Coding for a Non-symmetric Ternary Channel

Nicolas Bitouzé and Alexandre Graell i Amat
Department of Electronics, Institut TELECOM-TELECOM Bretagne, 29238 Brest, France
Email: nicolas.bitouze@telecom-bretagne.eu, alexandre.graell@telecom-bretagne.eu

***Abstract*—Non-symmetric ternary channels can be used to model the behavior of some memory devices. In this work, error correction coding for a non-symmetric ternary channel where some of the error transitions are not allowed, is considered. We study distance properties of ternary codes over this channel and define the maximum likelihood (ML) decoding rule. It is shown that the ML decoding rule is too complex, since it depends on the channel error probability. A simpler alternative decoding rule, called d_{A}-decoding, is then proposed. It is shown that d_{A}-decoding and ML decoding are equivalent for values of p under a certain threshold. Assuming d_{A}-decoding we characterize the error correction capabilities of ternary codes over the non-symmetric ternary channel. We also provide an upper bound and a constructive lower bound on the size of such codes given the code length and the minimum distance.**

I. Introduction

Electrically erasable programmable read-only memories (EEPROMs) are semiconductor memories that retain their data contents when power is off. They can be read and written to like standard RAM and are suitable for applications where storage of small amounts of data is critical and periodic writing of new data is required. Typical applications are radio frequency identification tag, smart dust, or automotive applications including car audio and multimedia, chassis and safety, and power train.

The communication channel underlying EEPROMs can be suitably modeled as a binary symmetric channel (BSC). Currently, very simple error correction codes based on the well-known Hamming codes combined with hard decoding are implemented on-chip to correct single bit errors [1]. However, next generation devices demand for more stringent requirements in terms of reliability as well as storage density. A suitable modification of the physics of EEPROM memories allows to store the information in three levels, thus higher densities can be achieved. A suitable model of the resulting channel is the discrete memoryless non-symmetric ternary channel depicted in Fig. 1, and denoted by $\mathcal{H}$. The channel is characterized by an input alphabet $\mathcal{X} = \{0, 1, 2\}$, an output alphabet $\mathcal{Y} = \{0, 1, 2\}$, and a set of conditional probabilities $\mathrm{P}(y|x)$ such that the symbol 0 is received correctly with probability $1 - p$ and received as a 1 or a 2 with crossover probability $p/2$, and symbols 1 and 2 are received as a 0 with crossover probability $p/2$ and received correctly with probability $1 - p/2$. Hence, transitions $1 \rightarrow 2$ and $2 \rightarrow 1$ are not allowed.

A. Graell i Amat was funded by a Marie Curie Intra-European Fellowship within the 6th European Community Framework Programme.

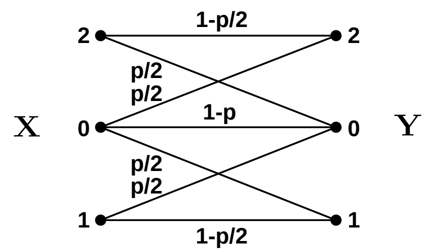

Fig. 1. Non-symmetric ternary channel $\mathcal{H}$.

In this paper we consider coding for the non-symmetric ternary channel of Fig. 1. We define the maximum likelihood (ML) decoding rule over this channel and show that its implementation becomes too complex, since it depends on p. As an alternative, a simpler decoding rule, called d_{A}-decoding, is proposed based on a more appropriate distance measure. It is shown that the proposed decoding rule is equivalent to ML decoding for the values of p of interest. We then study error correction capabilities of ternary codes under d_{A}-decoding rule. In particular, we derive an upper bound on the size of the code. We also derive a lower bound on the size of the code given the code length n and its minimum distance, which proves the existence of good codes. For small values of the minimum distance the lower bound and the upper bound are close.

II. Maximum Likelihood Decoding

For later use, let $\mathcal{C}$ be a ternary code of length n, and let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{C}$ and $\mathbf{y} = (y_1, \ldots, y_n)$ represent a codeword transmitted over channel $\mathcal{H}$, and a received vector, respectively.

Definition 1. *Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ two vectors in $\mathbb{F}_3^n$. The distance*[1] *$d_{\mathrm{ML}}(\mathbf{u}, \mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$ is defined as:*

$$d_{\mathrm{ML}}(\mathbf{u},\mathbf{v}) = \begin{cases} \infty & \text{if } \delta_1'(\mathbf{u},\mathbf{v}) > 0 \\ -\delta_0(\mathbf{u},\mathbf{v})\log(1-p) & \\ -\delta_1(\mathbf{u},\mathbf{v})\log(1-p/2) & \text{otherwise} \\ -\delta_0'(\mathbf{u},\mathbf{v})\log(p/2) & \end{cases} \tag{1}$$

[1]Note that d_{ML} is not formally a distance as it is not symmetric and as the identity of indiscernibles does not hold.

where

$$\begin{aligned}\delta_0(\mathbf{u},\mathbf{v}) &= |\{i \mid u_i = v_i = 0\}| \\ \delta_1(\mathbf{u},\mathbf{v}) &= |\{i \mid u_i = v_i \neq 0\}| \\ \delta'_0(\mathbf{u},\mathbf{v}) &= |\{i \mid u_i \neq v_i \wedge u_i v_i = 0\}| \\ \delta'_1(\mathbf{u},\mathbf{v}) &= |\{i \mid u_i \neq v_i \wedge u_i v_i \neq 0\}|\,.\end{aligned} \tag{2}$$

