GENERALIZED PERIODIC NON-UNIFORM SAMPLING OF NON-BANDLIMITED SIGNALS

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ABSTRACT

A new paradigm for the periodic non uniform sampling of a class of non bandlimited signals was recently proposed. The main idea is to generate the periodic non uniform samples by retaining a select group of samples from a larger set, obtained by oversampling the continuous time signal by an integer factor L. In this paper, we revisit the problem and consider the more challenging case where the sampler operates at L/M time the Nyquist rate with L and M being coprime integers. We derive a new multi-input multi-output (MIMO) discrete-time model, and then use this model to obtain a set of necessary conditions for perfect signal reconstruction with FIR filters. We also show that unlike the case of $M = 1$, the newly derived conditions are not sufficient for FIR perfect recovery.

Index Terms— Periodic Non-Uniform Sampling, Finite Rate of Innovations, Multirate DSP, Wavelets, Sylvester

1. INTRODUCTION

The classical sampling theorem derived by Shannon in 1949 [1] states that a σ-bandlimited continuous time signal $y_c(t)$ can be recovered *uniquely* from its uniform periodic samples $y(n) \triangleq y_c(nT)$ as long as $T \leq \pi/\sigma$ and is given by

$$y_c(t) = \sum_{n=-\infty}^{\infty} x(n)\phi(t - nT) \qquad (1)$$

where $\phi(t) = \sin(\pi t/T)/(\pi t/T)$ and $x(n) = y_c(nT)$. A nice interpretation of the reconstruction formula (1) was recently proposed in [2] by observing that any real bandlimited signal with support $[-\pi/T, \pi/T]$ has $1/T$ degrees of freedom per unit time, which is basically the number of time samples required to specify it. With this interpretation in mind, the extension of Shannon's sampling theorem for bandlimited signals to a larger class of continuous-time *non-bandlimited* signals $y_c(t)$, namely the class of signals that have *finite rate of innovations* [2], can be obtained. Consider therefore the class of continuous-time signals $y_c(t)$ that has the form (1) where, in this case, $x(n)$ is not necessarily equal to $y_c(nT)$ and $\phi(t)$ is not the sinc function but a *known* function of *compact support*. Popular examples of $\phi(t)$ found in the literature are the wavelet generating function proposed by Walter [3] and Mallat [4] in the context of multiresolution analysis (MRA) and spline functions proposed by Unser [5] for image processing applications. The sequence of coefficients $x(n)$ is typically assumed to have finite energy, i.e., $x(n) \in \ell_2$ and the generating function $\phi(t)$ decays faster than $1/t$ as given below

$$|\phi(t)| \leq \frac{K}{1 + |t|^{1+\epsilon}} \qquad (2)$$

for some $K > 0$ and $\epsilon > 0$ [6, 7]. The above properties are important to insure that the reconstruction of the signal $y_c(t)$ from its samples is pointwise stable, i.e., that a small perturbation in the samples values $y_c(nT)$ results in a reconstruction error that is also small pointwise in time. So, how is it possible to reconstruct $y_c(t)$ from the samples $y(n)$ even though $y_c(t)$ is not bandlimited and frequency aliasing occurs ? To answer this, we assume for simplicity purpose and without loss of generality that $T = 1$ (normalized time scale). Then, from (1), the uniform samples have the form $y(n) = \sum_{k=-\infty}^{\infty} x(k)\phi(n - k)$ which is a discrete time convolution sum. We can therefore recover the coefficients $x(n)$ from $y(n)$ by using the digital filter $1/\Phi(z)$ where $\Phi(z) = \sum_n \phi(n)z^{-n}$, provided the filter is realizable. For example, when $\phi(t)$ is $\beta^N(t)$, $\Phi(z)$ is FIR with zeros both inside and outside the unit circle and its inverse is implemented as a stable non causal IIR filter to achieve pointwise stability of the reconstruction process [5]. Although this works well for finite length signals like images (see [5] and the references therein), the implementation of non-causal IIR filters has limitations and therefore, reconstruction solutions with FIR filters are often desirable. The feasibility of perfect reconstruction by FIR filters using difference sampling was presented in [8] and a more general exposition that provides necessary and sufficient conditions for FIR reconstruction can be found in [9]. We consider here the periodic non uniform sampling of the class of non bandlimited signals that can be represented as in (1) with $\phi(t)$ having compact support. The periodic non uniform samples are obtained by retaining a select number of samples from $y(n) \triangleq y_c(nM/L)$. In this case, the sequence $y(n)$ is modeled as the output of the discrete time fractional interpolation filter of Fig. 1 where $\uparrow L$ denotes an upsampler of factor L, $\downarrow M$ denotes a downsampler of factor M and $F(z)$ is the FIR transfer function of the sequence $f(n) \triangleq \phi(n/L)$.

