Spectra and minimum distances of Repeat Multiple Accumulate codes

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***Abstract*— In this paper we consider ensembles of codes, denoted RA^m, obtained by a serial concatenation of a repetition code and m accumulate codes through uniform random interleavers. We analyze their average spectrum functions for each m showing that they are equal to 0 below a threshold distance ϵ_m and positive beyond it. One of our main results is to prove that these average spectrum functions form a not-increasing sequence in m converging uniformly to a limit spectrum function which is equal to the maximum between the average spectrum function of the classical linear random ensemble and 0. As a consequence the sequence ϵ_m converges to the Gilbert-Varshamov distance. A further analysis allows to conclude that the threshold distance ϵ_m is indeed the typical distance of the ensemble RA^m when the interleaver length goes to infinity. Combining the two results we are able to conclude that the typical distance of the ensembles RA^m converges to the Gilbert-Varshamov bound.**

I. INTRODUCTION

Repeat-Accumulate codes are a popular family of serially interconnected turbo codes which, for their simplicity and good performances, have received lots of attention in the past literature [3], [4]. They consist of a repetition code interconnected with a cascade of m simple accumulate codes and all interconnections are through random interleavers. The case when $m = 1$ represents an example of a classical serial turbo code (just two convolutional codes interconnected by an interleaver): for this coding scheme iterative decoding can be fruitfully used and this makes it interesting from an applicative point of view. On the other hand, as it happens for all classical serial turbo codes [6], minimum distances for such codes are not so good: typical minimum distances indeed grows only sublinearly in the length N of the interleaver. However, it was already remarked in [2] that instead, when $m = 2$, minimum distance present a linear growth in N. It was conjectured in [8] that distance performance should increase as m grows and in fact should reach the Gilbert-Varshamov bound when $m \to +\infty$. It is well known that repeat-accumulate codes for $m \geq 2$ are, at the moment, of little practical interest because of the fact that iterative decoding show in these cases very slow convergence. These codes are though of theoretical importance since the understanding of the ingredients which lead to linear distance may in principle open the possibility to find different codes with similar characteristics but, hopefully, more efficient decoding strategies.

In [9] a first analysis of the distance properties of repeat-accumulate codes for $m \geq 2$ is undertaken. The authors show that for fixed length N and letting $m \to +\infty$ the weight enumerating functions converge to the weight enumerating function of the linear coding ensemble. From this it follows that there exists a sequence of repeat-accumulate codes whose minimum distance converges to the Gilbert-Varshamov bound but it does not allow to conclude that the typical distance of all repeat-accumulate codes converges to the Gilbert-Varshamov bound. This difficulty is mathematically due to the fact that the two limits, for $m \to +\infty$ and $N \to +\infty$ can not be so simply interchanged.

In this paper we undertake a fundamental analysis of the average spectral shapes of repeat accumulate codes for any m. We prove that (Theorem 9), for each m, the average spectral shape is symmetric with respect to $1/2$, it is 0 in intervals $[0, \epsilon_m] \cup [1 - \epsilon_m, 1]$, where $\epsilon_m > 0$ for every $m \geq 2$, and it is strictly positive otherwise. Moreover we show that (Theorem 10), as m varies, the average spectral shapes form a decreasing uniformly convergent sequence of functions. Their limit point is the maximum between 0 and the average spectral shape of the linear coding ensemble. As a consequence the threshold sequence ϵ_m converges to the Gilbert-Varshamov distance. Since the spectral shapes are not negative but only equal to 0 before ϵ_m we can not conclude at this point anything on their typical minimum distances. However estimating weight enumerators using techniques proposed by Mc-Eliece, we are able to conclude (Theorem 11) that indeed for such ensembles distances grow linearly in N and the typical linear growth, for a specific m, is exactly given ϵ_m.

We now present a brief outline of this paper. Section II is a preliminary section containing all notation and classical results needed from coding theory. Section III is devoted to a formal description of repeat-accumulate codes and basic classical results. Section IV contains a summary of previous results, while Section V presents, in a formal way, all results presented in this paper. Section VI, VII, and VIII are the technical sections where results are proven.

II. PRELIMINARIES

We begin our work by fixing notations and reviewing some concepts about block encoders. In particular, we give a general definition of an encoder ensemble and of its weight structure as done in [5] and [8]. Moreover we suggest some techniques for the estimation of the minimum distance distribution.

A. Weight enumerators and spectral shapes for coding ensemble

Let $\mathbb{Z}_2$ be the usual binary field. A $\mathbb{Z}_2$-linear block encoder with rate R and length N is a $\mathbb{Z}_2$-linear map $\mathcal{E} : \mathbb{Z}_2^{\lfloor RN \rfloor} \to \mathbb{Z}_2^N$. Let $\mathcal{C}_\mathcal{E} = \mathrm{Im}(\mathcal{E})$ be the associated block code. Given a sequence $\boldsymbol{u} \in \mathbb{Z}_2^N$, denote with $w_H(\boldsymbol{u})$ its Hamming weight, namely the number of its non zero elements. The *minimum distance* of $\mathcal{E}$ (or of $\mathcal{C}_\mathcal{E}$) is defined as

$$d_{\min}(\mathcal{E}) = d_{\min}(\mathcal{C}_\mathcal{E}) = \min\{w_H(\boldsymbol{x}) : \ \boldsymbol{x} \in \mathcal{C}_\mathcal{E}\}.$$

For any block encoder $\mathcal{E}$ we denote

$$A_d(\mathcal{E}) = |\{\boldsymbol{u} \in \mathbb{Z}_2^{\lfloor RN \rfloor} : w_H(\mathcal{E}(\boldsymbol{u})) = d\}|,$$
$$A_{w,d}(\mathcal{E}) = |\{\boldsymbol{u} \in \mathbb{Z}_2^{\lfloor RN \rfloor} : w_H(\boldsymbol{u}) = w, w_H(\mathcal{E}(\boldsymbol{u})) = d\}|.$$

Often, we will also use the symbol $\mathbf{A}(\mathcal{E})$ to denote the $N \times N$ matrix whose (w, d) entry is given by $A_{w,d}(\mathcal{E})$.

Let now $\mathscr{E}$ be a set of $\mathbb{Z}_2$-linear encoders with rate R and length N. We can introduce a probabilistic structure on $\mathscr{E}$ by considering a random encoder chosen uniformly from this set. We then define the *average output weight enumerators* and the *average input-output weight enumerators* as follows

$$\overline{A}_d\left(\mathscr{E}\right) := \frac{1}{|\mathscr{E}|} \sum_{\mathcal{E} \in \mathscr{E}} A_d(\mathcal{E})$$
$$\overline{A}_{w,d}\left(\mathscr{E}\right) := \frac{1}{|\mathscr{E}|} \sum_{\mathcal{E} \in \mathscr{E}} A_{w,d}(\mathcal{E}).$$

We will also use the matrix notation $\overline{\mathbf{A}}\left(\mathscr{E}\right)$ as before.

Consider now a sequence $\overline{\mathscr{E}} = \{\mathscr{E}_N\}_{N \in \mathbb{N}}$, where each $\mathscr{E}_N$ is an ensemble of encoders of length N. For each ensemble $\mathscr{E}_N$ are well defined $\overline{A}_d(\mathscr{E}_N)$ and $\overline{A}_{w,d}(\mathscr{E}_N)$.

We define the *N-th spectral shape* of $\overline{\mathscr{E}}$ as

$$r_N(\delta; \overline{\mathscr{E}}) := \frac{1}{N} \ln \overline{A}_{\lfloor \delta N \rfloor}(\mathscr{E}_N), \quad \text{for } \delta \in [0, 1].$$

and the *asymptotic spectral shape* of $\overline{\mathscr{E}}$ as

$$\widehat{r}(\delta; \overline{\mathscr{E}}) := \limsup_{N \to \infty} r_N(\delta; \overline{\mathscr{E}}), \quad \text{for } \delta \in [0, 1]. \tag{1}$$

Whenever $\overline{\mathscr{E}}$ is clear from the context, spectral shapes will simply be denoted by $r_N(\delta)$ and $\widehat{r}(\delta)$, respectively.

Example 1 (Random linear encoder ensemble): Given $R > 0$ and fixed $N \in \mathbb{N}$, let $\mathscr{L}_N$ be the set of all possible $\mathbb{Z}_2$-linear encoder $\mathcal{E} : \mathbb{Z}_2^{\lfloor RN \rfloor} \to \mathbb{Z}_2^N$.