In (2) $\delta_0(\mathbf{u},\mathbf{v})$ and $\delta_1(\mathbf{u},\mathbf{v})$ denote the number of positions i where u_i and v_i are both equal, and $u_i = v_i = 0$ and $u_i = v_i \neq 0$, respectively. For symmetric channels, there is no need to distinguish between positions where $u_i = v_i = 0$ and positions where $u_i = v_i \neq 0$, since the probability of the transitions $0 \to 0$ is the same as the probability for any other non-error transition. However, for channel $\mathcal{H}$, this distinction is required. Also, $\delta'_0(\mathbf{u},\mathbf{v})$ and $\delta'_1(\mathbf{u},\mathbf{v})$ denote the number of positions i where u_i and v_i differ, and $u_i v_i = 0$ and $u_i v_i \neq 0$, respectively. Again, this distinction is not necessary for symmetric channels.

As the channel is memoryless, the probability $\mathrm{P}(\mathbf{y}|\mathbf{x})$ of receiving $\mathbf{y}$, being codeword $\mathbf{x}$ transmitted, is

$$\mathrm{P}(\mathbf{y}|\mathbf{x}) \;=\; \prod_{i=1}^{n} \mathrm{P}(y_i|x_i) \tag{3}$$

We can therefore relate $d_{\mathrm{ML}}(\mathbf{x},\mathbf{y})$ to $\mathrm{P}(\mathbf{y}|\mathbf{x})$:

$$-\log(\mathrm{P}(\mathbf{y}|\mathbf{x})) = \sum_{i=1}^{n} -\log(\mathrm{P}(y_i|x_i)) = d_{\mathrm{ML}}(\mathbf{x},\mathbf{y}). \tag{4}$$

Using d_{ML}, the ML decoding rule can be formulated as follows:

> *Given a received word* $\mathbf{y}$*, decode to the codeword* $\mathbf{x}$ *that minimizes the distance* $d_{\mathrm{ML}}(\mathbf{x},\mathbf{y})$*.*

Proof: It is sufficient to prove that for a given $\mathbf{y}$, when $\mathbf{x}$ varies among codewords, $d_{\mathrm{ML}}(\mathbf{x},\mathbf{y})$ increases when $\mathrm{P}(\mathbf{x}|\mathbf{y})$ decreases. For $d_{\mathrm{ML}}(\mathbf{x},\mathbf{y}) = \infty$, there is a position i such that the transition that goes from x_i to y_i is not allowed, therefore $\mathrm{P}(\mathbf{x}|\mathbf{y}) = 0$. We consider the case where $d_{\mathrm{ML}}(\mathbf{x},\mathbf{y}) < \infty$:

$$\mathrm{P}(\mathbf{x}|\mathbf{y}) = \frac{\mathrm{P}(\mathbf{x})}{\mathrm{P}(\mathbf{y})}\mathrm{P}(\mathbf{y}|\mathbf{x}) = \frac{\mathrm{P}(\mathbf{x})}{\mathrm{P}(\mathbf{y})}\exp\left(-d_{\mathrm{ML}}(\mathbf{x},\mathbf{y})\right). \tag{5}$$

Since $\mathrm{P}(\mathbf{x})$ does not depend on the transmitted codeword $\mathbf{x}$, we have:

$$\mathrm{P}(\mathbf{x}|\mathbf{y}) \;\propto\; \exp\left(-d_{\mathrm{ML}}(\mathbf{x},\mathbf{y})\right). \tag{6}$$

We conclude by monotonicity of the exponential. ■

Unfortunately, d_{ML} depends on the channel transition probability p. Therefore, ML decoding based on d_{ML} is cumbersome. To circumvent this drawback, a simple alternative decoding rule is proposed in the following.

Definition 2. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two vectors in* $\mathbb{F}_3^n$*. The distance* $d_{\mathrm{A}}(\mathbf{u},\mathbf{v})$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *is defined as:*

$$d_{\mathrm{A}}(\mathbf{u},\mathbf{v}) \;= \sum_{i=1}^{n} d_{\mathrm{A}}(u_i,v_i) \tag{7}$$

where

$$d_{\mathrm{A}}(u_i,v_i) = \begin{cases} 0 & \text{if } u_i = v_i, \\ 1 & \text{if } u_i \neq v_i \;\wedge\; u_i v_i = 0, \\ \infty & \text{if } u_i \neq v_i \;\wedge\; u_i v_i \neq 0. \end{cases} \tag{8}$$

Using this distance measure[2], we can define the following decoding rule which does not depend on p:

> *Given a received word* $\mathbf{y}$*, decode to the codeword* $\mathbf{x}$ *that minimizes the distance* $d_{\mathrm{A}}(\mathbf{x},\mathbf{y})$*.*

In the rest of the paper we shall refer to this decoding rule as to d_{A}-decoding. We denote by t_{A} the error correction capability of a code $\mathcal{C}$ over the channel $\mathcal{H}$ under d_{A}-decoding. Note that d_{A}-decoding does not necessarily minimize the probability of error. However, the following theorem gives an upper bound on p under which, if less than or equal to t_{A} errors occur, d_{A}-decoding is equivalent to ML decoding:

