

Approaching Capacity with Asymptotically Regular LDPC Codes

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Abstract—We present a family of protograph based LDPC codes that can be derived from permutation matrix based regular (J, K) LDPC convolutional codes by termination. In the terminated protograph, all variable nodes still have degree J but some check nodes at the start and end of the protograph have degrees smaller than K . Since the fraction of these stronger nodes vanishes as the termination length L increases, we call the codes *asymptotically regular*. The density evolution thresholds of these protographs are better than those of regular (J, K) block codes. Interestingly, this threshold improvement gets stronger with increasing node degrees (at a fixed rate) and it does not decay as L increases. Terminated convolutional protographs can also be derived from standard irregular protographs and may exhibit a significant threshold improvement.

I. INTRODUCTION

The performance of a belief propagation (BP) decoder for LDPC codes [1] is strongly influenced by the degrees of the different variable nodes and check nodes in the considered Tanner graph representation [2]. The original ensembles introduced by Gallager in [1] consisted of regular (J, K) LDPC codes with fixed variable node degree J and check degree K . One shortcoming of such regular ensembles is that a small K , which leads to short parity-check equations and improves the decoder performance, also implies a small J for a given rate of the code. For this reason, irregular code ensembles [3] [4] with a variety of different node degrees are usually used in practice. In these ensembles, the degrees of variable nodes and check nodes are considered as random variables that are characterized by their degree distributions $\lambda(x)$ and $\rho(x)$, respectively. Each coefficient in the polynomials $\lambda(x)$ and $\rho(x)$ corresponds to the fraction of edges connected to nodes of a certain degree. Gallager's regular (J, K) LDPC code ensembles correspond to the special case $\lambda(x) = x^{J-1}$ and $\rho(x) = x^{K-1}$.

For the binary erasure channel (BEC), a density evolution analysis of the BP decoder can be performed explicitly by means of the equation

$$p^{(i)} = \varepsilon \lambda \left(1 - \rho \left(1 - p^{(i-1)} \right) \right), \quad (1)$$

where ε denotes the erasure probability of the channel and $p^{(i)}$ the probability that a variable to check node message in decoding iteration i corresponds to an erasure, averaged over all codes of the ensemble. Due to this averaging, the

message probabilities are equal for all edges in the graph. The density evolution threshold of an ensemble, defined as the maximal value of the channel parameter ε for which $p^{(i)}$ converges to zero as i tends to infinity, directly follows from (1). Equation (1) is also the key to the design of degree distribution pairs (λ, ρ) for *capacity achieving sequences* of codes with a vanishing gap between the threshold and the Shannon limit $\varepsilon_{\text{sh}} = 1 - R$ [5]. Check-concentrated or even check-regular ensembles are known to provide a good trade-off between complexity (measured by the average node degrees) and gap to capacity.

A double exponential decrease of the decoding erasure probability with iterations implies that the probability of *erased frames* also converges to zero [6]. The lower bounds in [7] on the decoding complexity of general message passing decoders, obtained using sphere-packing arguments, also predict a double exponential reduction of the error probability with the number of iterations. A Taylor expansion of (1) reveals that the message probability $p^{(i)}$ converges to zero at least doubly exponentially with i if all nodes have a variable node degree of at least three. An analysis by means of the messages' Bhattacharyya parameter shows that this is also true for general binary-input output-symmetric memory-less channels [6]. For generalized LDPC codes, where the parity-check equations are replaced by stronger subcodes with minimum distance greater than two, it can be shown that a minimal variable node degree of two is sufficient. Unfortunately, unstructured irregular LDPC ensembles with thresholds close to capacity exhibit a non-vanishing fraction of degree two variable nodes.

LDPC code ensembles with a certain predefined structure can be constructed by means of protographs [8]. It has been observed that protograph ensembles often have better thresholds than unstructured irregular random ensembles with the corresponding degree distributions. Even codes with minimal variable node degree three may provide a good trade-off between distance and threshold [9]. Some codes that contain degree two variable nodes can also have a linear asymptotic minimum distance growth [10].