The average output weight enumerators for the linear encoder ensemble can be easily computed [1]:

$$\overline{A}_d\left(\mathscr{L}_N\right) = \binom{N}{d} \frac{2^{\lfloor RN \rfloor} - 1}{2^N}.$$

Consider now the sequence of ensembles $\overline{\mathscr{L}} = \{\mathscr{L}_N\}_{N \in \mathbb{N}}$: it can easily be verified that the asymptotic spectral shape has the following expression

$$\widehat{r}(\delta; \overline{\mathscr{L}}) = H(\delta) - (1 - R) \ln 2 \tag{2}$$

where $H(\delta) = -\delta \ln \delta - (1-\delta) \ln(1-\delta)$ is the natural entropy function.

We define the *relative Gilbert-Varshamov distance*

$$\delta_{GV}(R) = (\underline{H}|_{[0,1/2]})^{-1}[(1 - R) \ln 2]. \tag{3}$$

Notice that $\widehat{r}(\delta_{GV}(R); \overline{\mathscr{L}}) = 0$ and that the spectral shape is negative before this point.

B. Estimation of minimum distance distribution

One of the uses of the average weight enumerators and of the corresponding spectral shapes, is to obtain probabilistic information on the minimum distance for the encoders of the ensemble. Indeed simple probabilistic arguments lead to the estimation:

$$\mathbb{P}\left(d_{\min}(\mathscr{E}) < d\right) \leq \sum_{h=1}^{d-1} \overline{A}_h(\mathscr{E}), \tag{4}$$

where $d_{\min}(\mathscr{E})$ denotes the minimum distance as a random variable on the ensemble.

Consider now a sequence of encoder ensembles $\overline{\mathscr{E}} = \{\mathscr{E}_N\}_{N \in \mathbb{N}}$. Expressing in (4) the output weight enumerators as $\overline{A}_h(\mathscr{E}_N) = \exp\{N r_N(h/N; \overline{\mathscr{E}})\}$, it is easy to verify the following result:

Proposition 2: If there exists δ_0 such that

$$\sup_{\sigma \leq \delta} \widehat{r}(\sigma; \overline{\mathscr{E}}) < 0, \ \forall \delta < \delta_0$$

then, for any $\epsilon > 0$,

$$\mathbb{P}\left(d_{\min}(\mathscr{E}_N) < (\delta_0 - \epsilon)N\right) \overset{N \to \infty}{\longrightarrow} 0. \tag{5}$$

Example 3 (Random linear encoder ensemble): The use of Theorem 2 makes surprisingly easy the estimation of the minimum distance growth rate of a typical binary linear encoder, chosen uniformly from the set $\mathscr{L}_N$.

Notice that the asymptotic spectral shape given in (2) is negative for $\delta < \delta_{GV}(R)$, crosses zero at $\delta = \delta_{GV}(R)$ then is positive for some $\delta > \delta_{GV}(R)$. By Theorem 2 it follows that, for any $\epsilon > 0$,

$$\mathbb{P}\left(d_{\min}(\mathscr{L}_N) < (\delta_{GV}(R) - \epsilon)N\right) \overset{N \to \infty}{\longrightarrow} 0. \tag{6}$$

III. REPEAT-ACCUMULATE-m

In this section we describe the ensemble of Repeat-Accumulate-m codes and we state our main results.

A. Ensemble description

First, we formally introduce the repetition and the accumulate encoders. Given $q \in \mathbb{N}$ and $N \in q\mathbb{N}$, the repetition encoder $\mathrm{Rep}_N^q : \mathbb{Z}_2^{N/q} \to \mathbb{Z}_2^N$ repeats the information block q-times

$$\mathrm{Rep}_N^q([v_1, \ldots, v_{N/q}]) = \underbrace{[v_1, \ldots, v_{N/q}, \ldots, v_1, \ldots, v_{N/q}]}_{q \text{ times}}.$$

The accumulator $\mathrm{Acc}_N : \mathbb{Z}_2^N \to \mathbb{Z}_2^N$ is the block encoder defined by

$$\mathrm{Acc}_N([u_1, \ldots, u_N]) = [u_1, u_1 + u_2, \ldots, u_1 + \ldots + u_N].$$

Denote by S_N the group of permutation on N elements. Each $\sigma \in S_N$ can naturally be interpreted as a linear isomorphism $\sigma : \mathbb{Z}_2^N \to \mathbb{Z}_2^N$. Fixed $m \in \mathbb{N}$ and given $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m) \in S_N^m$, we can define the concatenated block encoder by the map composition (Fig. 1)

$$\mathrm{Acc}_N \circ \pi_m \circ \ldots \circ \mathrm{Acc}_N \circ \pi_1 \circ \mathrm{Rep}_N^q. \tag{7}$$

Let RA_N^m be the set of all serial encoders (7) generated by varying the vector of permutations $\boldsymbol{\sigma}$ over all possible elements in S_N^m.

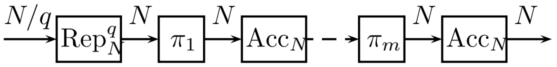

Fig. 1. Coding scheme: Repeat-Accumulate-m.

B. Weight enumerators and spectral shapes for RA^m

The average weight enumerators are computed for RA_N^m in [7] and [8] with combinatoric techniques. We have the following results.

Proposition 4:

$$\overline{\mathbf{A}}(RA_N^m) = \mathbf{A}(\mathrm{Rep}_N^q)\mathbf{P}(\mathrm{Acc}_N)^m \tag{8}$$

where $\mathbf{A}(\mathrm{Rep}_N^q)$ and $\mathbf{P}(\mathrm{Acc}_N)$ are matrices given by:

$$A_{w,d}(\mathrm{Rep}_N^q) = \mathbb{1}_{\{d=qw\}} \binom{N/q}{w},$$
$$P_{w,d}(\mathrm{Acc}_N) = \frac{\binom{N-d}{\lfloor w/2 \rfloor}\binom{d-1}{\lceil w/2 \rceil - 1}}{\binom{N}{w}}.$$

$P_{w,d}(\mathrm{Acc}_N)$ describes the probability that an input word of weight w is mapped, by the accumulator, to an output word of weight d. $\mathbf{P}(\mathrm{Acc}_N)$ is thus a stochastic matrix.

Given $\delta \in [0,1]$, define the interval $\Omega_\delta = [0, 2\delta \wedge (2 - 2\delta)]$ and consider the function $f(u,\delta)$ defined by

$$f(u,\delta) = -H(u) + (1-\delta)H\left(\frac{u}{2(1-\delta)}\right) + \delta H\left(\frac{u}{2\delta}\right) \tag{9}$$

if $\delta \in]0,1[$ and $u \in \Omega_\delta$, while $f(0,0) = f(0,1) = 0$. It is easy to see that this defines a continuous function in (u,δ).

It can be verified [8] that the asymptotic spectral shapes satisfy the iterative relation

$$\begin{array}{rcl} \widehat{r}^{(i)}(\delta) & = & \max_{u \in \Omega_\delta} \left\{ \widehat{r}^{(i-1)}(u) + f(u,\delta) \right\}, \quad i > 0 \\ \widehat{r}^{(0)}(\delta) & = & H(\delta)/q \end{array} \tag{10}$$

IV. PREVIOUS WORKS

Kahale and Urbanke show [6] that for $m = 1$ the typical minimum distance of such coding schemes grows sub-linearly in N with probability approaching one. Precisely, they prove the following result.

Theorem 5: For $q \geq 3$ and for every $\epsilon > 0$ we have

$$\lim_{N\to\infty} \mathbb{P}\left(d_{\min}(RA_N^1) < N^{1-1/\lceil q/2 \rceil - \epsilon}\right) = 0.$$

Notice that the repetition factor q plays a crucial role in the estimation of the minimum distance: the more the parameter q is large, the more the minimum distance growth rate is close to be linear with high probability.

For cases with $m \geq 2$ both analysis in [2] and [8] concern with the computation of the minimum distance distribution.

In particular, in [2] is proved that for $m = 2$ and $q \geq 3$ the typical minimum distance grows linearly in the interleaver length. In [8] an estimation of the linear growth rate is given. The result is summarized in the following theorem.