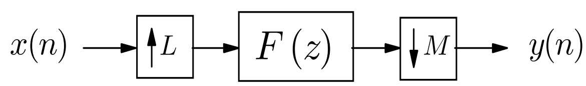

Fig. 1. A discrete-time multirate signal model

Since $y_c(t)$ is oversampled, we must have $L > M \geq 1$. The oversampling of $y_c(t)$ generates therefore a multirate DSP model and reconstruction of $x(n)$ (and consequently $y_c(t)$) from $y(n)$ can be achieved by FIR filtering. For $M = 1$, a number of necessary and sufficient conditions for retrieving $x(n)$ from a periodic non uniformly decimated version of $y(n)$ by FIR filtering is given in [10]. The main contribution of this paper is therefore the extension of the results of [10] to the more challenging fractional rate case, $M > 1$. Similar to [10], the work presented here is of theoretical nature and is organized as follows: using multirate DSP theory, we first obtain a novel discrete-time MIMO model for the fractional rate case. Using this structure, we then derive necessary conditions for FIR reconstruction of $x(n)$. Similar to the results of [10], these conditions depend on the relative primeness of the polyphase components of $F(z)$ but, unlike [10], are *not* sufficient for FIR reconstruction of $x(n)$. Numerical examples that illustrate the key findings of this work are also given.

2. THE DISCRETE TIME MODEL

For simplicity, we assume that $F(z)$ in Fig. 1 is a causal filter. We emphasize however that this assumption is not necessary in any of our derivations. The multirate discrete time model of Fig. 1 can be first converted to the structure of Fig. 2 [11].

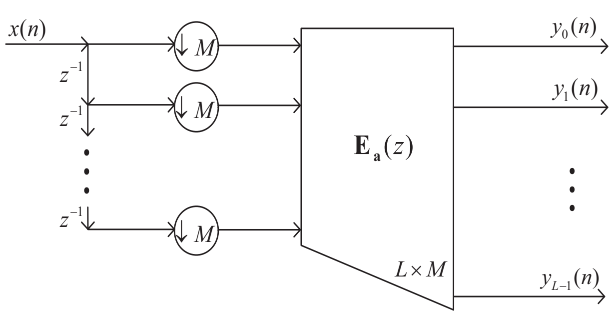

Fig. 2. An equivalent representation of Fig. 1

Let $F(z) = \sum_{i=0}^{L-1} F_i(z^L)z^i$, where $F_i(z)$, $i = 0, 1, \ldots, L$ are its L polyphase components. For the i^{th} row vector of the matrix transfer function $\mathbf{E_a}(z)$, the M elements are the M polyphase components of $H_i(z) \triangleq z^{il}F_i(z)$, where l is a factor determined by L and M. The elements of the matrix $E_a(z)$ are therefore the M polyphase components of the adequately shifted versions of the L polyphase components of $F(z)$. Starting from Fig. 2, we now derive a new and more generic Multi-Input Multi-Output (MIMO) model. Let $E_{a\ i,j}(z)$ denote the $(i, j)^{\text{th}}$ element of $\mathbf{E_a}(z)$. By expressing

