacency matrix of the shift if the shift is of finite type or sofic[3], [4]. However, we had to invent a method of calculating the capacity for each subshift if the shift is not sofic. Recently O. Milenkovic has shown that the DSV method, a method of calculating generating function of words defined by a context-free or slightly more general grammar, and a theory of analytical combinatorics can be employed in calculating capacities of subshifts defined by the context free grammar[5]. The method is systematic and we need no trick for each individual subshift other than elementary calculus.

Here we show that results by W. Krieger and K. Inoue can be derived by the DSV method and a result of the theory of analytical combinatorics. Then we apply the technique to the problem for calculating the capacity of a shift defined by a context free grammar and a subshift of finite type, e.g., a subshift consisting of Dyck paths satisfying a runlength limited constraints.

II. Preliminaries

For a set A, we define A^* to be a set of all finite sequences from A including the empty sequence and A^+ a set of all non-empty finite sequences from A.

Let Σ be a finite alphabet. A set of all bi-infinite sequences from Σ, or a set of all functions from the set $\mathbb{Z}$ of all integers to Σ, is called the full shift on Σ, denoted by $\Sigma^{\mathbb{Z}}$. For a set $\mathcal{F} \subseteq \Sigma^*$, we define $X_{\mathcal{F}}$ as follows

$$X_{\mathcal{F}} = \{x \in \Sigma^{\mathbb{Z}} : \forall w \in \mathcal{F}, w \text{ is not a subsequence of } x\}.$$

We call $X_{\mathcal{F}}$ a subshift and $\mathcal{F}$ a forbidden set. If a subshift S is given as $S = X_{\mathcal{F}}$ for some finite set $\mathcal{F}$, we call S a subshift of finite type(SFT). We define a map $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ by

$$(\sigma(x))_i = x_{i+1}, \qquad x \in \Sigma^{\mathbb{Z}}.$$

σ is called the shift map. If we introduce a distance $d(x,y)$ in the subshift by

$$d(x,y) = \sum_{i=-\infty}^{\infty} \frac{\overline{d}(x_i,y_i)}{2^{|i|}}$$

for sequences $x = (x_i)_{i\in\mathbb{Z}}$ and $y = (y_i)_{i\in\mathbb{Z}}$. where

$$\overline{d}(a,b) = \begin{cases} 0 & \text{if } a = b; \\ 1 & \text{if } a \neq b. \end{cases}$$

Let X and Y be two subshifts. For a function $f : X \to Y$, f is continuous and commutes with the shift map, that is,

$$f(\sigma(x)) = \sigma(f(x)) \quad \forall x \in X,$$

if and only if there exists a map $\phi : \Sigma^n \to \Sigma$ and integers n and l such that

$$(f(x))_i = \phi(x_{-l+i}x_{-l+1+i}\cdots x_{n+i}), \qquad \forall i \in \mathbb{Z},$$

[6]. The function f is called block map. If there is a surjective block map from X to Y, we say Y is a factor of X. If X is of finite type, then we say Y is a sofic shift. A forbidden set of a sofic shift is not necessarily a finite set.

Let X be a subshift. Let $\mathcal{B}_n(X)$ be the set of words of length n appearing as a subsequence of some sequence in X. Let $\mathcal{B}(X) = \cup_n \mathcal{B}_n(X)$. The topological entropy $h(X)$ of X or the capacity $C(X)$ of X is defined by

$$h(X) = C(X) = \lim_{n\to\infty} \frac{\log \mathcal{B}_n(X)}{n}.$$

See [4] or [3]. The base of the logarithm is usually 2.

We can define a measure theoretical entropy of a subshift for a measure on the shift, which was introduced as an invariant of a measure theoretical dynamical system by extending the concept of the entropy or the entropy rate of an information source introduced by Shannon[7]. The measure theoretical entropy of an SFT with a measure coincides with the entropy rate of a Markov chain corresponding to the SFT with the same measure. The topological entropy of an SFT is the maximum of the measure theoretical entropy over possible measures for the SFT. This fact was found by Shannon[7] and then rediscovered by Parry[8]. For an irreducible sofic shift, the maximal entropy measure is unique. But this does not hold for an irreducible non-sofc shift[1].

III. Dyck Shift and Motzkin Shift

A class of subshifts of finite type is simplest and well-studied in symbolic dynamics. A class of sofic shift is wider than the class of SFTs and corresponds to a class of regular languages in computational language theory or automata theory. The class has different names in other fields. There is a dictionary for translating technical terms among different fields[9].

A set of finite words is said to be regular if the set of generated by a regular grammar. If the set is generated by a context free grammar then the set is said to be context free. Let L be a language (a set of words of finite length) defined by a context free grammar. We can define a subshift whose forbidden set is $\Sigma^+ \backslash L$. A class of such subshift is properly wider than the class of sofic shifts[1]. In this draft subshifts in the class are said to be context free. There are many textbooks for computational language theory, e.g., [10].