Theorem 6: For $q \geq 3$ and for any $\overline{\delta} < (4\mathrm{e}^{8/q})^{-1}$

$$\mathbb{P}\left(d_{\min}(RA_N^2) < \overline{\delta}N\right) \overset{N\to\infty}{\longrightarrow} 0.$$

If we serially concatenate any encoder, whose minimum distance is growing like $\overline{\delta}N$, with an accumulate encoder through a Uniform Random Interleaver, the minimum distance of the new encoder must grow faster than $\overline{\delta}N/2$ as $P_{h,d}(\mathrm{Acc}_N)$ is zero for every $d \leq \lceil h/2 \rceil$. Then Theorem 6 implies that if the minimum distance behaves linearly in N for $m = 2$ then so must hold for every $m \geq 2$:

$$\mathbb{P}\left(d_{\min}(RA_N^m) < \frac{\overline{\delta}N}{2^{m-2}}\right) \overset{N\to\infty}{\longrightarrow} 0.$$

Although this argument allows us to conclude that the typical minimum distance is growing linearly in N, it leads us to think that the linear growth rate should decrease monotonically with the number of accumulators. Instead, simulation results in [9] show that the linear growth rate is increasing monotonically with m. Moreover the authors, with arguments from the spectral theory of stochastic matrices applied to $\mathbf{P}(\mathrm{Acc}_N)$, prove that

Theorem 7 (Theorem 3 in [9]): Fixed $N \in q\mathbb{N}$

$$\lim_{m\to\infty} \overline{A}_d\left(RA_N^m\right) = \begin{cases} 1, & \text{if } d = 0 \\ \binom{N}{d}\frac{2^{RN}-1}{2^N-1}, & \text{if } d \geq 1. \end{cases} \tag{11}$$

As a consequence, we have the following result

Corollary 8: There exists $\{m_N\}_{N\in q\mathbb{N}}$ such that

$$\lim_{N\to\infty} r_N^{(m_N)}(\delta) = H(\delta) - (1-R)\ln 2, \;\; \delta \in [0,1]. \tag{12}$$

This result is very encouraging, as it puts into evidence that there exists a subsequence of encoders in $\{RA_N^{m_N}\}_{N\in q\mathbb{N}}$ behaving asymptotically like all the $\mathbb{Z}_2$-linear random encoders. In particular the minimum distance of such codes becomes close to the normalized Gilbert-Varshamov distance with high probability.

Notice however that this argument does not give any information about the minimum distance distribution for the case of a finite number of accumulators.

V. SUMMARY OF OUR RESULTS

In the following theorems are summarized our main results.

Theorem 9: There exists a sequence of points $\{\epsilon_m\}_{m\in\mathbb{N}}$ (with $\epsilon_1 = 0$) non-decreasing in m such that

$$\begin{aligned} \widehat{r}^{(m)}(\delta) = 0 \quad \forall\delta \in [0, \epsilon_m] \cup [1-\epsilon_m, 1], \\ \widehat{r}^{(m)}(\delta) > 0 \quad \forall\delta \in (\epsilon_m, 1-\epsilon_m). \end{aligned}$$

Theorem 10: The sequence of asymptotic spectral shapes $\{\widehat{r}^{(m)}\}_{m\in\mathbb{N}} : [0,1] \to \mathbb{R}^+$ is monotonically strictly decreasing in m and converges uniformly for $m \to \infty$ to

$$\widehat{r}^{(\infty)}(\delta) = \begin{cases} H(\delta) - (1-R)\ln 2 & \text{if } \delta \in (\delta_{GV}, 1-\delta_{GV}) \\ 0 & \text{otherwise} \end{cases}$$

The spectral shapes are visualized in Figure 2 with $q = 3$ for $m = 1, 2, 3$ and compared to that of all $\mathbb{Z}_2$-linear random codes.

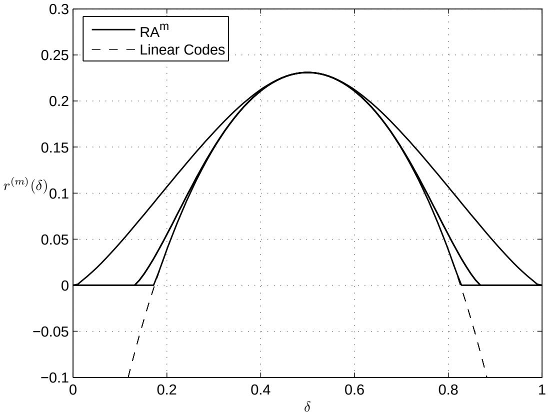

Fig. 2. Asymptotic spectral shapes for ensembles of Repeat-Accumulate-m.

Unfortunately, the floor of the spectral shapes is zero and we can not apply Theorem 2, in order to estimate the minimum distance distribution.

Nevertheless we shall prove the following theorem.

Theorem 11: We have that $\forall\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}(d_{\min}(RA_N^m) \le (\epsilon_m - \epsilon)N) = 0.$$

These theorems guarantee together that the typical minimum distance of such coding schemes grows linearly in N with probability close to one. Moreover the minimum distance growth rate increases monotonically with m and converges to the limit implied by GVB when m tends to infinity. In Table I the normalized minimum distances ϵ_m are listed for $m = 2, 3, 4$ and compared to the GV-distance. Notice that convergence looks quite fast: it is sufficient a small number of accumulate codes to get very close to the limit.

Summarizing, our results generalize those in [9] and improve the earlier estimations of the growth rates in [2] and [8] for $m = 2$.

In sections VI, VII and VIII we shall prove respectively Theorem 9, 10, 11 through intermediate steps.

R	ϵ_2	ϵ_3	ϵ_4	δ_{GV}
1/2	0.0352	0.1055	0.1106	0.1106
1/3	0.1357	0.1759	0.1759	0.1759
1/4	0.1960	0.2161	0.2161	0.2161
1/5	0.2312	0.2462	0.2462	0.2462
1/6	0.2563	0.2663	0.2663	0.2663
1/7	0.2764	0.2814	0.2814	0.2814
1/8	0.2915	0.2965	0.2965	0.2965
1/9	0.3065	0.3065	0.3065	0.3065
1/10	0.3166	0.3166	0.3166	0.3166

TABLE I

NUMERICAL VALUES OF LINEAR GROWTH RATES ϵ_m FOR $m = 2, 3, 4$ AND COMPARISON TO THE NORMALIZED GILBERT-VARSHAMOV DISTANCE.

VI. SPECTRAL SHAPE ANALYSIS

This section is devoted to the study of the asymptotic spectral shapes for a fixed number of accumulators m.

Proposition 12: The following facts are true

1) $\widehat{r}^{(m)}(\delta) = \widehat{r}^{(m)}(1-\delta)$;
2) $\widehat{r}^{(m)}(\delta) : [0,1] \to \mathbb{R}^+$ is continuous;
3) $\widehat{r}^{(m)}(\delta) \ge 0$, $\forall\delta \in [0,1]$;
4) $\widehat{r}^{(m)}(\delta)$ is increasing in $\delta \in [0, 1/2]$ and $\widehat{r}^{(m)}\left(\frac{1}{2}\right) = \frac{\ln 2}{q}$.

Proof: 1) The assertion can be verified trivially.

2) The asymptotic spectral shape $\widehat{r}^{(1)}$ is continuous as the entropy function and the function f defined in (9) are continuous. Then proceed by induction on m.

3) We have $\widehat{r}^{(1)}(\delta) \ge f(0,\delta) + H(0)/q = 0$. The general case can be proven again by induction on m.

4) The assertion follows by the fact that $f(u,\delta)$ is strictly increasing in $\delta < 1/2$ and $f(u, 1/2) = 0$. ■

Proposition 13: The sequence of functions $\{\widehat{r}^{(m)}(\delta)\}_{m\ge 1}$ is decreasing in m.

Proof: We prove the assertion by induction on m.

Consider first the case $m = 1$. We prove that

$$\widehat{r}^{(1)}(\delta) \le \widehat{r}^{(0)}(\delta) \quad \forall\delta \in [0,1] \tag{13}$$

and the equality holds if and only if $\delta = 0, 1/2$.