Theorem 1. *Let* $\mathcal{C}$ *be a ternary code, and let* $\mathcal{H}$ *be the ternary channel of Fig. 1 with transition error probability* p*.* d_{A}*-decoding and ML decoding of vectors transmitted with less than or equal to* t_{A} *errors over* $\mathcal{H}$ *are equivalent for all codes* $\mathcal{C}$ *of length* n *if and only if:*

$$\frac{p/2}{1-p} \;<\; \left(\frac{1-p}{1-p/2}\right)^{\left\lfloor\frac{n-1}{2}\right\rfloor}\;. \tag{9}$$

Proof:

- *Direct implication:*
  We assume that (9) does not hold. For n odd, we write $n = 2m+1$. Consider the code $\mathcal{C} = \{\mathbf{0}^n, \mathbf{1}^n\}$ consisting of two codewords, the all-zero codeword and the all-one codeword, and the received vector $\mathbf{y} = \mathbf{0}^{m+1}\mathbf{1}^m$ consisting of $m+1$ zeros and m ones. Clearly, d_{A}-decoding decodes $\mathbf{y}$ to the all-zero codeword $\mathbf{0}^n$. On the other hand,
  $$\begin{aligned}\mathrm{P}(\mathbf{y}|\mathbf{0}^n) &= (1-p)^{m+1}(p/2)^m\;, \\ \mathrm{P}(\mathbf{y}|\mathbf{1}^n) &= (p/2)^{m+1}(1-p/2)^m.\end{aligned} \tag{10}$$
  Using the hypothesis, we obtain $\mathrm{P}(\mathbf{y}|\mathbf{0}^n) \leq \mathrm{P}(\mathbf{y}|\mathbf{1}^n)$. Therefore, ML decoding will not necessarily decode to $\mathbf{0}^n$.
  If n is even, we use the same argument considering the same vectors with an extra zero appended at the end.
- *Converse:*
  Only the sketch of the proof is given here.
  We prove that for a vector and $\mathbf{y}$ of weight w, and another vector $\mathbf{x}$ in $\mathbb{F}_3^n$ such that $d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) = d$, if d is finite, then:
  $$\mathrm{P}(\mathbf{y}|\mathbf{x}) \;\leq\; \begin{cases} \left(\frac{p}{2}\right)^d\left(1-\frac{p}{2}\right)^w(1-p)^{n-w-d} & \text{if } d \leq n-w, \\ \left(\frac{p}{2}\right)^d\left(1-\frac{p}{2}\right)^{n-d} & \text{otherwise.}\end{cases} \tag{11}$$
  We denote this upper bound by $B^+(n,w,d)$. We find a similar lower bound that we call $B^-(n,w,d)$, such that

[2]Note that in this case the identity of indiscernibles holds and the distance is symmetric. However, the triangular inequality does not hold anymore.

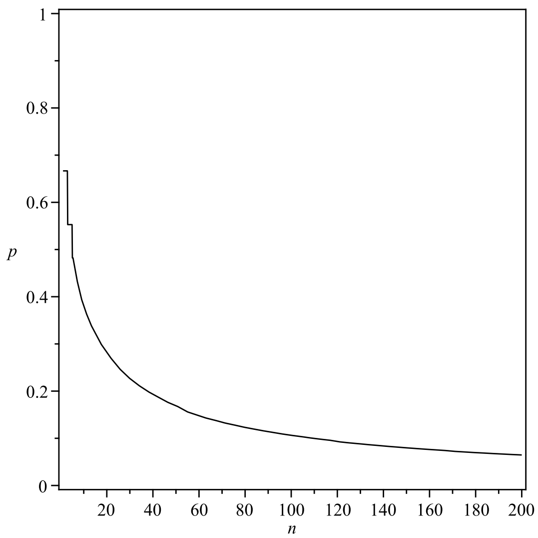

Fig. 2. Maximum value of p for equivalence between ML decoding and d_{A}-decoding rules.

for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}_3^n$, if w denotes the weight of $\mathbf{y}$ and d the d_{A}-distance between $\mathbf{x}$ and $\mathbf{y}$,

$$B^-(n,w,d) \le \mathrm{P}(\mathbf{y}|\mathbf{x}) \le B^+(n,w,d). \tag{12}$$

Then, we show that for all n, for all $w \le n$ and for all $d \le n-1$, the following inequality is equivalent to (9):

$$B^-(n,w,d) > B^+(n,w,d+1). \tag{13}$$

Thus, if (9) holds and if the minimization of $d_{\mathrm{A}}(\mathbf{x},\mathbf{y})$ yields a unique $\mathbf{x}$ (which happens whenever less than or equal to t_{A} errors occur), the $\mathbf{x}$ obtained is the same as the one obtained by maximizing $\mathrm{P}(\mathbf{x},\mathbf{y})$, i.e. the two decoding methods are equivalent.

■

The upper bound on p given by (9) is depicted in Fig. 2. d_{A}-decoding and ML decoding are equivalent for all values of p under the curve. For reasonable values of n, i.e. $n < 100$, the equivalence holds for values of $p < 0.1$ compatible with the error rate in EEPROM memories. Therefore, the d_{A}-decoding rule can be considered instead of the more complex ML decoding rule with no loss in performance.