Example 1: [3] Let $\Sigma = \{a, b, c\}$. We consider a constraint that if block ab^mc^ka appears then we should have $m = k$. A forbidden set for the constraint is

$$\mathcal{F}_1 = \{ab^mc^ka : m \neq k\}.$$

This set is a language defined by the following grammar

$$\begin{aligned} E &\to bEC|\varepsilon \\ B &\to b|bB \\ C &\to c|cC \\ D &\to aBEa|aECa \end{aligned}$$

A shift defined by $\mathcal{F}_1$ is not sofic, which can be proved by the pumping lemma.

If L is a regular language then $\Sigma^+ \backslash L$ is also regular. However, there is a context free language L such that $\Sigma^+ \backslash L$ is not a context free language. Thus, for a subshift X if X is an SFT then the forbidden set of X can be a finite set but if X is a context free shift then $\mathcal{B}(X)$ is context free.

A. Dyck shift

We consider points on the 2 dimensional plane whose coordinates are integers and a set of paths start from the origin and move forward along only directions $(1,1)$ and $(1,-1)$. A path in the set is said to be a Dyck path if it does not cross the X-axis. The Dyck path correspond to a sequence of brackets in which the number of right brackets is not greater than that of the left bracket. However, we note that a forbidden set of the subshift is empty because every finite sequence of directions $(1,1)$ and $(1,-1)$ can appear in some Dyck path. Hence the subshift is the full shift of 2 symbols.

Next we consider another constraint which is similar to the above constraint but whose alphabet consists of pairs of brackets of n kinds. We consider a set of sequences satisfying a constraint that brackets are always balanced, that is, an left bracket should be closed with a right bracket of the same kind. For example, if $n = 2$ then the following sequences are allowed

$$[(())][(())()]([([][][[]])]), \quad [[(())]]([])[][[]]([()])()[]$$

We also consider a set of left subsequences(prefixes) of sequences satisfying the above constraint. By these sets of sequences, we can define a subshift.

A formal definition of this subshift is given as follow [1]: An alphabet is defined as

$$\Sigma_D = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n\}.$$

Roughly speaking, α_i and β_i are left and right brackets of the i-th kind, respectively, for $i = 1, 2, \ldots, n$. We define an operation "$\cdot$" on $\Sigma_D \cup \{0, 1\}$ by the following rules

$$\begin{array}{ll} \alpha_i \cdot \beta_j = 1, & \text{if } i = j, \\ \alpha_i \cdot \beta_j = 0, & \text{if } i \neq j, \\ \eta \cdot 1 = 1 \cdot \eta = \eta, & \eta \in \Sigma_D \cup \{1\}, \\ 0 \cdot \eta = \eta \cdot 0 = 0, & \eta \in \Sigma_D \cup \{1\}, \\ 0 \cdot 0 = 0. & \end{array}$$

We also define a function red_D on Σ_D^* as follows,

$$red_D(a) = a_1 \cdot a_2 \cdots a_m, \quad a = a_1a_2\cdots a_m \in \Sigma_D^m.$$

For the empty sequence ε, we define

$$red_D(\varepsilon) = 1.$$

We define the Dyck shift D_n with brackets of n kinds by

$$D_n = \{x \in \Sigma_D^{\mathbb{Z}} : red_D(x_i x_{i+1} \cdots x_j) \neq 0, \text{ for } \forall i \leq \forall j\}.$$

For n with $n \geq 2$, D_n is not the full shift with $2n$ symbols.

This shift was introduced by Krieger and he proved that D_n is irreducible but the entropy maximizing measure is not unique and the entropy of D_n is $\log(n+1)$. In [11] we can find a method of calculating the topological entropy of D_n by counting possible paths without any measure theoretical discussion.

We can prove that D_n is not a sofic shift if $n \geq 2$ as follows(its proof is essentially the pumping Lemma). Suppose that D_n is a sofic shift. Then there should exist an SFT X and a block map π such that $D_n = \pi(X)$. We can assume that X is a set of bi-infinite paths(sequences of edges) of a directed finite graph G and π is a labeling of edges of G. There are paths in D_n which generates a sequence containing a finite sequence

$$\underbrace{(((\cdots(}_{k \text{ times}}$$

for any positive integer k. Therefore X should contain a cycle $a = a_1 \cdots a_l$ such that $\pi(a^\infty) = (^\infty$ where a^∞ means a path repeating a infinitely many times. We can also find a cycle $b = b_1 \cdots b_k$ such that $\pi(b^\infty) =]^\infty$. Let c be a path from the initial state of a to the initial state of b. We consider a sequence which can be generated by a path $a^m c b^n$

$$z = \pi(a^m)\ \pi(c)\ \pi(b^n), \quad ml > nk + \lg(c`)$$

where $\lg(y)$ is the length of y. Although z would be generated by G, we note that $red_D(z) = 0$. This means G generates a sequence which is not a Dyck path. This is a contradiction.