In this paper, we derive asymptotically regular protographs from convolutional protographs by termination. These protographs, which are described in Section III, have a constant variable node degree of at least three and at the same time

corresponding regular code. But when the symbols at $t = 1$ and $t = L$ are perfectly known, the connected edges can be removed from the protograph, which results in the shortened protograph $\mathbf{B}_{[2,L-1]}$, as illustrated in Fig. 2(b). It follows now by induction that the messages eventually converge to zero at all times $t = 1, \dots, L$ for arbitrary values L . A proof of Theorem 1 can be found in [16].

C. Protograph LDPC codes for arbitrary J and K

If we want to construct convolutional protographs for arbitrary rates, we have to face the problem that, in the above described construction, $m_s = a - 1$ becomes zero if J and K are relatively prime. This results in a sequence of disconnected protographs \mathbf{B}_0 , each defining a standard regular block code. An essential property of a convolutional protograph is that edges from variable nodes at time t are spread among check nodes at *different* times $t, t + 1, \dots, t + m_s$, as illustrated for the case $m_s = 1$ in Fig. 3. Starting from an arbitrary (J, K) block protograph with base matrix \mathbf{B} we can achieve such an *edge spreading* by dividing the entries of \mathbf{B} among various matrices $\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_{m_s}$. This procedure ensures that the degrees of variable nodes and check nodes of the resulting convolutional protograph are the same as those of the original block protograph.

Example 2: Consider the construction of a convolutional protograph from a $(5, 6)$ LDPC block code, defined by a 5×6 all-one base matrix \mathbf{B} . A convolutional protograph, defining a rate $R = Mb/(Mc) = 1/6$ code with $m_s = 1$, follows from (6) with the component base matrices

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

which can be obtained from \mathbf{B} by edge spreading. The threshold of the terminated protographs $\mathbf{B}_{[1,L]}$ approaches $\varepsilon^* = 0.829$ as L increases. This value is remarkably close to the Shannon limit $\varepsilon_{\text{sh}} = 0.833$ for rate $R_\infty = 1/6$. \square

The convolutional protograph in Fig. 2 can also be constructed using this procedure with

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The convolutional code has now rate $R = 3M/(6M)$ and memory $m_s = 1$ instead of $R = M/(2M)$ and $m_s = 2$. As a consequence, the nodes of the two protographs have different time instants associated with them, but otherwise the structure of the two graphs is the same.

There are many ways of spreading the edges among component base matrices, and different assignments can lead to different thresholds. Even for $m_s > 0$ there exist assignments that result in a sequence of disconnected subgraphs, e.g., if all-zero columns or rows exist in the component base matrices. A good threshold value is expected when the checks at time $t = 1$

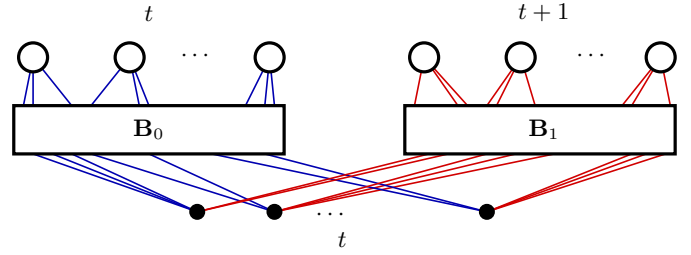


Fig. 3. The protograph connections from variable nodes at time t , represented by the component base matrices \mathbf{B}_0 and \mathbf{B}_1 for the case $m_s = 1$.

have low degree (but at least degree two). The convolutional protograph in Example 2 is designed in such a way that all rows in \mathbf{B}_0 have weight two and the entries are spread among all rows and columns. Note that, by symmetry, if we reverse the order of the component base matrices (e.g., exchanging \mathbf{B}_0 and \mathbf{B}_1 in Example 2), the convolutional protograph is simply mirrored horizontally and the threshold is consequently the same. Simple row or column permutations, applied simultaneously to all component base matrices, also do not affect the graph structure.