III. Error correction capabilities

In this section we analyze distance properties and error correction capabilities of ternary codes over the ternary channel $\mathcal{H}$ under d_{A}-decoding rule defined in the previous section. To do so, instead of using directly the distance measure d_{A}, we require the definition of another distance measure:

Definition 3. *Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{F}_3^n$. The distance $d_{\mathrm{B}}(\mathbf{u},\mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$ is defined as:*

$$d_{\mathrm{B}}(\mathbf{u},\mathbf{v}) = \min_{\mathbf{w}\in\mathbb{F}_3^n} \left(d_{\mathrm{A}}(\mathbf{u},\mathbf{w}) + d_{\mathrm{A}}(\mathbf{w},\mathbf{v})\right). \tag{14}$$

It is easy to check that d_{B} is such that, for u_i and v_i in $\mathbb{F}_3$:

$$d_{\mathrm{B}}(u_i,v_i) = \begin{cases} 0 & \text{if } u_i = v_i, \\ 1 & \text{if } u_i \ne v_i \wedge u_i v_i = 0, \\ 2 & \text{if } u_i \ne v_i \wedge u_i v_i \ne 0. \end{cases} \tag{15}$$

Definition 4. *Let $\mathbf{x}$ and $\tilde{\mathbf{x}}$ be two codewords of $\mathcal{C}$. We define the minimum d_{B}-distance of a code $\mathcal{C}$, $d_{\mathrm{B,min}}$, as*

$$d_{\mathrm{B,min}} = \min_{\substack{\mathbf{x},\tilde{\mathbf{x}}\in\mathcal{C} \\ \mathbf{x}\ne\tilde{\mathbf{x}}}} d_{\mathrm{B}}(\mathbf{x},\tilde{\mathbf{x}}). \tag{16}$$

Then, assuming d_{A}-decoding, the error correction capability t_{A} of a code over the channel $\mathcal{H}$ is given by the following proposition:

Proposition 1. *The error correction capability t_{A} of a code $\mathcal{C}$ over the ternary channel $\mathcal{H}$ is:*

$$t_{\mathrm{A}} = \left\lfloor \frac{d_{\mathrm{B,min}} - 1}{2} \right\rfloor. \tag{17}$$

Proof: Let $\mathbf{x}$ and $\mathbf{y}$ be the transmitted and the received vectors, respectively. If a decoder implementing the d_{A}-decoding rule erroneously decodes $\mathbf{y}$ to $\tilde{\mathbf{x}} \ne \mathbf{x}$, then

$$d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) \ge d_{\mathrm{A}}(\tilde{\mathbf{x}},\mathbf{y}) \tag{18}$$

Using (18) and Definition 3,

$$\begin{aligned} 2d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) &\ge d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) + d_{\mathrm{A}}(\mathbf{y},\tilde{\mathbf{x}}) \ge d_{\mathrm{B}}(\mathbf{x},\tilde{\mathbf{x}}) \\ &\ge d_{\mathrm{B,min}} > 2\left\lfloor \frac{d_{\mathrm{B,min}} - 1}{2} \right\rfloor. \end{aligned} \tag{19}$$

Therefore, we successfully d_{A}-decode $\mathbf{y}$ if

$$d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) \le \left\lfloor \frac{d_{\mathrm{B,min}} - 1}{2} \right\rfloor, \tag{20}$$

where $d_{\mathrm{A}}(\mathbf{x},\mathbf{y})$ is the number of errors that occurred during the transmission of $\mathbf{x}$.
Conversely, by Definitions 3 and 4, there exist two codewords $\mathbf{x}$ and $\tilde{\mathbf{x}}$ and a vector $\mathbf{y} \in \mathbb{F}_3^n$ such that

$$\begin{aligned} d_{\mathrm{B}}(\mathbf{x},\tilde{\mathbf{x}}) &= d_{\mathrm{B,min}} \\ d_{\mathrm{B}}(\mathbf{x},\tilde{\mathbf{x}}) &= d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) + d_{\mathrm{A}}(\mathbf{y},\tilde{\mathbf{x}}). \end{aligned} \tag{21}$$

Therefore, if $d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) > t_{\mathrm{A}}$, then

$$d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) \ge \frac{d_{\mathrm{B,min}}}{2} = \frac{d_{\mathrm{A}}(\mathbf{x},\mathbf{y}) + d_{\mathrm{A}}(\mathbf{y},\tilde{\mathbf{x}})}{2}, \tag{22}$$

thus $d_{\mathrm{A}}(\mathbf{y},\tilde{\mathbf{x}}) \le d_{\mathrm{A}}(\mathbf{x},\mathbf{y})$, and the d_{A}-decoder may fail to decode $\mathbf{y}$ to $\mathbf{x}$. ■

IV. A sphere packing bound

In this section we give a simple upper bound on the size of codes over the ternary channel $\mathcal{H}$ assuming d_{A}-decoding. The bound we introduce is a sphere-packing bound. However, its formulation is harder than for the case of symmetric channels. Due to the fact that transitions between symbols 1 and 2 are not allowed, the ternary space we deal with is not isotropic and has the shape of an hypercube of dimension n, centered on