$$E_{a\ i,j}(z) = \sum_{k=0}^{Q-1} E_{a\ i,j}^{(k)}(z^Q)z^{-k}$$

for some integer Q, we replace each scalar element $E_{a\ i,j}(z)$ by its blocked version, the $Q \times Q$ pseudocirculant matrix [12]

$$\begin{bmatrix} E_{a\ i,j}^{(0)}(z) & E_{a\ i,j}^{(1)}(z) & \cdots & E_{a\ i,j}^{(Q-1)}(z) \\ z^{-1}E_{a\ i,j}^{(Q-1)}(z) & E_{a\ i,j}^{(0)}(z) & \cdots & E_{a\ i,j}^{(Q-2)}(z) \\ \vdots & \vdots & \vdots & \vdots \\ z^{-1}E_{a\ i,j}^{(1)}(z) & z^{-1}E_{a\ i,j}^{(2)}(z) & \cdots & E_{a\ i,j}^{(0)}(z) \end{bmatrix}$$

to get the $LQ \times MQ$ matrix $\mathbf{E_b}(z)$. By further interchanging the rows of $\mathbf{E_b}(z)$ to get $\mathbf{E_c}(z)$ and by invoking the so called polyphase identity [12], we get the equivalent discrete-time model of Fig. 3. The signals $y_k(n)$, $k = 0, 1, \ldots, LQ - 1$, in Fig. 3 are the LQ polyphase components of $y(n)$.

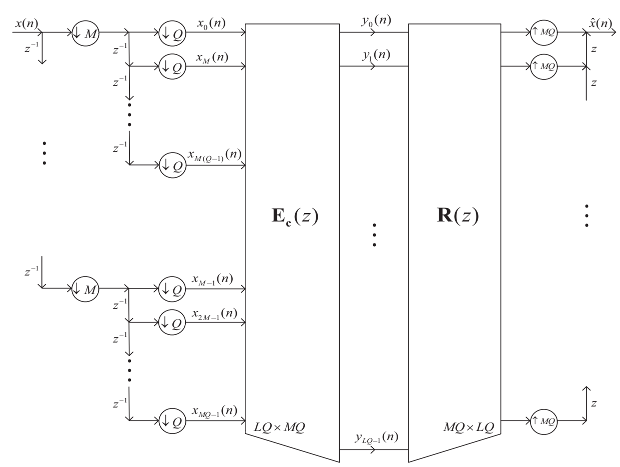

Fig. 3. The new discrete-time model

3. PERFECT RECONSTRUCTION BY FIR FILTERS

From Fig. 3, since $x(n)$ is the interleaved version of $x_k(n)$, $k = 0, 1, \ldots, MQ - 1$, perfect reconstruction is achieved if and only if the MQ signals $x_k(n)$ can be reconstructed from a subset of MQ signals out of the LQ sequences $y_k(n)$. Let

$$\begin{bmatrix} Y_{k_0}(z) \\ Y_{k_1}(z) \\ \vdots \\ Y_{k_{MQ-1}}(z) \end{bmatrix} = \mathbf{E}(z) \begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{MQ-1}(z) \end{bmatrix}$$

where the $MQ \times MQ$ matrix $\mathbf{E}(z)$ is carefully selected out of $\mathbf{E_c}(z)$ (alternatively $\mathbf{E_b}(z)$). It therefore follows that a necessary and sufficient condition for perfect signal recovery is that $\mathbf{E}(z)$ is non singular. Furthermore, for FIR reconstruction, the determinant of $\mathbf{E}(z)$ has to be a pure delay.

Construction of E. Assume that N_i, $i = 0, 1, \ldots, L-1$ is the highest order of $E_{a\ i,j}(z)$, $j = 0, 1, \ldots, M-1$, and let $Q = \sum_{i=0}^{L-1} N_i$. We can then form the $LQ \times MQ$ matrices $\mathbf{E_b}(z)$ and $\mathbf{E_c}(z)$. By carefully forsaking Q rows which contain the factor of z^{-1} in $\mathbf{E_c}(z)$ ($\mathbf{E_b}(\mathbf{z})$), we can get an $(L-1)Q \times MQ$ *scalar* matrix $\mathbf{E}$. Since $L > M$, this new matrix has at least MQ rows, and we therefore have the following result.