From the expression (9) we have

$$f(u,\delta) = u\ln\left(2\sqrt{\delta(1-\delta)}\right) - \frac{2\delta-u}{2}\ln\left(\frac{2\delta-u}{2\delta}\right) + \\ + (1-u)\ln(1-u) - \frac{2-2\delta-u}{2}\ln\left(\frac{2-2\delta-u}{2-2\delta}\right).$$

Jensen's inequality and the fact that $g(u) = u\ln u$ is strictly convex imply that

$$(1-u)\ln(1-u) = g\left(\frac{2\delta-u}{2\delta}\delta + \frac{2-2\delta-u}{2(1-\delta)}(1-\delta)\right) \le \\ \le \delta g\left(\frac{2\delta-u}{2\delta}\right) + (1-\delta)g\left(\frac{2-2\delta-u}{2(1-\delta)}\right)$$

by which

$$f(u,\delta) \le u\ln\left(2\sqrt{\delta(1-\delta)}\right) \tag{14}$$

and the equality holds if and only if $u = 0$ or $\delta = 1/2$.

By (14) we get that

$$\widehat{r}^{(1)}(\delta) \le \max_{0\le u\le 1}\left\{\frac{H(u)}{q}+\frac{u}{q}\ln\left(2\sqrt{\delta(1-\delta)}\right)^q\right\}. \tag{15}$$

Deriving this expression respect to variable u we have that the optimizing value in the computation is

$$\tilde{u}=\frac{\left(2\sqrt{\delta(1-\delta)}\right)^q}{1+\left(2\sqrt{\delta(1-\delta)}\right)^q}. \tag{16}$$

If we replace it in (15) we get

$$\widehat{r}^{(1)}(\delta) \le R_q(\delta)=\frac{1}{q}\ln\left(1+2\sqrt{\delta(1-\delta)}\right)^q \tag{17}$$

In order to prove (13), it is now sufficient to show that $R_q(\delta)\le\widehat{r}^{(0)}(\delta)$.

Define the auxiliary function

$$F_q(\delta)=q[\widehat{r}^{(0)}(\delta)-R_q(\delta)]=H(\delta)-\ln\left[1+\left(2\sqrt{\delta(1-\delta)}\right)^q\right].$$

The sequence of functions $\{R_q(\delta)\}_{q\ge 2}$ is strictly decreasing in q. So it is sufficient to verify that $R_2(\delta)\le\widehat{r}^{(0)}(\delta)$ or equivalently $F_2(\delta)\ge 0,\ \forall\delta\in[0,1]$.

We have $F_2(0)=0$ and $F_2(1/2)=0$. Deriving we get

$$\lim_{\delta\to 0}F_2'(\delta)=\infty \qquad F_2'(1/2)=0.$$

Moreover it can be shown that

$$\begin{aligned}
F_2''(\delta)>0 &\iff 1/2-1/\sqrt{6}<\delta<1/2\\
F_2''(\delta)=0 &\iff \delta=1/2-1/\sqrt{6},\ \delta=1/2\\
F_2''(\delta)<0 &\iff 0<\delta<1/2-1/\sqrt{6}.
\end{aligned}$$

Therefore we conclude that $F_2(\delta)>0$ for $0<\delta<1/2$ and that it has a minimum point at $\delta=1/2$. This completes the proof for the case with $m=1$.

The inductive step can be verified trivially. ■

It is easy to prove that we can strengthen the properties 2) and 3) of Proposition 12 in the case $m=1$. Indeed, we have that $\widehat{r}^{(1)}(\delta)>0,\ \forall\delta\in(0,1)$ and $\widehat{r}^{(1)}$ is differentiable. In particular, from the bound in (17) we have

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\widehat{r}^{(1)}(\delta)\Big|_{\delta=0}\begin{cases}=0 & \text{for } q\ge 3\\ \le 2 & \text{for } q=2\end{cases} \tag{18}$$

Define the sequence of points $\{\epsilon_m\}_{m\ge 1}$ such that

$$\epsilon_m=\max\{\epsilon\in[0,1/2):\ \widehat{r}^{(m)}(\delta)=0\ \forall\ \delta\le\epsilon\}. \tag{19}$$

Notice that $\epsilon_1=0$.

Proposition 14: The sequence of points $\{\epsilon_m\}_{m\ge 1}$ is non-decreasing and $\epsilon_2>0$.

Proof: From Proposition 13 we know that $\epsilon_{m+1}\ge\epsilon_m$. Consider $G^{(2)}(u,\delta)=\widehat{r}^{(1)}(u)+f(u,\delta)$.

Notice that

$$f_u(u,\delta)\le\frac{1}{2}\ln(2\delta)+\frac{1}{2}\ln 2+o_\delta(1)\quad \delta\to 0.$$

and, by (18), it follows that there exists $\epsilon>0$ such that

$$G_u^{(2)}(u,\delta)<0\quad\forall\delta<\epsilon,\ \forall u\in(0,2\delta).$$

Being $G^{(2)}(0,\delta)=0$, we conclude that $\widehat{r}^{(2)}(\delta)=0\ \forall\delta<\epsilon$. This completes the thesis. ■

It can be proven that the sequence of points $\{\epsilon_m\}_{m\ge 1}$ is strictly increasing. The details will be given elsewhere.

Theorem 9 follows trivially from Proposition 12 and 14.

Proposition 15:

$$\widehat{r}^{(m)}(\delta)\ge H(\delta)-(1-R)\ln 2\qquad\forall m.$$

Proof: We prove it by induction on m.

Consider the case with $m=1$

$$\begin{aligned}
\widehat{r}^{(1)}(\delta) &\ge q^{-1}H(2\delta(1-\delta))+f(2\delta(1-\delta),\delta)\\
&=q^{-1}H(2\delta(1-\delta))+H(\delta)-H(2\delta(1-\delta))\\
&\ge-(1-q^{-1})\ln 2+H(\delta).
\end{aligned}$$

Suppose now that the inequality holds true for m. We have

$$\widehat{r}^{(m+1)}(\delta)\ge\widehat{r}^{(m)}(2\delta(1-\delta))+f(2\delta(1-\delta),\delta).$$

Using the inductive assumption on $\widehat{r}^{(m)}$ and again the fact that

$$f(2\delta(1-\delta),\delta)=H(\delta)-H(2\delta(1-\delta))$$

we prove that the inequality also holds for $m+1$. The proof is thus complete. ■

VII. ASYMPTOTIC ANALYSIS

In the previous section we have studied the properties of the asymptotic spectral shape for a fixed number of accumulators m. We are devoting now to the study of the spectral shape evolution. The lower bound derived in Proposition 15 together with Proposition 13 guarantee that the sequence $\{r^{(m)}\}_{m\in\mathbb{N}}$ has limit when m tends to infinity.

The recursive expression in (10) allows us to track the evolution of the spectral shape as it passes through each accumulate encoder. In this case the spectral function in the new iteration can be expressed through a dynamical system. Through some techniques of non smooth analysis and the study of fixed points of the dynamical system, we will see that these functions converge uniformly when $m\to\infty$ to that of all $\mathbb{Z}_2$-linear random codes where it is positive. This section, together with Proposition 13, completes the proof of Theorem 10.

A. Dynamical system formulation

We start by considering the operator

$$\begin{aligned}
\Psi:\quad & C\left([0,1]\right)\longrightarrow C\left([0,1]\right)\\
\Psi[g](\delta) &=\max_{u\in\Omega_\delta}\{g(u)+f(u,\delta)\},\quad\forall\delta\in[0,1].
\end{aligned} \tag{20}$$

Given $\widehat{r}^{(0)}$ as initial condition, the sequence of asymptotic spectral shapes can be obtained recursively by

$$\widehat{r}^{(t+1)}=\Psi\left[\widehat{r}^{(t)}\right]. \tag{21}$$

In order to describe the evolution (21), we study now some properties of Ψ.

We start with some simple properties:

Lemma 16: Let $g,h\in C([0,1])$, then,

1) $\|\Psi[g]-\Psi[h]\|_\infty \leq \|g-h\|_\infty$.
2) If $g(\delta) \leq h(\delta)$ $\forall\delta \in [0,1]$, then $\Psi[g](\delta) \leq \Psi[h](\delta)$ $\forall\delta \in [0,1]$.
3) $\Psi[g+C] = C + \Psi[g]$, for any $C \in \mathbb{R}$.

Proof: 1): The result is an immediate consequence of the following fact

$$\Psi[g](\delta) \leq \max_{u\in\Omega_\delta}[g(u)-h(u)] + \max_{u\in\Omega_\delta}[h(u)+f(u,\delta)] = \\ = \max_{u\in\Omega_\delta}[g(u)-h(u)] + \Psi[h](\delta).$$

2) and 3) are obvious. ■

B. Fixed points analysis

We say that $g \in C([0,1])$ is a *fixed point* for Ψ if $g = \Psi[g]$. It follows from (3) of Lemma 16 that, if g is a fixed point for Ψ, then the same holds for $g+C$. Another interesting way to modify fixed points is illustrated in the following result.