IV. IMPROVING THRESHOLDS OF IRREGULAR PROTOGRAPHS

The edge spreading procedure described above is not restricted to regular protographs, but can be applied to any conventional protograph, including those with multiple edges between a pair of nodes. Starting from an arbitrary block protograph, defined by an $M_P \times N_P$ base matrix \mathbf{B} , we divide the edges among times $t, t + 1, \dots, t + m_s$. For a given target memory m_s , any set of component base matrices $\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_{m_s}$ which satisfies the condition

$$\sum_{i=0}^{m_s} \mathbf{B}_i = \mathbf{B} \quad (10)$$

corresponds to a possible assignment of edges, resulting in a convolutional protograph with the same variable and check node degrees as the original block protograph. The corresponding convolutional base matrix $\mathbf{B}_{[-\infty, \infty]}$ follows from (6). Termination of such a convolutional protograph, after an arbitrary number of time instants L , results in a block protograph $\mathbf{B}_{[1,L]}$ with LN_P variable nodes and $(L + m_s)M_P$ check nodes, corresponding to a design rate

$$R_L = 1 - \left(\frac{L + m_s}{L} \right) \frac{M_P}{N_P} = 1 - \left(1 + \frac{m_s}{L} \right) (1 - R_B), \quad (11)$$

where $R_B = 1 - M_P/N_P$ is the design rate of codes obtained from the original block protograph.

Example 3: Consider the protograph of an accumulate-repeat-by-4-jagged-accumulate (AR4JA) code [10], as depicted in Fig. 4. The base matrix of this code is equal to

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \end{bmatrix}. \quad (12)$$

The variable nodes corresponding to the second column in \mathbf{B}

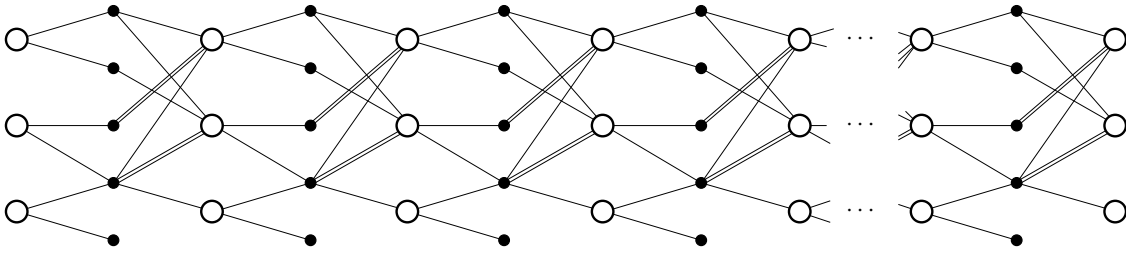


Fig. 5. A terminated convolutional protograph obtained from the AR4JA code in Fig. 4.

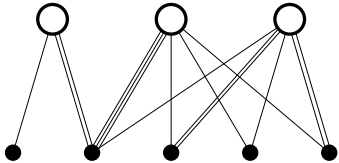


Fig. 4. The protograph of an AR4JA code.

are punctured, resulting in a design rate equal to $R_B = 1/2$ and a threshold of $\varepsilon^* = 0.4387$. Using the edge spreading procedure, we derive the base matrices

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix}.$$

The resulting terminated convolutional protograph $\mathbf{B}_{[1,L]}$ is illustrated in Fig. 5. Its threshold approaches $\varepsilon^* = 0.4996$ as L increases, which is very close to the Shannon limit $\varepsilon_{sh} = 0.5$ for rate $R_\infty = 1/2$. \square

V. CONCLUSION

We presented a technique for the construction of asymptotically regular protographs with thresholds close to the Shannon limit. These protographs can be derived by termination from convolutional protographs, which were obtained from regular (J, K) protographs by means of an edge spreading technique, where we assume that $J > 2$. Since all variable nodes have degree greater than two, asymptotically the error probability converges at least doubly exponentially with decoding iterations and the minimum distance grows linearly with the length of the codes. As a result we obtain sequences of asymptotically good LDPC codes with fast convergence rates and thresholds close to capacity. The construction can also be generalized to arbitrary irregular protographs. Although we restricted the discussion to the BEC, the results for $(J, 2J)$ codes in [15] indicate that a similar behavior can be expected for the additive white Gaussian noise channel.

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