Theorem 1. Perfect FIR reconstruction of $x(n)$ is possible if the matrix $\mathbf{E}$ has full column rank, i.e., $rank(\mathbf{E}) = MQ$.

We emphasize that perfect FIR reconstruction is, in general, *not unique*. In fact, the FIR reconstructing system is unique only if $L = M + 1$. We can further explore the structure of the matrix $\mathbf{E}$ by rewriting it in the following form

$$\left[\begin{array}{cccccc} e_{0,0} & \cdots & e_{0,n_0} & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & e_{0,0} & \cdots & e_{0,n_0} \\ \hline & & \vdots & & & \\ \hline e_{L-1,0} & \cdots & e_{L-1,n_{L-1}} & 0 & \cdots & 0 \\ & & \ddots & & & \\ 0 & \cdots & 0 & e_{L-1,0} & \cdots & e_{L-1,n_{L-1}} \end{array}\right]$$

The matrix $\mathbf{E}$ is a generalized Sylvester matrix with L basic building blocks [13]. By defining $P_i(z) = \sum_{j=0}^{n_i} e_{i,j} z^{-j}$ ($i = 0, 1, \ldots, L-1$) as the leading polynomials of every block, we can derive the main result of this paper.

Theorem 2. A necessary condition for the matrix $\mathbf{E}$ to have full column rank is that its leading polynomials are coprime.

Proof. Define the $(L-1)Q \times L$ polynomial matrix $\mathbf{S}(z)$ as

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ z^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ z^{-(Q-N_0-1)} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & z^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & z^{-(Q-N_1-1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & z^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & z^{-(Q-N_{L-1}-1)} \end{pmatrix}$$

$$\begin{aligned} \mathbf{p}(z) = \mathbf{S}(z) \cdot \begin{pmatrix} P_0(z) \\ P_1(z) \\ \vdots \\ P_{L-1}(z) \end{pmatrix} &= \left(\begin{array}{c} P_0(z) \\ z^{-1}P_0(z) \\ \vdots \\ z^{-(Q-N_0-1)}P_0(z) \\ \hline \vdots \\ \hline P_{L-1}(z) \\ z^{-1}P_{L-1}(z) \\ \vdots \\ z^{-(Q-N_{L-1}-1)}P_{L-1}(z) \end{array}\right) \\ &= \mathbf{e}_0 + \mathbf{e}_1 z^{-1} + \cdots + \mathbf{e}_{MQ-1} z^{-(MQ-1)}, \end{aligned}$$

where $\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_{MQ-1}$ are just the MQ column vectors of the matrix $\mathbf{E}$. Suppose $P_0(z), P_1(z), \cdots, P_{L-1}(z)$ have a common zero at $z = z_0$. Then, we have

$$\begin{aligned} \mathbf{p}(z_0) = \mathbf{S}(z_0) \cdot \begin{pmatrix} P_0(z_0) \\ P_1(z_0) \\ \vdots \\ P_{L-1}(z_0) \end{pmatrix} &= \mathbf{S}(z_0) \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{L\times 1} \\ = \mathbf{0} &= \mathbf{e}_0 + \mathbf{e}_1 z_0^{-1} + \cdots + \mathbf{e}_{MQ-1} z_0^{-(MQ-1)} \end{aligned}$$

Clearly, the MQ columns of the matrix $\mathbf{E}$ are linearly dependent, i.e., $rank(\mathbf{E}) < MQ$. Hence, the coprimeness of the leading polynomials $P_0(z), P_1(z), \cdots, P_{L-1}(z)$ is a necessary condition for the matrix $\mathbf{E}$ to have full column rank. ■

The above theorem can be therefore interpreted as the analogue result to Theorem 1 in [10]. We note however that, unlike the case of $M = 1$, the converse of this theorem is not true, i.e., the coprimeness of the leading polynomials in this case is not sufficient for FIR perfect reconstruction. Note also, that similar to the case of $M = 1$ [10], the choice of Q is in general **not unique**. For example, for $Q = L \max N_i$, the same results are obtained. The following examples illustrate the above ideas by examining a number of possible scenarios.