B. Motzkin shift

A Motzkin shift can be regarded as a Dyck shift with 'neutral symbols.' A formal definition of the shift is given as follows[1]. An alphabet Σ_M of the shift is

$$\Sigma_M = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n, 1_1, 1_2, \ldots, 1_m\}.$$

We define a binary operation " $\circ$ " on $\Sigma_M \cup \{0,1\}$ by the following rules:

$$\begin{array}{ll}
\alpha_i \circ \beta_j = 1, & \text{if } i = j, \\
1_i \circ 1_j = 1, & \text{for } 1 \leq i \leq n, 1 \leq j \leq m \\
\alpha_i \circ \beta_j = 0, & \text{if } i \neq j, \\
\eta \circ 1 = 1 \circ \eta = \eta, & \text{for } \eta \in \Sigma_M \cup \{1\} \\
1_i \circ \eta = \eta \circ 1_i = \eta, & \text{for } 1 \leq i \leq N, \eta \in \Sigma_M \cup \{1\} \\
0 \circ \eta = \eta \circ 0 = 0, & \\
0 \circ 0 = 0. &
\end{array}$$

We also define the function red_M on sequences of this alphabet with respect to the operation " $\circ$ "as well as red_D. Then $M_{n,m}$ is defined by

$$M_{n,m} = \{x \in \Sigma_M^{\mathbb{Z}} : red_M(x_i x_{i+1} \cdots x_j) \neq 0 \text{ for } \forall i \leq \forall j\}$$

Inoue has shown that the topological entropy of $M_{n,m}$ is $\log(n+m+1)$ [2].

IV. DSV Theory

Schützenberger[12] introduced a systematic method of calculating a generating function of words generated by a grammar given as a set of rewriting rules. This method was extended to attribute grammars. We mean both of these methods by the DSV method[5]. The DSV method are explained in [5].

In this section we calculate a generating function of Dyck paths which terminate on the X-axis as an example of an application of the DSV method.

Example 2: [5] The following set of rules determine the Dyck paths which start from the origin and terminate on the X-axis.

$$\begin{array}{lcl}
D & \to & \varepsilon, \\
D & \to & z \cdot D \cdot \bar{z} \cdot D.
\end{array}$$

In the second rule, z means the unit move $(1,1)$ and $\bar{z}$ means $(1,-1)$, respectively. Symbol ε means the empty sequences.

Next we take sums of rewriting rules for the same terminal symbols. Since we have only one terminal symbol D in this example, we have

$$D \to \varepsilon + z \cdot D \cdot \bar{z} \cdot D.$$

We assume that all addition and multiplications are commutative. Under this assumption, we get

$$D \to \varepsilon + z\bar{z}D^2.$$

We transform all variables into a single variable t and ε into 1. We replace symbol '$\to$' with symbol '$=$'. Then we get the following for our example.

$$D(t) = 1 + t^2 D^2(t).$$

By solving this equation with respect to $D(t)$, we get

$$D(t) = \frac{1 \pm \sqrt{1-4t^2}}{2t^2}$$

by multiplying $1 \mp \sqrt{1-4t^2}$,

$$= \frac{1 - (1-4t^2)}{2t^2(1 \mp \sqrt{1-4t^2})}$$

$$= \frac{2}{1 \mp \sqrt{1-4t^2}}$$

From the definition of the generating function, we should have $\lim_{t \to 0} D(t) = 1$,

$$\begin{array}{rcl}
D(t) & = & \dfrac{2}{1+\sqrt{1-4t^2}} \\
& = & \dfrac{1-\sqrt{1-4t^2}}{2t^2}.
\end{array} \qquad (1)$$

This is equal to the generating function of the Catalan number.

Example 3: Consider the following set of blocks

$$E = \{a^i b^j : 0 \leq i \leq j\}.$$

The blocks in E are generated by the following rewriting rules,

$$\begin{aligned} S &\rightarrow ab+aSb, \\ B &\rightarrow \varepsilon+bB, \\ E &\rightarrow B+SBE. \end{aligned}$$

We can show that a subshift defined by this set of rewriting rules is no sofic. From these rules we have

$$S(t)=\frac{t^2}{1-t^2}, B(t)=\frac{1}{1-t}.$$

Since we have

$$E(t)=B(t)+B(t)S(t)E(t),$$

$E(t)$ is given by

$$E(t)=\frac{1-t^2}{(1-t)(1-t^2)-t^2}.$$

The denominator of this vanishes at 0.554958132.

A sofic shift can be represented by a directed graph with a right resolving labeling of its edges and the capacity of the shift is the logarithm of the largest real eigen value of the adjacency matrix of the graph. Although we can not apply this method to non sofic shifts, we can calculate the capacity of a non sofic shift by using the following result when we can find a generating function of the shift.

When a function $f(z)$ is expanded as

$$f(z)=\sum_{n\geq 0} f_n z^n,$$

by $[z^n]f(z)$ we mean the n-th coefficient in this expansion.