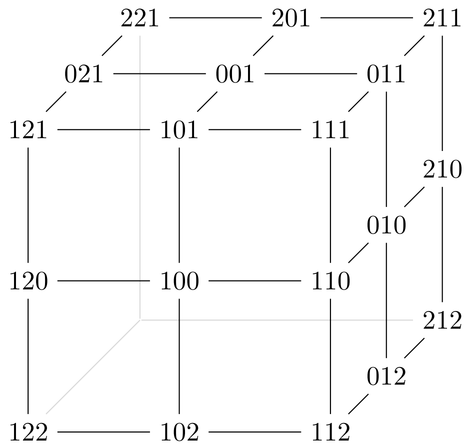

Fig. 3. $\mathbb{F}_3^3$ with distance d_{B}

the all-zero vector (see Fig. 3 for $n = 3$). Therefore, spheres have smaller volumes if they are closer to the vertices of the hypercube. The goal here is to find how many spheres of a given radius can be placed in the ternary space. We give a lower bound on the volume of the spheres:

Proposition 2. *We denote by $\mathcal{S}(\mathbf{u}, r)$ the sphere with center $\mathbf{u}$ and radius r in F_3^n:*

$$\mathcal{S}(\mathbf{u}, r) \;=\; \{\mathbf{v} \in \mathbb{F}_3^n \,|\, d_{\mathrm{B}}(\mathbf{u}, \mathbf{v}) \le r\}. \tag{23}$$

The volume $|\mathcal{S}(\mathbf{u}, r)|$ of $\mathcal{S}(\mathbf{u}, r)$ is such that:

$$|\mathcal{S}(\mathbf{u}, r)| \;\ge\; \sum_{d=0}^{r} \sum_{e_2=0}^{\lfloor d/2 \rfloor} \binom{n}{e_2} \binom{n-e_2}{d-2e_2} \tag{24}$$

Proof: We first prove that the smallest spheres are the ones centered in words of maximum weight (the vertices of the hypercube).
Let n and r be two integers. For $w \le n$, let $\mathbf{u}_w$ be a vector of $\mathbb{F}_3^n$ of weight w. The volume of $\mathcal{S}(\mathbf{u}_w, r)$ is independent from the choice of $\mathbf{u}_w$. We denote it by $\mathcal{V}(n, w, r)$. For $n > 0$, we denote by $\mathbf{u}'_w$ the vector of $\mathbb{F}_3^{n-1}$ obtained by removing the last symbol of $\mathbf{u}_w$:

$$\begin{aligned} \mathcal{V}(n, w, r) \;&=\; |\{\mathbf{v} \in \mathbb{F}_3^n \,|\, d_{\mathrm{B}}(\mathbf{u}_w, \mathbf{v}) \le r\}| \\ &=\; |\{\mathbf{w}0 \,|\, \mathbf{w} \in \mathbb{F}_3^{n-1} \wedge d_{\mathrm{B}}(\mathbf{u}'_w, \mathbf{w}) \le r\}| \\ &\quad + |\{\mathbf{w}1 \,|\, \mathbf{w} \in \mathbb{F}_3^{n-1} \wedge d_{\mathrm{B}}(\mathbf{u}'_w, \mathbf{w}) \le r-1\}| \\ &\quad + |\{\mathbf{w}2 \,|\, \mathbf{w} \in \mathbb{F}_3^{n-1} \wedge d_{\mathrm{B}}(\mathbf{u}'_w, \mathbf{w}) \le r-1\}| \\ &=\; \mathcal{V}(n-1, w, r) + 2\mathcal{V}(n-1, w, r-1) \end{aligned} \tag{25}$$

where for $\mathbf{w} \in \mathbb{F}_3^{n-1}$, $\mathbf{w}0$ denotes the vector of $\mathbb{F}_3^n$ obtained by appending a 0 at the end of $\mathbf{w}$.
Similarly, we show that for $w \le n-1$,

$$\begin{aligned} \mathcal{V}(n, w+1, r) \;&=\; \mathcal{V}(n-1, w, r) + \mathcal{V}(n-1, w, r-1) \\ &\quad + \mathcal{V}(n-1, w, r-2). \end{aligned} \tag{26}$$

Therefore, if $n > 0$ and $w \le n-1$,

$$\begin{aligned} &\mathcal{V}(n, w, r) - \mathcal{V}(n, w+1, r) = \\ &\qquad = \mathcal{V}(n-1, w, r-1) - \mathcal{V}(n-1, w, r-2) \ge 0. \end{aligned} \tag{27}$$

From (27) it follows that the spheres of minimal volume are the one centered on words of maximum weight.
Now, we give an expression for $\mathcal{V}(n, n, r)$. We consider the all-one vector $\mathbf{1}^n$. Let $\mathbf{v} \in \mathbb{F}_3^n$: $\mathbf{v}$ is in $\mathcal{S}(\mathbf{1}^n, r)$ if and only if $d_{\mathrm{B}}(\mathbf{1}^n, \mathbf{v}) \le r$. We denote by d this distance, and by e_2 the number of positions i where $v_i = 2$. The number of positions j where $v_j = 0$ is $d - 2e_2$. The number of vectors $\mathbf{v}$ that match a given d and e_2 is therefore:

$$\binom{n}{e_2} \binom{n-e_2}{d-2e_2}.$$

We conclude by summing over all possible d and e_2:

$$|\mathcal{S}(\mathbf{1}^n, r)| \;=\; \sum_{d=0}^{r} \sum_{e_2=0}^{\lfloor d/2 \rfloor} \binom{n}{e_2} \binom{n-e_2}{d-2e_2} \tag{28}$$

■

It is now possible to formulate the sphere-packing bound for our channel:

Theorem 2. *Let $\mathcal{C}$ be a code of length n and minimum d_{B}-distance $d_{\mathrm{B,min}}$ over the ternary channel $\mathcal{H}$. If $|\mathcal{C}|$ denotes the size of $\mathcal{C}$,*

$$|\mathcal{C}| \;\le\; \frac{3^n}{\displaystyle\sum_{d=0}^{t_{\mathrm{A}}} \sum_{e_2=0}^{\lfloor d/2 \rfloor} \binom{n}{e_2} \binom{n-e_2}{d-2e_2}}. \tag{29}$$

Proof: Let $\mathbf{x}$ and $\tilde{\mathbf{x}}$ be two codewords of $\mathcal{C}$. Since $d_{\mathrm{B}}(\mathbf{x}, \tilde{\mathbf{x}}) \ge 2t_{\mathrm{A}} + 1$, the spheres $\mathcal{S}(\mathbf{x}, t_{\mathrm{A}})$ and $\mathcal{S}(\tilde{\mathbf{x}}, t_{\mathrm{A}})$ are non-intersecting. This implies that

$$\begin{aligned} \left| \bigcup_{\mathbf{x} \in \mathcal{C}} \mathcal{S}(\mathbf{x}, t_{\mathrm{A}}) \right| \;&=\; \sum_{\mathbf{x} \in \mathcal{C}} |\mathcal{S}(\mathbf{x}, t_{\mathrm{A}})| \\ &\ge\; |\mathcal{C}| \sum_{d=0}^{t_{\mathrm{A}}} \sum_{e_2=0}^{\lfloor d/2 \rfloor} \binom{n}{e_2} \binom{n-e_2}{d-2e_2}. \end{aligned} \tag{30}$$

Furthermore,

$$\left| \bigcup_{\mathbf{x} \in \mathcal{C}} \mathcal{S}(\mathbf{x}, t_{\mathrm{A}}) \right| \;\le\; |\mathbb{F}_3^n| \;=\; 3^n. \tag{31}$$

Therefore, we conclude that

$$|\mathcal{C}| \;\le\; \frac{3^n}{\displaystyle\sum_{d=0}^{t_{\mathrm{A}}} \sum_{e_2=0}^{\lfloor d/2 \rfloor} \binom{n}{e_2} \binom{n-e_2}{d-2e_2}}.$$

■

Nit that this bound is less tighter as larger $d_{\mathrm{B,min}}$ are considered, since the tightness of the lower bound on the volume of the spheres given by Proposition 2 also decays when $d_{\mathrm{B,min}}$ increases. This explains the results of Table I presented at the end of the article.

V. Constructive lower bound

In this section, we give a constructive lower bound on the size of codes over channel $\mathcal{H}$ and show the existence of good codes. We define mappings that are applied to binary codes to generate a set of codewords of $\mathbb{F}_3^n$ that respects a given minimum d_{B}-distance $d_{\mathrm{B,min}}$.

A. Mappings and their topological properties

Let $\mathbf{u}$ be a vector of $\mathbb{F}_2^n$ and $w_{\mathbf{u}}$ its Hamming weight. We denote by $g_{\mathbf{u}}(j)$ ($1 \le j \le w_{\mathbf{u}}$) the j-th non-zero coordinate of $\mathbf{u}$. We define the mapping $\varphi_{\mathbf{u}}$ such that:

$$\begin{array}{rccc} \varphi_{\mathbf{u}}: & \mathbb{F}_3^{w_{\mathbf{u}}} & \longrightarrow & \mathbb{F}_3^n \\ & \hat{\mathbf{u}} & \longmapsto & \sum_{j=1}^{w_{\mathbf{u}}} \hat{u}_j e_{g_{\mathbf{u}}(j)} \end{array} \quad (32)$$

where $(e_i)_{1\le i\le n}$ is the canonical basis of $\mathbb{F}_3^n$.

We also call $\mathcal{E}_{\mathbf{u}}$ the subspace of $\mathbb{F}_3^n$ defined by $\mathcal{E}_{\mathbf{u}} = \varphi_{\mathbf{u}}(\mathbb{F}_3^{w_{\mathbf{u}}})$. For instance, for $\mathbf{u} = 10011000$, $\varphi_{\mathbf{u}}(201) = 20001000$ and the elements of $\mathcal{E}_{\mathbf{u}}$ are the vectors of the form $a00bc000$ for $a, b, c \in \mathbb{F}_3$.

We define another mapping ψ that transforms a binary word into a ternary word with no 0 coordinate by changing the symbols 0 into 1's and the symbols 1 into 2's:

$$\begin{array}{rccc} \psi: & \mathbb{F}_2^n & \longrightarrow & \mathbb{F}_3^n \\ & \mathbf{u} & \longmapsto & \sum_{i=1}^{n} (u_i+1) e_i . \end{array} \quad (33)$$

These mappings have several topological properties regarding d_{B}:

Proposition 3. *Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{F}_2^n$, and ψ the mapping defined in (33). It follows:*

$$d_{\mathrm{B}}(\psi(\mathbf{u}), \psi(\mathbf{v})) = 2 d_{\mathrm{B}}(\mathbf{u}, \mathbf{v}) \quad . \quad (34)$$