Example 1. Assume $L = 3$, $M = 2$, and

$$\begin{aligned} F(z) = 3 + z^{-2} + 5z^{-3} + 6z^{-5} + z^{-6} \\ + 2z^{-7} + 7z^{-10} + 2z^{-11} \end{aligned}$$

$$\Longrightarrow \mathbf{E_a}(z) = \begin{bmatrix} 3 + 5z^{-1} & 1 \\ 1 & 6 + 2z^{-1} \\ 2z^{-1} & 7 \end{bmatrix}$$

Let $Q = \sum_{i=0}^{2} N_i = 3$, we have

$$\mathbf{E_b}(z) = \begin{bmatrix} 3 & 5 & 0 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 5z^{-1} & 0 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 6 & 2 & 0 \\ 0 & 1 & 0 & 0 & 6 & 2 \\ 0 & 0 & 1 & 2z^{-1} & 0 & 6 \\ 0 & 2 & 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 & 7 & 0 \\ 2z^{-1} & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

$$\Longrightarrow \mathbf{E} = \begin{bmatrix} 3 & 5 & 0 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 0 & 6 & 2 & 0 \\ 0 & 1 & 0 & 0 & 6 & 2 \\ 0 & 2 & 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 & 7 & 0 \end{bmatrix}$$

The three leading polynomials $3 + 5z^{-1} + z^{-3}$, $1 + 6z^{-3} + 2z^{-4}$ and $2z^{-1} + 7z^{-3}$ are coprime and $rank(\mathbf{E}) = 6$.

Example 2. Assume $L = 3$, $M = 2$, and

$$F(z) = 3 + 2z^{-2} + 3z^{-3} - 5z^{-4} + 2z^{-5} + 3z^{-6} + 5z^{-7} + 2z^{-11}$$

$$\Longrightarrow \mathbf{E_a}(z) = \begin{bmatrix} 3 + 3z^{-1} & 3 \\ 2 & 2 + 2z^{-1} \\ -5 & 5 \end{bmatrix}$$

Let $Q = \sum_{i=0}^{2} N_i = 2$, we have

$$\mathbf{E_b}(z) = \begin{bmatrix} 3 & 3 & 3 & 0 \\ 3z^{-1} & 3 & 0 & 3 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 2z^{-1} & 2 \\ -5 & 0 & 5 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

$$\Longrightarrow \mathbf{E} = \begin{bmatrix} 3 & 3 & 3 & 0 \\ 2 & 0 & 2 & 2 \\ -5 & 0 & 5 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix}$$

The three leading polynomials $3 + 3z^{-1} + 3z^{-2}$, $2 + 2z^{-2} + 2z^{-3}$ and $-5 + 5z^{-2}$ are relatively prime. However, $rank(\mathbf{E}) = 3 < 4$. Therefore, this example clearly shows that the coprimeness of the leading polynomials is *not sufficient* for the matrix $\mathbf{E}$ to have full column rank.