Definition *1:* We define a relation '$\bowtie$' by

$$a_n \bowtie K^n \quad \Leftrightarrow \quad \limsup_n |a_n|^{1/n} = K$$

This is equivalent to

$$\limsup_n \frac{|a_n|}{K^n}=1.$$

Definition *2:* We say that the function $f(z)$ is analytic if $f(z)$ can be expanded at z.

Theorem *1:* (Exponential Growth Formula) [13] We assume that $f(z)$ is analytic at $z=0$ and define R by

$$R=\sup\{r\geq 0 : f \text{ is analytic at } z \text{ with } |z|<r\} \qquad (2)$$

Then we have

$$[z^n]f(z)=f_n \bowtie \left(\frac{1}{R}\right)^n.$$

V. Topological Entropy of Dyck Shift

A. Case: $n=1$

We have shown that the capacity of the Dyck shift is 1 when $n=1$. Here we consider generating function of paths starting from the origin and terminating on the X-axis. The generating function is given by (1). It is well-known that by expanding the function and taking the $2n$-th coefficient of the expansion we can see that the number of such paths of length $2n$ is

$$\frac{1}{n+1}\binom{2n}{2},$$

which is the Catalan number C_n. We approximate C_n by using the stirling formula and then we obtain

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

The exponential growth rate of the number is

$$\lim_{n\to\infty}\frac{1}{2n}\log C_n = \lim_{n\to\infty}\frac{1}{2n}\log 2^{2n}=1.$$

This corresponds to the fact that the Dyck shift D_1 with one pair of brackets is the full shift. But Krieger shown that this does not hold for $n\geq 2$ and calculated the capacity of D_n [1]. We will explain this by using the DSV method.

B. Case: $n\geq 2$

A set $\mathcal{B}_N(D_n)$, a set of N-blocks appearing in some sequence in D_n, is larger than a set of balanced N-blocks $\mathcal{D}_N(D_n)$,

$$\mathcal{D}_N(D_n)=\{a_1\cdots a_N : red_D(a_1\cdots a_N)=1\}.$$

For example, if (((()((()[]))))))) then $\alpha\in\mathcal{B}_{18}(D_2)$ but $\alpha\notin\mathcal{D}_{18}(D_2)$.

On the other hand, such blocks correspond to periodic points in D_n. These blocks are defined formally as follows:

$$\mathcal{D}_L(D_n)=\{u\in\mathcal{B}(D_n) : red_D(u)\in A^+\}.$$

where $A=\{\alpha_1,\cdots,\alpha_n\}$. A similar set of blocks can be defined as follows

$$\mathcal{D}_R(D_n)=\{u\in\mathcal{B}(D_n) : red_D(u)\in B^+\},$$

where $B=\{\beta_1,\cdots,\beta_n\}$. Intuitively, $\mathcal{D}_L(D_n)$ and $\mathcal{D}_R(D_n)$ can be considered as a set of all postfixes of blocks in $\mathcal{D}_N(D_n)$ and a set of all prefixes of blocks in $\mathcal{D}_N(D_n)$, respectively. The remaining blocks are given by

$$\mathcal{D}_B(D_n)=\{u\in\mathcal{B}(D_n) : red_D(u)=B^+A^+\}.$$

Then we have

$$\mathcal{B}(D_n)=\mathcal{D}(D_n)\cup\mathcal{D}_L(D_n)\cup\mathcal{D}_R(D_n)\cup\mathcal{D}_B(D_n),$$

where $\mathcal{D}(D_n)=\cup_{i=0}^{\infty}\mathcal{D}_N(D_n)$. Note that four sets in the right hand side are disjoint. Rewriting rules for these sets of blocks are given as follows.

$$\begin{aligned} D &\rightarrow \sum_{i=1}^{n}\alpha_i\cdot D\cdot\beta_i\cdot D+\varepsilon \\ D_L &\rightarrow \sum_{i=1}^{n} D\cdot\beta_i\cdot D+\sum_{i=1}^{n} D_L\cdot\beta_i\cdot D \\ D_R &\rightarrow \sum_{i=1}^{n} D\cdot\alpha_i\cdot D+\sum_{i=1}^{n} D\cdot\alpha_i\cdot D_R \\ D_B &\rightarrow \sum_{i=1}^{n} D_L\cdot\alpha_i\cdot D+\sum_{i=1}^{n} D_B\cdot\alpha_i\cdot D \end{aligned}$$

where we write rules having the same nonterminal symbol on the left hand side as a single sum.