Proof: Since $d_{\mathrm{B}}(1,2) = 2 d_{\mathrm{B}}(0,1)$ and $d_{\mathrm{B}}(a,a) = 0$ for all $a \in \mathbb{F}_3$, we have

$$\begin{aligned} d_{\mathrm{B}}(\psi(\mathbf{u}), \psi(\mathbf{v})) &= \sum_{i=1}^{n} d_{\mathrm{B}}((\psi(\mathbf{u}))_i, (\psi(\mathbf{v}))_i) \\ &= \sum_{i=1}^{n} d_{\mathrm{B}}(u_i+1, v_i+1) \\ &= \sum_{i=1}^{n} 2 d_{\mathrm{B}}(u_i, v_i) = 2 d_{\mathrm{B}}(\mathbf{u}, \mathbf{v}) \ . \end{aligned} \quad (35)$$

■

Proposition 4. *Let $\mathbf{u} \in \mathbb{F}_2^n$. For $\hat{\mathbf{u}}, \hat{\mathbf{u}}' \in \mathbb{F}_3^{w_{\mathbf{u}}}$ it follows,*

$$d_{\mathrm{B}}(\varphi_{\mathbf{u}}(\hat{\mathbf{u}}), \varphi_{\mathbf{u}}(\hat{\mathbf{u}}')) = d_{\mathrm{B}}(\hat{\mathbf{u}}, \hat{\mathbf{u}}'). \quad (36)$$

Proof:

$$\begin{aligned} d_{\mathrm{B}}(\varphi_{\mathbf{u}}(\hat{\mathbf{u}}), \varphi_{\mathbf{u}}(\hat{\mathbf{u}}')) &= d_{\mathrm{B}}\left(\sum_{j=1}^{w_{\mathbf{u}}} \hat{u}_j e_{g_{\mathbf{u}}(j)}, \sum_{j=1}^{w_{\mathbf{u}}} \hat{u}'_j e_{g_{\mathbf{u}}(j)} \right) \\ &= \sum_{j=1}^{w_{\mathbf{u}}} d_{\mathrm{B}}(\hat{u}_j, \hat{u}'_j) = d_{\mathrm{B}}(\hat{\mathbf{u}}, \hat{\mathbf{u}}') \ . \end{aligned} \quad (37)$$

■

Proposition 5. *For $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^n$, and $\hat{\mathbf{u}} \in \mathbb{F}_3^{w_{\mathbf{u}}}$ and $\hat{\mathbf{v}} \in \mathbb{F}_3^{w_{\mathbf{v}}}$ both with no 0 coordinate, the following inequality holds:*

$$d_{\mathrm{B}}(\varphi_{\mathbf{u}}(\hat{\mathbf{u}}), \varphi_{\mathbf{v}}(\hat{\mathbf{v}})) \ge d_{\mathrm{B}}(\mathbf{u}, \mathbf{v}) \quad . \quad (38)$$

Proof: We denote the vector of $\mathbb{F}_3^m$ with all coordinates at 1 by $\mathbf{1}^m$. As $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ have no 0 coordinate:

$$d_{\mathrm{B}}(\varphi_{\mathbf{u}}(\hat{\mathbf{u}}), \varphi_{\mathbf{v}}(\hat{\mathbf{v}})) \ge d_{\mathrm{B}}(\varphi_{\mathbf{u}}(\mathbf{1}^{w_{\mathbf{u}}}), \varphi_{\mathbf{v}}(\mathbf{1}^{w_{\mathbf{v}}})) = d_{\mathrm{B}}(\mathbf{u}, \mathbf{v}) \ . \quad (39)$$

■

B. Construction and lower bound

The aim of this section is to build, with n and minimum d_{B}-distance $d_{\mathrm{B,min}}$ given, say $d_{\mathrm{B,min}} = d$, an $[n, M, d]$ code for the ternary channel $\mathcal{H}$ with reasonable M, where $M = |\mathcal{C}|$ is the cardinality of the code, starting from elementary binary codes.

Let $\bar{\mathcal{C}}$ be an (n, k, d) binary code with minimum (Hamming) distance $d_{\mathrm{H,min}}$ equal to d and denote by A_h its weight enumerator, the number of codewords of weight h, $0 \le h \le n$. For all h such that $A_h \ne 0$, let $\bar{\mathcal{C}}_h$ be a $(h, k_h, \lceil d/2 \rceil)$ binary code. We consider the following ternary code:

$$\mathcal{C} = \bigcup_{\bar{\mathbf{x}} \in \bar{\mathcal{C}}} \varphi_{\bar{\mathbf{x}}} \left(\psi(\bar{\mathcal{C}}_{w_{\bar{\mathbf{x}}}}) \right) \quad (40)$$

Proposition 6. *The cardinality of code $\mathcal{C}$ satisfies:*

$$|\mathcal{C}| = \sum_{h=0}^{n} A_h |\bar{\mathcal{C}}_h| \quad (41)$$

Proof: Since for all $\bar{\mathbf{x}}$, $\varphi_{\bar{\mathbf{x}}}$ and ψ are trivially injective, it is enough to prove that the union $\bigcup_{\bar{\mathbf{x}} \in \bar{\mathcal{C}}} \varphi_{\bar{\mathbf{x}}} \left(\psi(\bar{\mathcal{C}}_{w_{\bar{\mathbf{x}}}}) \right)$ is disjoint.