Proposition 17: If g is a fixed point for Ψ, then $g_+ = 0 \vee g$ is a fixed point for Ψ.

Proof: Consider the subset of maximizing points

$$\Gamma^+(\delta) = \operatorname*{argmax}_{u\in\Omega_\delta}\,[g_+(u)+f(u,\delta)].$$

For each $\delta \in [0,1]$ choose $u^+(\delta) \in \Gamma^+(\delta)$. We have

$$\begin{aligned}\Psi[g_+](\delta) &= g_+(u^+(\delta)) + f(u^+(\delta),\delta) = \\ &= f(u^+(\delta),\delta) \vee [g(u^+(\delta)) + f(u^+(\delta),\delta)] \leq \\ &\leq f(u^+(\delta),\delta) \vee \left\{\max_{u\in\Omega_\delta}[g(u)+f(u,\delta)]\right\} = \\ &= f(u^+(\delta),\delta) \vee g(\delta). \end{aligned} \tag{22}$$

Suppose now that $\delta \in [0,1]$ is such that $g(\delta) > 0$. Then, from (22) we have

$$0 \leq \Psi[g_+](\delta) \leq f(u^+(\delta),\delta) \vee g(\delta) \leq 0.$$

We conclude that $\Psi[g_+](\delta) = 0 = g_+(\delta)$.

If instead δ is such that $g(\delta) \leq 0$, we have

$$g(u) \leq g_+(u) \Longrightarrow g = \Psi[g] \leq \Psi[g_+].$$

As f is negative, it follows that

$$g(\delta) \leq \Psi[g_+](\delta) \leq f(u^+(\delta),\delta) \vee g(\delta) = g(\delta),$$

and we conclude that $\Psi[g_+](\delta) = g(\delta) = g_+(\delta)$. This completes the proof. ■

Proposition 18: The following functions are fixed points for Ψ, for any arbitrary constant C:

1) $g(\delta) = C$;
2) $g(\delta) = H(\delta) + C$.

Proof: 1) The result follows trivially by noticing that $g = 0$ is a fixed point for Ψ as $f(u,\delta)$ is negative.

2) Consider

$$\begin{aligned} L(u,\delta) &= H(u) + f(u,\delta) \\ &= \delta H\left(\tfrac{u}{2\delta}\right) + (1-\delta) H\left(\tfrac{u}{2(1-\delta)}\right) \end{aligned} \tag{23}$$

Since, for any fixed δ, $L(u,\delta)$ is concave in u it is maximized at the only stationary point

$$u_{\max} = 2\delta(1-\delta). \tag{24}$$

It is straightforward to verify that $L(u_{\max}(\delta),\delta) = H(\delta)$. ■

An important consequence of Proposition 17 and 18 is that both $H(\delta) - (1-R)\ln 2$ and $[H(\delta)-(1-R)\ln 2]_+$ are fixed points for Ψ.

The following is the key technical result of this section: proof will be given in the appendix B.

Lemma 19: We have

$$|\widehat{r}^{(m)}(\delta_2) - \widehat{r}^{(m)}(\delta_1)| \leq K|\delta_2 - \delta_1| \qquad \forall\delta_1,\delta_2,\forall m.$$

Theorem 20: The sequence $\{\widehat{r}^{(m)}\}_{m\geq 1}$ converges uniformly to the limit $\widehat{r}^{(\infty)}$.

Proof: Since the sequence of functions $\{\widehat{r}^{(m)}\}_{m\geq 1}$ is decreasing in m and is lower bounded, it converges to the limit function $\widehat{r}^{(\infty)}$. Let

$$a_m = \max_{\delta\in[0,1]}[\widehat{r}^{(m)}(\delta) - \widehat{r}^{(\infty)}(\delta)]\,.$$

Then, a_m is monotonically decreasing in m and has a limit when $m \to \infty$.

From Proposition 12 and Lemma 19 the family $\{\widehat{r}^{(m)}\}_{m\geq 1}$ consists of uniformly bounded Lipschitz functions. Therefore Ascoli Arzelà's theorem guarantees that there exists a subsequence $\{m_j\}_{j\in\mathbb{N}}$ such that $a_{m_j} \to 0$. For uniqueness of limit we conclude that $a_m \to 0$. ■

Corollary 21: $\widehat{r}^{(\infty)}(\delta)$ is a fixed point for Ψ.

Proof: It follows from Theorem 20 and Lemma 16. ■

C. Analysis of limit function $\widehat{r}^{(\infty)}(\delta)$

As we know the family $\{\widehat{r}^{(m)}\}_{m\geq 1}$ consists in a sequence of continuous and non negative functions converging uniformly to the limit function $\widehat{r}^{(\infty)}$. The next proposition characterizes some properties of it.

Proposition 22: The following facts are true

1) $\widehat{r}^{(\infty)}(\delta) = \widehat{r}^{(\infty)}(1-\delta)$;
2) $\widehat{r}^{(\infty)}(\delta) : [0,1] \to \mathbb{R}^+$ is continuous;
3) there exists $\epsilon_\infty > 0$ such that $\widehat{r}^{(\infty)}(\delta) = 0$, $\forall\delta \leq \epsilon_\infty$;
4) $\widehat{r}^{(\infty)}(\delta)$ is increasing in $\delta \in [0,1/2]$ and $\widehat{r}^{(\infty)}(1/2) = \frac{\ln 2}{q}$.

Proof: They are trivial consequences of Proposition 12 and, for the only case of continuity, also of Theorem 20. ■

Notice that we already know a fixed point of Ψ satisfying all properties stated in Proposition: it is the function $[H(\delta) - (1-R)\ln 2]_+$. For the moment we only know that, for any $\delta \in [0,1]$, $\widehat{r}^{(\infty)}(\delta) \geq [H(\delta) - (1-R)\ln 2]_+$. In the rest of this section we will prove that they are in fact equal.

Consider g such that $\Psi[g] = g$ and such that it satisfies all properties listed in Proposition 22. Let

$$\Gamma_g(\delta) = \operatorname*{argmax}_{u\in\Omega_\delta}\,[g(u)+f(u,\delta)]\,.$$

Then, for any $u \in \Gamma_g(\delta)$ it clearly holds

$$g(\delta) = g(u) + f(u,\delta)\,. \tag{25}$$

We start with a technical result.

Lemma 23: The following facts are true.

1) For any $\delta \in [0,1/2[$, $\Gamma_g(\delta) \subseteq [0,1/2]$.
2) If $\delta_1 < \delta_2 \leq 1/2$ and $u_i \in \Gamma_g(\delta_i)$ $(i=1,2)$, then $u_1 \leq u_2$;
3) For any $\delta \in]\epsilon_\infty, 1/2[$ and $u \in \Gamma_g(\delta)$, we have $u \geq \delta$. Moreover, $\delta \in \Gamma_g(\delta)$ if and only if $\delta \in \{0, 1/2\}$.
4) If $\delta_n \overset{n\to\infty}{\longrightarrow} \delta_\infty$ and, $u_n \in \Gamma_g(\delta_n)$ is such that $u_n \overset{n\to\infty}{\longrightarrow} u_\infty$, then $u_\infty \in \Gamma_g(\delta_\infty)$

Proof: 1) For any fixed $\delta \neq 1/2$, it can be checked that $f(u,\delta)$ is strictly increasing in $u \in \Omega_\delta$. The result then immediately follows from the symmetry 1 of Proposition 22.

2) Notice first that since $f_{\delta u}(u,\delta) > 0$ for any $\delta < 1/2$ and every $u \in \Omega_\delta$, it follows that

$$\frac{\partial}{\partial u}[f(u,\delta_2) - f(u,\delta_1)] = \int_{\delta_1}^{\delta_2} f_{\delta u}(u,\delta)\,\mathrm{d}\delta > 0\,. \tag{26}$$

We now prove the result by contradiction. Suppose that $u_2 < u_1$. Using (26) we can write,

$$\begin{aligned} g(\delta_2) &= g(u_2) + f(u_2,\delta_2) + f(u_2,\delta_1) - f(u_2,\delta_1) \\ &< g(u_2) + f(u_2,\delta_1) + f(u_1,\delta_2) - f(u_1,\delta_1) \\ &\leq g(u_1) + f(u_1,\delta_1) + f(u_1,\delta_2) - f(u_1,\delta_1) \\ &= g(u_1) + f(u_1,\delta_2). \end{aligned}$$

by which $u_2 \notin \Gamma_g(\delta_2)$.