We calculate generating functions of $D(t)$, $D_L(t)$, $D_R(t)$ and $D_B(t)$ with respect to the block length by the DSV method. Although $D(t)$ is one of solutions of a quadratic equation, we can get the following results from the fact that $\lim_{t\to 0} D(t) = 1$.

$$\begin{aligned} D(t) &= \frac{2}{1+\sqrt{1-4nt^2}} \\ &= \frac{1-\sqrt{1-4nt^2}}{2nt^2}, \qquad (3) \\ D_L(t) &= D_R(t) = \frac{tD(t)^2}{1-tnD(t)}, \qquad (4) \\ D_B(t) &= \frac{t^2D(t)^3}{(1-ntD(t))^2}. \end{aligned}$$

Equation (3) is very similar to (1). Therefore we can apply the same method of calculating the Catalan number to (3). The result is

$$\#\mathcal{B}_{2N}(D_n) = [t]^{2N}D(t) = \frac{1}{N+1}\binom{2N}{N}n^N.$$

Therefore we have

$$\lim_{N\to\infty}\frac{1}{2N}\log\#\mathcal{B}_{2N}(D_n) = 1+\frac{1}{2}\log n.$$

There is another method of obtaining this value. In (3), if $1-4nt^2$ is negative then $D(t)$ is not analytic. Therefore

$$\frac{1}{2\sqrt{n}} = \sup\{r \geq 0 : (3) \text{ is analytic } z \text{ with } |z| < r\}.$$

Thus we can conclude that

$$\limsup_{n\to\infty}\frac{1}{n}\log[t^n]D(t) = 1+\frac{1}{2}\log n.$$

Next, we evaluate a denominator of $D_L(t)$, $D_R(t)$ and $D_B(t)$ at $\bar{t} = 1/(n+1)$ to calculate exponential growth rate of $[t^n]D_L(t)$, $[t^n]D_R(t)$ and $[t^n]D_B(t)$.

$$\begin{aligned} 1-\bar{t}nD(\bar{t}) &= 1-\bar{t}\frac{1-\sqrt{1-4n\bar{t}^2}}{2n\bar{t}^2} \\ &= 1-\frac{1-\sqrt{1-\frac{4n}{(n+1)^2}}}{\frac{2}{n+1}} \\ &= 1-\frac{1-\sqrt{\frac{(n-1)^2}{(n+1)^2}}}{\frac{2}{n+1}} \\ &= 0. \end{aligned}$$

This means that none of $D_L(t), D_R(t), D_B(t)$ is analytic at $\bar{t}$. We not that $2\sqrt{n} < n+1$ for integer n with $n \geq 2$. Therefore

$$\frac{1}{n+1} = \sup\{r \geq 0 : \widetilde{D}(t) \text{ is analytic } z \text{ with } |z| < r\}$$

where $\widetilde{D}(t) = D(t)+D_L(t)+D_R(t)+D_B(t)$. From Theorem 1 we note that the exponential growth rate $[t^n]\widetilde{D}(t)$ is $\log(n+1)$. This result coincides with the result by Krieger [1].

The above argument can be applied to the case $n=1$. If $n=1$ then

$$\limsup_{n\to\infty}\frac{1}{n}\log[t^n]D(t) = 1+\frac{1}{2}\log n = 1.$$

The denominator of (4) is 0 at $\bar{t} = 1/2$ when $n=1$ and we note that the denominator is not 0 if $0 < t < 1/2$. Therefore we can also conclude that the entropy of D_1 is $\log 2 = 1$ from Theorem 1.

Remark *1:* For the case $n=1$, Dyck paths are defined to be sequences in D_L in [5]. We can define a set of periodic points in D_n from the Dyck paths and get D_n by taking the topological closure of the set. Therefore there is no essential difference between a definition of Dyck paths in [5] and a definition of Dyck shift in [1].

According to the definition, we must calculate the generating function of $\mathcal{B}(D_n)$. The above example means that we can calculate the function step by step, that is, $D(t), D_L(t)$ and $D_B(t)$. For the Dyck shift we have shown

$$\limsup_{n\to\infty}\frac{1}{n}\log[t^n]D(t) \neq \limsup_{n\to\infty}\frac{1}{n}\log[t^n]D_L(t).$$

But this is not a common property of a class of context free shift.

Example 4: We consider Example 3 again. Let Y be a subshift defined by the set of rewriting rules in the example. First we describe rewriting rules for a set of prefixes of blocks in E.

$$\begin{aligned} E_R &\rightarrow EAS \\ A &\rightarrow a|aA \end{aligned}$$

Let $A(t)$ and $E_R(t)$ be generating functions of A and E_R, respectively. Then we have

$$\begin{aligned} A(t) &= \frac{1}{1-t}, \\ E_R(t) &= E(t)A(t)S(t) = \frac{1-t^2}{(1-t)(1-t^2)-t^2}\frac{1}{1-t} \end{aligned}$$

We note that every postfix of a block in E is again in E. Hence, $E_L = E$ and we have

$$\mathcal{B}(Y) = E + E_R.$$

Therefore we get the generating function of $\mathcal{B}(t)$ as

$$\frac{1-t^2}{(1-t)(1-t^2)-t^2}\left(1+\frac{t^3}{(1-t)(1-t^2)}\right).$$

The denominator of this function also vanishes at 0.554958132. Therefore, from Theorem 1 we can conclude that

$$\limsup_{n\to\infty}\frac{1}{n}\log[t^n]E(t) = \limsup_{n\to\infty}\frac{1}{n}\log[t^n]E_L(t).$$

VI. Entropy of Motzkin Shift

We can calculate entropies of Motzkin shifts as well as Dyck shifts.