Example 3. Assume $L = 5$, $M = 2$, and

$$F(z) = 3 + 2z^{-4} - 5z^{-10} - 7z^{-12} - 6z^{-13} + 8z^{-15} + z^{-16} + z^{-17} + 6z^{-18} - 2z^{-19} + 3z^{-21}$$

$$\Longrightarrow \mathbf{E_a}(z) = \begin{bmatrix} 3 - 5z^{-1} & 8z^{-1} \\ 2 & -2z^{-1} \\ 6z^{-1} & -6 \\ -7 & 1 \\ 1 & 3 \end{bmatrix}$$

Let $Q = \sum_{i=0}^{4} N_i = 3$, we have

$$\mathbf{E_b}(z) = \begin{bmatrix} 3 & -5 & 0 & 0 & 8 & 0 \\ 0 & 3 & -5 & 0 & 0 & 8 \\ -5z^{-1} & 0 & 3 & 8z^{-1} & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & -2z^{-1} & 0 & 0 \\ 0 & 6 & 0 & -6 & 0 & 0 \\ 0 & 0 & 6 & 0 & -6 & 0 \\ 6z^{-1} & 0 & 0 & 0 & 0 & -6 \\ -7 & 0 & 0 & 1 & 0 & 0 \\ 0 & -7 & 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{E} = \left[\begin{array}{cccccc} 3 & -5 & 0 & 0 & 8 & 0 \\ 0 & 3 & -5 & 0 & 0 & 8 \\ 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ 0 & 6 & 0 & -6 & 0 & 0 \\ 0 & 0 & 6 & 0 & -6 & 0 \\ \hline -7 & 0 & 0 & 1 & 0 & 0 \\ 0 & -7 & 0 & 0 & 1 & 0 \\ 0 & 0 & -7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \triangleq \left[\begin{array}{c} \mathbf{A_1} \\ \hline \mathbf{A_2} \end{array}\right]$$

Since $L - 1 > M$, we may have several possible choices for synthesizing the reconstruction scheme. In this case, both $\mathbf{A_1}$ and $\mathbf{A_2}$ are non-singular and we can design at least two distinct FIR recovery systems for the same given $F(z)$.

Example 4. Let $L = 3$, $M = 2$, and

$$F(z) = 1 + z^{-2} + z^{-3} + z^{-5} + z^{-6} + z^{-7} + z^{-11}$$

$$\Longrightarrow \mathbf{E_a}(z) = \begin{bmatrix} 1 + z^{-1} & 1 \\ 1 & 1 + z^{-1} \\ 0 & 1 \end{bmatrix}$$

Assume now that $Q = L \max N_i = 3$, we have

$$\mathbf{E_b}(z) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ z^{-1} & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & z^{-1} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \mathbf{E} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The three leading polynomials $1 + z^{-1} + z^{-3}$, $1 + z^{-3} + z^{-4}$ and z^{-3} are coprime, as implied by $rank(\mathbf{E}) = 6$.

Example 5. Let $L = 3$, $M = 2$, and

$$F(z) = 1 + z^{-2} + z^{-3} - z^{-4} + z^{-5} + z^{-6} + z^{-7} + z^{-11}$$

$$\Longrightarrow \mathbf{E_a}(z) = \begin{bmatrix} 1 + z^{-1} & 1 \\ 1 & 1 + z^{-1} \\ -1 & 1 \end{bmatrix}$$

Assume again that $Q = L \max N_i = 3$, we have

$$\mathbf{E_b}(z) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ z^{-1} & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & z^{-1} & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \mathbf{E} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

The three leading polynomials $1 + z^{-1} + z^{-3}$, $1 + z^{-3} + z^{-4}$ and $-1 + z^{-3}$ are coprime. Nevertheless, $rank(\mathbf{E}) = 5 < 6$. The example therefore illustrates that the relative primeness of the leading polynomials is also not sufficient for the matrix $\mathbf{E}$ to have full column rank for this alternative choice of Q.

4. CONCLUDING REMARKS

The results presented here raise a number of issues for future research. To start with, sufficient conditions for FIR reconstruction for the fractional rate case are not currently known. Second, given that the choice of Q is not unique, is there a specific value of Q that produces necessary and sufficient conditions for perfect FIR recovery in the fractional rate case? Moreover, since the FIR reconstruction solution is in general not unique, how do you choose the particular FIR recovery system ? Finally, these results are derived for the deterministic noiseless case. What are the results when noise is present?

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