3) Notice first that if $\delta \in]\epsilon_\infty, 1/2[$, for sure $0 \notin \Gamma_g(\delta)$. Since $f(u,\delta) < 0$ for any $\delta \neq 1/2$ and $u \neq 0$, it follows from (25) that, necessarily, $g(\delta) < g(u)$. It now follows by property 4) of Proposition 22, that, $\delta \geq u$. Finally notice that (25) holds with $u = \delta$ if and only if $f(\delta,\delta) = 0$ and this happens if and only if $\delta \in \{0, 1/2\}$.

4) Let $\tilde{u} \in [0,1]$. It holds,

$$g(\tilde{u}) + f(\tilde{u},\delta_n) \leq g(u_n) + f(u_n,\delta_n)\,.$$

By letting $n\to\infty$ and by continuity of g and f we get

$$g(\tilde{u}) + f(\tilde{u},\delta_\infty) \leq g(u_\infty) + f(u_\infty,\delta_\infty)\,.$$

This yields the result.

■

Theorem 24: Let g_1 and g_2 be fixed points of Ψ satisfying the properties listed in Proposition 22, then $g_1(\delta) = g_2(\delta)$, $\forall \delta$.

Proof: Let

$$\bar{\epsilon}_i = \max\{\epsilon < 1/2 : \; g_i(\delta) = 0, \; \forall \delta \leq \epsilon\}$$

For any $\delta \in [0,1]$, choose arbitrarily $\tilde{u}_i(\delta) \in \Gamma_{g_i}(\delta)$ with the only constraint that $\tilde{u}_1(1/2) = \tilde{u}_2(1/2) = 1/2$. For any δ, we can estimate as follows

$$\begin{aligned} g_2(\delta) &= g_2(\tilde{u}_2(\delta)) + f(\tilde{u}_2(\delta),\delta) + g_1(\tilde{u}_2(\delta)) - g_1(\tilde{u}_2(\delta)) \leq \\ &\leq g_2(\tilde{u}_2(\delta)) + f(\tilde{u}_1(\delta),\delta) + g_1(\tilde{u}_1(\delta)) - g_1(\tilde{u}_2(\delta)) = \\ &= g_2(\tilde{u}_2(\delta)) + g_1(\delta) - g_1(\tilde{u}_2(\delta)) \end{aligned}$$

$$\Longrightarrow g_2(\delta) - g_1(\delta) \leq g_2(\tilde{u}_2(\delta)) - g_1(\tilde{u}_2(\delta)).$$

Repeating the argument k times we get

$$g_2(\delta) - g_1(\delta) \leq g_2(\tilde{u}_2^{(k)}(\delta)) - g_1(\tilde{u}_2^{(k)}(\delta)). \tag{27}$$

In the same way we get that

$$\begin{aligned} g_2(\delta) &\geq g_2(\tilde{u}_1(\delta)) + f(\tilde{u}_1(\delta),\delta) + g_1(\tilde{u}_1(\delta)) - g_1(\tilde{u}_1(\delta)) \geq \\ &\geq g_2(\tilde{u}_1(\delta)) + g_1(\delta) - g_1(\tilde{u}_1(\delta)) \end{aligned}$$

Iterating the argument k times, we have

$$g_2(\delta) - g_1(\delta) \geq g_2(\tilde{u}_1^{(k)}(\delta)) - g_1(\tilde{u}_1^{(k)}(\delta)). \tag{28}$$

Fix now $\delta \in [\bar{\epsilon}_1 \vee \bar{\epsilon}_2, 1/2]$ and consider the recursive systems, for $i = 1,2$:

$$\delta_i^{k+1} = \tilde{u}_i(\delta_i^k) \quad \delta_i^0 = \delta \tag{29}$$

By the way $\tilde{u}_i$ have been constructed and from points 1) and 2) of Lemma 23 we know that both sequences $\{\delta_i^k\}_{k\in\mathbb{N}}$, $i = 1,2$ are upper bounded by $1/2$ and increasing in k. Using 3) of Lemma 23, it follows that they both converge to $1/2$.

From inequalities in (27) and (28) we have

$$\lim_{k\to\infty} g_2(\delta_1^k) - g_1(\delta_1^k) \leq g_2(\delta) - g_1(\delta) \leq \lim_{k\to\infty} g_2(\delta_2^k) - g_1(\delta_2^k).$$

As the functions g_2 and g_1 are continuous

$$0 = g_2\left(\frac{1}{2}\right) - g_1\left(\frac{1}{2}\right) \leq g_2(\delta) - g_1(\delta) \leq g_2\left(\frac{1}{2}\right) - g_1\left(\frac{1}{2}\right) = 0$$

and we conclude that $g_2(\delta) = g_1(\delta)$ for every $\delta \in [\bar{\epsilon}_1 \vee \bar{\epsilon}_2, 1/2]$. Proof is completed using the properties of Proposition 22.

■

Corollary 25:

$$r^{(\infty)}(\delta) = [H(\delta) - (1-R)\ln 2]_+.$$

VIII. ESTIMATION OF MINIMUM DISTANCE DISTRIBUTION

As we have already noticed, the floor of the spectral shapes is zero (see Figure 2) and we can not apply Theorem 2, in order to estimate the minimum distance distribution. This means that $\forall \epsilon > 0$ the minimum distance is upper bounded by $d_{\min}(RA_N^m) \leq (\epsilon_m - \epsilon)N$ with a probability that does not decay exponentially in N.

We will prove that this probability decreases to zero polinomially in the interleaver length N. Inspired by asymptotic techniques devised in [5], we split the computation of the probability in two parts. The first part considers the contribution of the codewords with small input weight in the last accumulate encoder $h < h_N$ and the second part those with weight $h \geq h_N$. The sequence $\{h_N\}_{N\in\mathbb{N}}$ can be chosen in such a way that the first term dominates the behavior of the overall probability.

Lemma 26: Let $\{h_N\}_{N\in\mathbb{N}}$ be a sequence of integers such that for any arbitrary $\eta > 0$

$$\lim_{N\to\infty} \frac{h_N}{N^\eta} = 0, \qquad \lim_{N\to\infty} \frac{\ln N}{h_N} = 0. \tag{30}$$

Then

$$\sum_{h=1}^{h_N-1} \overline{A}_h \left(RA_N^m\right) = O\left(N^{\beta_m+\eta}\right) \tag{31}$$

where $\beta_m = 1 - \sum_{i=1}^{m} \lceil q/2^m \rceil$.

Proof: A complete proof is given in [7]. ■

Lemma 27:

$$r_N^{(m)}(\delta) \leq 2m\frac{\ln(N+1)}{N} + \widehat{r}^{(m)}(\delta)$$

Proof: By using the inequalities (35b) in Appendix A the assertion can be proven by induction on m. ■

Theorem 28: Let $\{\epsilon_m\}_{m\geq 2}$ be the sequence defined by (19). We have that $\forall\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}(d_{\min}(RA_N^m) \leq (\epsilon_m - \epsilon)N) = 0.$$

Proof: Fix $\epsilon > 0$ and put $d_N = (\epsilon_m - \epsilon)N$. Pick a sequence of integers $\{h_N\}_{N\in\mathbb{N}}$ satisfying conditions (30).

From (4) and (8) we have

$$\begin{aligned}
\mathbb{P}(d_{\min}(RA_N^m) \leq d_N) &\leq \sum_{d=1}^{d_N}\sum_{h=1}^{2d} \overline{A}_h \left(RA_N^{m-1}\right) P_{h,d}(\mathrm{Acc}_N) \\
&\leq \sum_{h=1}^{2d_N}\sum_{d=1}^{d_N} \overline{A}_h \left(RA_N^{m-1}\right) P_{h,d}(\mathrm{Acc}_N) \\
&\leq \sum_{h=1}^{h_N-1} \overline{A}_h \left(RA_N^{m-1}\right) \sum_{d=1}^{d_N} P_{h,d}(\mathrm{Acc}_N) + \\
&+ \sum_{h=h_N}^{2d_N}\sum_{d=1}^{d_N} \overline{A}_h \left(RA_N^{m-1}\right) P_{h,d}(\mathrm{Acc}_N) \\
&\leq \sum_{h=1}^{h_N-1} \overline{A}_h(RA_N^{m-1}) + \\
&+ \underbrace{\sum_{h=h_N}^{2d_N}\sum_{d=1}^{d_N} \overline{A}_h(RA_N^{m-1}) P_{h,d}(\mathrm{Acc}_N)}_{S^m(N)}\ .
\end{aligned}$$

Put $G^{(m)}(x,y) = \widehat{r}^{(m-1)}(x) + f(x,y)$.