First we define sets of blocks corresponding to periodic points of a Motzkin shift

$$\begin{aligned} M &= \{u \in \mathcal{B}(M_{n,m}) : red_M(u) = 1\}, \\ M_L &= \{u \in \mathcal{B}(M_{n,m}) : red_M(u) \in A^*\}, \\ M_R &= \{u \in \mathcal{B}(M_{n,m}) : red_M(u) \in B^*\}, \end{aligned}$$

where $A = \{\alpha_1, \ldots, \alpha_n\}$ and $B = \{\beta_1, \ldots, \beta_n\}$. The remaining blocks are

$$M_B = \{u \in \mathcal{B}(M_{n,m}) : red_M(u) \in B^+A^+\}.$$

These four sets are disjoint. Blocks in these sets are generated by the following rewriting rules.

$$\begin{aligned} M &\rightarrow \varepsilon + \sum_{i=1}^{n} \alpha_i \cdot M \cdot \beta_i \cdot M + \sum_{j=1}^{m} 1_j \cdot M, \\ M_R &\rightarrow \sum_{i=1}^{n} M \cdot \alpha_i \cdot M_R + \sum_{i=1}^{n} M \cdot \alpha_i \cdot M, \\ M_L &\rightarrow \sum_{i=1}^{n} M_L \cdot \beta_i \cdot M + \sum_{i=1}^{n} M \cdot \beta_i \cdot M, \\ M_B &\rightarrow \sum_{i=1}^{n} M_L \cdot \alpha_i \cdot M + \sum_{i=1}^{n} M_B \cdot \alpha_i \cdot M. \end{aligned}$$

The generating function $M(t)$ corresponding to a nonterminal symbol M is given by solving an equation

$$M(t) = 1 + nt^2 M(t)^2 + mtM(t).$$

So we have

$$M(t) = \frac{1 - mt \pm \sqrt{(mt-1)^2 - 4nt^2}}{2nt^2}.$$

We must have $\lim_{t \to 0} M(t) = 1$ from the definition of the generating function. Therefore we can conclude that

$$M(t) = \frac{1 - mt - \sqrt{(mt-1)^2 - 4nt^2}}{2nt^2}.$$

Similarly, generating functions $M_L(t)$ and $M_R(t)$ corresponding to M_L and M_R, respectively, are given as follows,

$$M_L(t) = M_R(t) = \frac{ntM(t)^2}{1 - tnM(t)}.$$

$M_B(t)$ is

$$M_B(t) = \frac{nM(t)M_L(t)}{1 - tnM(t)} = \frac{n^2 tM(t)^3}{(1 - tnM(t))^2}.$$

If $(mt-1)^2 < 4nt^2$ then $M(t)$ is not analytic. Therefore

$$\frac{1}{m+2n} = \sup\{r \geq 0 : M(t) \text{ is analytic at } z \text{ with } |z| < r\}.$$

From Theorem 1 we have

$$\limsup_{n \to \infty} \frac{1}{n} \log[t^n]M(t) = \log(m+2n).$$

We can show that denominators of $M_L(t)$,$M_R(t)$ and $M_B(t)$ vanish at $t = 1/(n+m+1)$. Let $\bar{t} = 1/(n+m+1)$.

$$\begin{aligned} (m\bar{t}-1)^2 - 4n\bar{t}^2 &= \left(\frac{m-m-n-1}{n+m+1}\right) - \frac{4n}{(n+m+1)^2} \\ &= \frac{n^2 - 2n + 1}{(n+m+1)^2} \\ &= \frac{(n-1)^2}{(n+m+1)^2} \\ M(\bar{t}) &= \frac{n+m+1}{n} \\ 1 - \bar{t}nM(\bar{t}) &= 1 - \frac{n+m+1}{n} \cdot n \cdot \frac{1}{n+m+1} = 0. \end{aligned}$$

If $n \geq 2$ then we have

$$\frac{1}{n+m+1} < \frac{1}{m+2n}.$$

Thus we have

$$\sup\{r \geq 0 : \widetilde{M}(t) \text{ is analytic at } z \text{ with } |z| < r\} = \frac{1}{n+m+1}$$

where $\widetilde{M}(t) = M(t) + M_L(t) + M_R + M_B(t)$. Therefore, from Theorem 1 we note that the exponential growth rate of $[t^n]M_R(t)$ is $\log(n+m+1)$. This result coincides with the result by Hamachi and Inoue[14].

VII. Approximation of Dyck Shift from Inside and Coding

Hamachi and Inoue gave necessary and sufficient conditions that an SFT is embedded into a Dyck shift [14].

For a subshift X, we mean a set of all periodic points of X by $P(X)$. We define $P(D_n)^-$ by

$$P(D_n)^- = \{x \in P(D_n) : red(x_i \cdots x_j) \in A^+, \forall i \leq \forall j\},$$

where A is the set of left brackets. We note that $P(D_n)^-$ corresponds to M_L in the previous section. Then we have

Theorem *2 (T. Hamachi and K. Inoue):* [14, Theorem 6.3] Suppose that $n > 2$. A necessary and sufficient condition that X is embedded into D_n is that there is an injective map from $P(X)$ to $P(D_n)^-$ and $h(X) < \log(n+1)$.