From Lemma 27, from the inequalities (35b) in Appendix A, and using the fact that $f(u,\delta)$, for fixed u is increasing in $\delta < 1/2$ we can estimate as follows:

$$\begin{aligned}
S^m(N) &\leq (N+1)^{2m+1} \sum_{h=h_N}^{2d_N}\sum_{d=1}^{d_N} \mathrm{e}^{NG^{(m)}(h/N,d/N)} \\
&\leq d_N(N+1)^{2m+1} \sum_{h=h_N}^{2d_N} \mathrm{e}^{N \max_{y\in[1/N,d_N/N]} G^{(m)}(h/N,y)} \\
&\leq (\epsilon_m - \epsilon)(N+1)^{2m+2} \sum_{h=h_N}^{2d_N} \mathrm{e}^{NG^{(m)}(h/N,\epsilon_m-\epsilon)}
\end{aligned}$$

Moreover for $h_N \leq h \leq 2d_N$

$$NG^{(m)}(h/N,\epsilon_m - \epsilon) \leq hL_{\max}^{(m)}$$

where

$$L_{\max}^{(m)} = \max_{h_N/N<u\leq 2(\epsilon_m-\epsilon)} \frac{\widehat{r}^{(m-1)}(u) + f(u,\epsilon_m - \epsilon)}{u}$$

Being $\widehat{r}^{(m-1)}(u) = 0 \ \forall u \leq \epsilon_{m-1}$, we can compute the optimizing value, by splitting the computation of $L_{\max}^{(m)}$ as follows

$$L_{\max}^{(m)} = \max\left(L_1^{(m)}, L_2^{(m)}\right)$$

where

$$\begin{aligned}
L_1^{(m)} &= \max_{u\in[h_N/N,\epsilon_{m-1}]} \frac{f(u,\epsilon_m - \epsilon)}{u} \\
L_2^{(m)} &= \max_{u\in(\epsilon_{m-1},2(\epsilon_m-\epsilon)]} \frac{\widehat{r}^{(m-1)}(u) + f(u,\epsilon_m - \epsilon)}{u}
\end{aligned}$$

As $f(u,\epsilon_m - \epsilon)$ is decreasing in u, and $h_N/N \to 0$ when $N \to +\infty$, we can conclude that

$$\begin{aligned}
L_1^{(m)} &= \frac{f(h_N/N,\epsilon_m - \epsilon)}{h_N/N} \leq \lim_{N\to\infty} \frac{f(h_N/N,\epsilon_m - \epsilon)}{h_N/N} = \\
&= f_u(0,\epsilon_m - \epsilon) < 0\,.
\end{aligned} \tag{32}$$

Using the fact that $\widehat{r}^{(m-1)}$ and f are both continuous and $f(u,\delta)$ is strictly increasing in $\delta < 1/2$, by the way ϵ_m has been defined (19), we obtain that

$$\widehat{r}^{(m-1)}(u) + f(u,\epsilon_m - \epsilon) < 0 \quad \text{for } u \in [\epsilon_{m-1}, 2(\epsilon_m - \epsilon)].$$

Hence,

$$L_2^{(m)} = \max_{u\in]\epsilon_{m-1},2(\epsilon_m-\epsilon)]} \frac{\widehat{r}^{(m-1)}(u) + f(u,\epsilon_m - \epsilon)}{u} < 0\,. \tag{33}$$

Put $\tau = \min(L_1^{(m)}, L_2^{(m)}) < 0$, from (32) and (33) we get

$$\begin{aligned}
S^m(N) &\leq (\epsilon_m - \epsilon)(N+1)^{2m+2} \sum_{h=h_N}^{2d_N} \mathrm{e}^{\tau h} \leq \\
&\leq (\epsilon_m - \epsilon)(N+1)^{2m+2} \frac{\mathrm{e}^{\tau h_N}}{1 - e^{\tau}}.
\end{aligned}$$

It follows that

$$S^m(N) \leq C \exp\left[(2m+2)\ln(N+1) + \tau h_N\right] \xrightarrow{N\to\infty} 0$$

with $C = \frac{(\epsilon_m - \epsilon)}{1 - e^{\tau}}$.

From (VIII) and from Lemma 26 we conclude that $\forall\eta > 0$

$$\mathbb{P}(d_{\min}(RA_N^m) \leq (\epsilon_m - \epsilon)N) = O\left(N^{\beta_{m-1}+\eta}\right), \tag{34}$$

by which

$$\lim_{N\to\infty} \mathbb{P}(d_{\min}(RA_N^m) \leq (\epsilon_m - \epsilon)N) = 0$$

for $m \geq 3$ and $q \geq 2$ or $m = 2$ and $q \geq 3$. ■

IX. CONCLUDING REMARKS

A. Summary

This paper leaves a number of open problems to study. Our analysis focused on repeat accumulate codes. However, we expect many of the properties to hold for more general constituent encoder. Replacing the repetition code with any other encoder is a simple and straightforward generalization: we get the same exact results. Much more subtle is the role of the accumulator since the dynamical system Ψ we have defined depends exclusively on the accumulator and it would be different with another convolutional encoder. Encouraged by the results in [9] which did hold for any choice of the convolutional encoder (as long it was not the identity), we believe that the dynamical systems analysis should holds true for all convolutional encoders (recursive or not recursive). Instead for the final part on the estimation of distances, the role of recursivity must necessarily come up since, if it is not recursive can not certainly exhibit linear growth of minimum distances.

APPENDIX

A. *Some useful inequalities [8, Appendix 3A]*

For any integers $1 \le k \le n$ the following inequalities hold true:

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{n\mathrm{e}}{k}\right)^k \tag{35a}$$

$$\frac{\mathrm{e}^{nH(k/n)}}{n+1} \le \binom{n}{k} \le \mathrm{e}^{nH(k/n)} \tag{35b}$$

B. *Proof of Lemma 30*

In order to prove Lemma 19 we need to establish some intermediate results. Lemma 29 allows us to get some information about the monotony of non-smooth functions. The result is surely not original but we give the assertion, as we don't have any reference.

Lemma 29: Let $y \in L^\infty(\mathbb{R})$ and let $y : \mathbb{R} \to \mathbb{R}$ be Lipschitz. If

$$\limsup_{\eta \to 0} \frac{y(x+\eta) - y(x)}{\eta} \le 0 \tag{36}$$

then $y(x)$ is a decreasing function.

For every δ, define the following set

$$\Gamma^{(m)}(\delta) = \operatorname*{argmax}_{u \in \Omega_\delta} \{\widehat{r}^{(m-1)}(u) + f(u,\delta)\} \tag{37}$$

and choose $u^{(m)}(\delta) \in \Gamma^{(m)}(\delta)$. Moreover it is easy to prove that in the case $m = 1$ that for any fixed δ the maximizing value $u^{(1)}(\delta)$ is unique.