A condition for the case $n = 2$ is slightly different from that for the case $n > 2$ [14, Theorem 5.3].

As an application of these results we may construct a code by which we encode a free binary sequence into a DNA or RNA sequence satisfying a constraint modeled as a Dyck or Motzkin path. The condition on the number of periodic points may not be satisfied for some cases. Moreover, an encoder for the coding problem may not be needed to be a block map but it will be sufficient that we are able to construct a finite state encoder with a sliding block decoder. Consequently, we use the following fact to construct such an encoder.

$$\log(n+1) = \sup\{h(W) : W \subset D_n,\ W \text{ is an irreducible SFT}\}. \tag{5}$$

This is the fact proved in the proof of the above theorem [14, Remark 4.5]. From (5), we can take a sequence $(W_i)_{i\geq 0}$ of SFT's such that

$$\lim_{i\to\infty} h(W_i) = \log(n+1), \quad W_i \subset D_n.$$

Using this fact we can construct a finite state encoder with a sliding block decoder for D_n as follows.

1) We assume that the number of data symbols is K. We take nonnegative integers p and q such that $K^p < (n+1)^q$.
2) We find an SFT $W_i \subset D_n$ with
$$\frac{p}{q}\log K \leq h(W_i) < (n+1).$$
There should be such an SFT from the fact mentioned above.
3) Since we have $p\log K \leq qh(W_i) = h(W_i^q)$, we can apply the code construction method given in [15] to W_i^q and we can get a sliding block decodable finite state encoder from the full shift with K^p symbols to W_i^q where W_i^q is the q-th power shift of W_i.

A resulting encoder may depend on W_i^q which we take in Step 2). Therefore, we must investigate the structure of a constraint defined by a context free grammar in order to construct an encoder for the constraint.

VIII. Dyck shift and Run-length constraint

Dyck shifts and Motzkin shifts can be regarded as mathematical models for constraints on DNA and RNA sequences. Pairs of brackets correspond to bonds of Watson-Crick pairs. In some operations on genetic sequences the number of G(guanine) in a sequence should be equal to that of C(cytosine). A ratio of four symbols determines the temperature at which some chemical operation can be applied. These facts mean we may need some constraints other than the secondary structure of genetic sequences. Therefore we consider sequences which satisfy the Dyck constraint and a constraint of finite type simultaneously. We show that the entropy of a shift consisting such sequences can also be calculated systematically.

We assume that the alphabet is $\{(,),[,]\}$ and consider a $(0,1)$-RLL constraint as an additional constraint which requires that there should be at most 1 left bracket between consecutive two left parenthesis and there should be at most 1 right bracket between consecutive two right parenthesis. We mean a shift satisfying this constraint by $D_{0,1}$. This shift can be considered as an example of shifts investigated by Krieger and Matsumoto [16].

In calculating the capacities of D_n and $M_{n,m}$ we first tried to obtain generating functions for allowable blocks for these constraints. We also calculate the capacity of $D_{0,1}$ in a similar way.

Rewriting rules for this constraints can be given as follows:

$$\begin{aligned} D &\rightarrow \varepsilon + (\cdot D\cdot)\cdot D + [\cdot E\cdot]\cdot D, \\ E &\rightarrow (\cdot D\cdot) + \varepsilon. \end{aligned}$$

Generating functions $D(t)$ and $E(t)$ corresponding D and E, respectively, are

$$\begin{aligned} D(t) &= 1 + t^2D^2(t) + t^2E(t)D(t), \\ E(t) &= t^2D(t) + 1. \end{aligned}$$

We have the following equation for $D(t)$

$$0 = 1 + (t^2-1)D(t) + t^2(t^2+1)D^2(t).$$

By solving this with respect to $D(t)$, we get

$$D(t) = \frac{1-t^2 \pm \sqrt{(t^2-1)^2 - 4t^2(t^2+1)}}{2t^2(t^2+1)}.$$

Since we should have $\lim_{t\to 0} D(t) = 1$, we conclude that

$$\begin{aligned} D(t) &= \frac{1-t^2 - \sqrt{(t^2-1)^2 - 4t^2(t^2+1)}}{2t^2(t^2+1)} \\ &= \frac{2}{1-t^2 + \sqrt{(t^2-1)^2 - 4t^2(t^2+1)}}. \end{aligned}$$

We have

$$(t^2-1)^2 - 4t^2(t^2)+1) = 0$$

when $t = 0.3933198\ldots$.

We can also define a set D_R in $D_{0,1}$ corresponding to D_R in D_n. Rewriting rules of D_R can be given by

$$\begin{aligned} D_R \rightarrow\ & (\cdot D_R + [\cdot(\cdot D_R + (\cdot D + [\cdot E \\ & +(\cdot D\cdot D_R + [\cdot E\cdot D_R \end{aligned}$$

A generating function $D_R(t)$ corresponding to D_R must satisfy

$$\begin{aligned} D_R(t) =\ & tD_R(t) + t^2D_R(t) + tD(t) + tE(t) \\ & tD(t)D_R(t) + tE(t)D_R(t) \end{aligned}$$

By solving this equation we get

$$D_R(t) = \frac{tD(t) + tE(t)}{1-t^2-t-tD(t)-tE(t)} \tag{6}$$

By numerical calculation, we know that the denominator of (6) vanishes at $t_0 = 0.3364712$ and is positive for $0 \leq t < t_0$. Thus, we have

$$\limsup_{n\to\infty} \frac{1}{n}\log[t^n]D_R(t) = \log_2 \frac{1}{0.3364712} = 1.57144507.$$

We note that $D_L(t) = D_R(t)$. Since

$$D_B \rightarrow D_L \cdot D \cdot D_R,$$

we have $D_B(t) = D(t)D_R(t)^2$. Thus we can conclude that the entropy of the shift is approximately $\log_2 1/0.3364712 = 1.51768881531$.

Next we consider a shift defined as a product of D_2 and a $(1,2)$ RLL constraint, say $D_{1,2}^{RLL}$. A set of rewriting rules for the shift can be given as follows.

$$\begin{aligned} U &\rightarrow \varepsilon + [\cdot[\cdot U_0\cdot]\cdot]\cdot U + [\cdot U_1\cdot]\cdot U \\ &\qquad (\cdot(\cdot V_0\cdot)\cdot)\cdot U + (\cdot V_1\cdot)\cdot U \\ U_0 &\rightarrow (\cdot U_1\cdot) + \varepsilon \\ U_1 &\rightarrow (\cdot V_0\cdot) + [\cdot V_1\cdot] + \varepsilon \\ V_0 &\rightarrow [\cdot V_1\cdot] + \varepsilon \\ V_1 &\rightarrow [\cdot U_0\cdot] + (\cdot U_1\cdot) + \varepsilon \end{aligned}$$

From these rewriting rules we get

$$\begin{aligned} U_1(t) &= V_1(t) = \frac{t^2+1}{1-t^2(t^2+1)}, \\ U_0(t) &= V_0(t) = \frac{1}{1-t^2(t^2+1)}, \\ U(t) &= \frac{1}{1-2t^2U_0(t)-2tU_1(t)} \end{aligned}$$

The denominator of $U(t)$ vanishes at 0.361103. Let S_R be a set of prefixes of U. The rewriting rules for S_R can be given as follows

$$\begin{aligned} S_R &\to [\cdot[\cdot P_0 + [\cdot P_1 + (\cdot(\cdot B_0 + (\cdot B_1 \\ &\quad +[\cdot[\cdot P_0 \cdot S_R + [\cdot P_1 \cdot S_R + (\cdot(\cdot B_0 \cdot S_R + (\cdot B_1 \cdot S_R \\ S_1 &\to (\cdot E_0 \cdot) + [\cdot E_1 \cdot] + \varepsilon \\ S_0 &\to (\cdot S_1 \cdot) + \varepsilon \\ E_0 &\to [\cdot E_1 \cdot] + \varepsilon \\ E_1 &\to [\cdot S_0 \cdot] + (\cdot S_1 \cdot) + \varepsilon \\ P_1 &\to (\cdot B_0 + [\cdot B_1 + S_1 \\ P_0 &\to (\cdot P_1 + S_0 \\ B_0 &\to [\cdot B_1 + E_0 \\ B_1 &\to [\cdot P_0 + (\cdot P_1 + E_1 \end{aligned}$$

From these rules, we have

$$\begin{aligned} S_R(t) &= \frac{2t^2P_0(t)+2tP_1(t)}{1-2t^2P_0(t)-2tP_1(t)} \\ P_1(t) &= B_1(t) = \frac{t^3S_1(t)+t+1}{1-t^2-t} \\ P_0(t) &= B_0(t) = tP_1(t)+t^2S_1(t)+1 \\ S_1(t) &= E_1(t) = \frac{t^2+1}{1-t^2-t^4} \end{aligned}$$

The denominator of $S_R(t)$ vanishes at 0.334389419. We can show that S_L, the set of all postfixes of blocks in S, has the same generating function. If we define S_B by

$$S_B \to S_L \cdot S \cdot S_R,$$

we get a disjoint decomposition of $\mathcal{B}(D^{RLL}_{1,})$

$$\mathcal{B}(D^{RLL}_{1,}) = S + S_R + S_L + S_B.$$

A generating function of S_B is

$$S_B(t) = S_R(t)^2 S(t).$$

Therefore we can conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log[t^n]G(t) = \log 0.334389419$$

where $G(t)$ is a generating function of $\mathcal{B}(X)$. Therefore capacities of $D_{0,1}$ and $D^{RLL}_{1,2}$ are different.

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