Lemma 30: For any arbitrary $\epsilon \in]0, 1/2]$, we have:

1) if $u^{(m)}(\delta) \le 2\delta(1-\delta)$ and $\widehat{r}^{(m)}(\delta)$ is Lipschitz in $\delta \in [\epsilon, 1/2]$, then $\widehat{r}^{(m)}(\delta) - H(\delta)$ is decreasing in $\delta \in [\epsilon, 1/2]$;
2) if $\widehat{r}^{(m)}(\delta) - H(\delta)$ decreases in $\delta \in [\epsilon, 1/2]$, then $u^{(m+1)}(\delta) \le 2\delta(1-\delta)$ and $\widehat{r}^{(m+1)}(\delta)$ is Lipschitz in $\delta \in [\epsilon, 1/2]$;

Proof: 1) From the hypothesis we know that $u^{(m)}(\delta) \le 2\delta(1-\delta)$ and we can write that for any arbitrary $\eta > 0$

$$\widehat{r}^{(m)}(\delta+\eta) = \max_{0 \le u \le 2(\delta+\eta)(1-\delta-\eta)} [\widehat{r}^{(m-1)}(u) + f(u, \delta+\eta)].$$

Using the fact that $f_{\delta\delta}(u,\delta) \le 0$, and $f_{u\delta}(u,\delta) \ge 0 \; \forall \delta \le 1/2$, $\forall u \in \Omega_\delta$, we can estimate, for $u \le 2(\delta+\eta)(1-\delta-\eta)$,

$$\begin{aligned} f(u,\delta+\eta) &\le f(u,\delta) + f_\delta(u,\delta)\eta \\ &\le f(u,\delta) + f_\delta(2(\delta+\eta)(1-\delta-\eta),\delta)\eta \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{r}^{(m)}(\delta+\eta) \le \max_{0 \le u \le 2(\delta+\eta)(1-\delta-\eta)} &[\widehat{r}^{(m-1)}(u) + f(u,\delta) + \\ &+ f_\delta(2(\delta+\eta)(1-\delta-\eta),\delta)\eta] \\ \le \widehat{r}^{(m)}(\delta) &+ f_\delta(2(\delta+\eta)(1-\delta-\eta),\delta)\eta, \end{aligned}$$

where the last inequality follows by the fact that $u^{(m)}(\delta) \le 2\delta(1-\delta)$. So we have

$$\frac{\widehat{r}^{(m)}(\delta+\eta) - \widehat{r}^{(m)}(\delta)}{\eta} \le f_\delta(2(\delta+\eta)(1-\delta-\eta),\delta)$$

and

$$\limsup_{\eta \to 0} \frac{\widehat{r}^{(m)}(\delta+\eta) - \widehat{r}^{(m)}(\delta)}{\eta} \le f_\delta(2\delta(1-\delta),\delta) = H'(\delta).$$

From Lemma 29 we conclude that $\widehat{r}^{(m)}(\delta) - H(\delta)$ decreases in $\delta \in [\epsilon, 1/2]$.

2) We prove it by contradiction.

If we assume that, for some $\delta \in [\epsilon, 1/2]$, it holds $u^{(m+1)}(\delta) > 2\delta(1-\delta)$, then

$$\begin{aligned} \widehat{r}^{(m+1)}(\delta) &= \widehat{r}^{(m)}(u^{(m+1)}(\delta)) + f(u^{(m+1)}(\delta),\delta) = \\ &= \widehat{r}^{(m)}(u^{(m+1)}(\delta)) - H(u^{(m+1)}(\delta)) + \\ &\quad + H(u^{(m+1)}(\delta)) + f(u^{(m+1)}(\delta),\delta). \end{aligned}$$

From the hypothesis and from (24) it can be upper bounded as follows

$$\begin{aligned} \widehat{r}^{(m+1)}(\delta) &< \widehat{r}^{(m)}(2\delta(1-\delta)) - H(2\delta(1-\delta)) + \\ &\quad + H(2\delta(1-\delta)) + f(2\delta(1-\delta),\delta) = \\ &= \widehat{r}^{(m)}(2\delta(1-\delta)) + f(2\delta(1-\delta),\delta) \end{aligned}$$

This is absurd by the definition of $\widehat{r}^{(m+1)}$.

We now prove the second part of 2). Let $\delta_1 < \delta_2 \in [\epsilon, 1/2]$. We have

$$u^{(m+1)}(\delta_2) \in [0, 2\delta_2(1-\delta_2)].$$

As $f(u,\delta)$ is Lipschitz in $\delta \in [\epsilon, 1/2]$ uniformly in $u \in [0, 2\delta(1-\delta)]$ there exists a constant $K \in \mathbb{R}$ such that

$$\begin{aligned} \widehat{r}^{(m+1)}(\delta_2) &= \max_{0 \le u \le 2\delta_2(1-\delta_2)} \{\widehat{r}^{(m)}(u) + f(u,\delta_2)\} \\ &\le \max_{0 \le u \le 2\delta_1(1-\delta_1)} \{\widehat{r}^{(m)}(u) + f(u,\delta_1)\} + K|\delta_2 - \delta_1| \\ &= \widehat{r}^{(m+1)}(\delta_1) + K|\delta_2 - \delta_1|\,. \end{aligned}$$

Similarly, we can estimate,

$$\widehat{r}^{(m+1)}(\delta_2) \ge \widehat{r}^{(m+1)}(\delta_1) - K|\delta_2 - \delta_1|$$

We conclude that

$$|\widehat{r}^{(m+1)}(\delta_2) - \widehat{r}^{(m+1)}(\delta_1)| \le K|\delta_2 - \delta_1| \qquad \forall \delta_1, \delta_2 \in [\epsilon, 1/2].$$

Notice that the constant K only depends on f and ϵ, and not on m. ■

Lemma 18: We have

$$|\widehat{r}^{(m)}(\delta_2) - \widehat{r}^{(m)}(\delta_1)| \le K|\delta_2 - \delta_1| \qquad \forall \delta_1, \delta_2, \forall m.$$

Proof: We first consider the case $m = 1$. Let $u_q^{(1)}(\delta) = \underset{u\in\Omega_\delta}{\operatorname{argmax}}\,[H(u)/q + f(u,\delta)]$.

If we consider the case $q = 2$, we find the analytical expression

$$u_2^{(1)}(\delta) = \frac{3-\sqrt{9-32\delta(1-\delta)}}{4} \qquad \forall\delta \in [0, 1/2],$$

by which $u_2^{(1)}(\delta) \leq 2\delta(1-\delta)$ and $u_2^{(1)}(\delta) = 2\delta(1-\delta) \iff \delta = 0$ or $\delta = 1/2$.

We prove now that $\{u_q^{(1)}(\delta)\}_{q\in\mathbb{N}}$ is a decreasing sequence of functions in q. Supposing ab absurdo that $u_q^{(1)}(\delta) < u_{q+1}^{(1)}(\delta)$,

$$\begin{aligned}
\widehat{r}_q^{(1)}(\delta) \quad &= \tfrac{H(u_q^{(1)}(\delta))}{q} + f(u_q^{(1)}(\delta),\delta) + \\
&\qquad + \tfrac{H(u_q^{(1)}(\delta))}{q+1} - \tfrac{H(u_q^{(1)}(\delta))}{q+1} \leq \\
&\leq \tfrac{H(u_q^{(1)}(\delta))}{q} - \tfrac{H(u_q^{(1)}(\delta))}{q+1} + \\
&\qquad + f(u_{q+1}^{(1)}(\delta),\delta) + \tfrac{H(u_{q+1}^{(1)}(\delta))}{q+1} \leq \\
&\leq \tfrac{H(u_{q+1}^{(1)}(\delta))}{q} - \tfrac{H(u_{q+1}^{(1)}(\delta))}{q+1} \\
&\qquad + f(u_{q+1}^{(1)}(\delta),\delta) + \tfrac{H(u_{q+1}^{(1)}(\delta))}{q+1} = \\
&= f(u_{q+1}^{(1)}(\delta),\delta) + \tfrac{H(u_{q+1}^{(1)}(\delta))}{q}
\end{aligned}$$

we get that $u_q^{(1)}(\delta) \neq \underset{u\in\Omega_\delta}{\operatorname{argmax}}\,[r^{(1)}(u) + f(u,\delta)]$.

So we have $u_q^{(1)}(\delta) \leq u_2^{(1)}(\delta) \leq 2\delta(1-\delta),\ \forall q \in \mathbb{N}$.

Notice that $\widehat{r}^{(1)}(\delta)$ is differentiable and $u^{(1)}(\delta) \leq 2\delta(1-\delta)$. Applying, inductively, Lemma 30 for some $\epsilon \in (0, \epsilon_2)$ we obtain that $\widehat{r}^{(m)}(\delta)$ are all Lipschitz in $\delta \in [\epsilon, 1/2]$, $\forall m$. As $\widehat{r}^{(m)}(\delta)$ is symmetric respect to axis $\delta = 1/2$ and $\widehat{r}^{(m)}(\delta) = 0$ $\forall\delta \leq \epsilon$, $\widehat{r}^{(m)}(\delta)$ is Lipschitz in every point in $[0, 1]$, $\forall m$.

Notice that the Lipschitz's constant K is the same for every spectral shape. ■

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