For $\bar{\mathbf{x}}, \bar{\mathbf{z}} \in \bar{\mathcal{C}}$ such that $\bar{\mathbf{x}} \ne \bar{\mathbf{z}}$, let $\mathbf{x} \in \varphi_{\bar{\mathbf{x}}}(\psi(\bar{\mathcal{C}}_{w_{\bar{\mathbf{x}}}}))$ and $\mathbf{z} \in \varphi_{\bar{\mathbf{z}}}(\psi(\bar{\mathcal{C}}_{w_{\bar{\mathbf{z}}}}))$. By Proposition 5, $d_{\mathrm{B}}(\mathbf{x}, \mathbf{z}) \ge d_{\mathrm{H}}(\bar{\mathbf{x}}, \bar{\mathbf{z}}) > 0$, thus $\mathbf{x} \ne \mathbf{z}$. ■

Corollary 1.

$$|\mathcal{C}| = \sum_{h=0}^{n} A_h 2^{k_h} \quad (42)$$

Proposition 7. *Let $\mathbf{x}$ and $\mathbf{z}$ be two codewords of $\mathcal{C}$. Then $d_{\mathrm{B}}(\mathbf{x}, \mathbf{z}) \ge d$.*

Proof: Let $\mathbf{x}$ and $\mathbf{z}$ be two distinct codewords of $\mathcal{C}$. We denote by $\bar{\mathbf{x}}$ the codeword of $\bar{\mathcal{C}}$ and $\bar{\mathbf{x}}'$ the codeword of $\bar{\mathcal{C}}_{w_{\bar{\mathbf{x}}}}$ such that $\mathbf{x} = \varphi_{\bar{\mathbf{x}}}(\psi(\bar{\mathbf{x}}'))$ (the unicity is proved in the proof of Proposition 6). Likewise, we define $\bar{\mathbf{z}}$ and $\bar{\mathbf{z}}'$ with respect to $\mathbf{z}$. We consider two cases:

TABLE I

CONSTRUCTIVE LOWER BOUND AND UPPER BOUND (IN BRACKETS) ON THE SIZE ($\log_2 M$) OF CODES, AS A FUNCTION OF n AND $d_{\mathrm{B,min}}$

$d_{\mathrm{B,min}} \backslash n$	8	16	32	64	128
2	11 (12)	24 (25)	49 (50)	100 (101)	201 (202)
4	7 (9)	19 (21)	43 (45)	93 (95)	193 (195)
6		12 (18)	34 (41)	82 (90)	180 (189)
8	4 (5)	11 (15)	28 (38)	75 (85)	172 (184)
10				65 (81)	159 (179)
12			21 (32)	61 (78)	151 (174)
14				52 (74)	140 (170)
16		5 (7)	17 (27)	46 (71)	132 (165)
20					118 (157)
22				36 (62)	106 (154)
24				36 (59)	98 (150)
28				30 (54)	86 (143)
32			6 (12)	28 (49)	78 (137)
44					64 (119)
48					60 (114)
56					50 (104)
64				7 (22)	43 (94)
128					8 (41)

- *Case* $\bar{\mathbf{x}} = \bar{\mathbf{z}}$: In this case $\bar{\mathbf{x}}'$ and $\bar{\mathbf{z}}'$ are two different codewords of $\bar{\mathcal{C}}_{w_{\bar{\mathbf{x}}}}$ (otherwise $\mathbf{x} = \mathbf{z}$). Thus, by choice of the code $\bar{\mathcal{C}}_{w_{\bar{\mathbf{x}}}}$ it follows that $d_{\mathrm{B}}(\bar{\mathbf{x}}', \bar{\mathbf{z}}') \geq \lceil d/2 \rceil$. By Propositions 3 and 4,

$$\begin{aligned} d_{\mathrm{B}}(\mathbf{x}, \mathbf{z}) &= d_{\mathrm{B}}(\varphi_{\bar{\mathbf{x}}}(\psi(\bar{\mathbf{x}}')), \varphi_{\bar{\mathbf{z}}}(\psi(\bar{\mathbf{z}}'))) = d_{\mathrm{B}}(\psi(\bar{\mathbf{x}}'), \psi(\bar{\mathbf{z}}')) \\ &= 2 d_{\mathrm{B}}(\bar{\mathbf{x}}', \bar{\mathbf{z}}') \geq 2\lceil d/2 \rceil \geq d \ . \end{aligned} \tag{43}$$

- *Case* $\bar{\mathbf{x}} \neq \bar{\mathbf{z}}$: By choice of $\bar{\mathcal{C}}$ it follows that $d_{\mathrm{B}}(\bar{\mathbf{x}}, \bar{\mathbf{z}}) \geq d$. Now, by Proposition 5,

$$d_{\mathrm{B}}(\mathbf{x}, \mathbf{z}) = d_{\mathrm{B}}(\varphi_{\bar{\mathbf{x}}}(\psi(\bar{\mathbf{x}}')), \varphi_{\bar{\mathbf{z}}}(\psi(\bar{\mathbf{z}}'))) \geq d_{\mathrm{B}}(\bar{\mathbf{x}}, \bar{\mathbf{z}}) = d. \tag{44}$$

In both cases, $d_{\mathrm{B}}(\mathbf{x}, \mathbf{z}) \geq d$, which concludes the proof. ■

C. Results

The constructive method proposed above gives a lower bound on the cardinality of ternary codes over $\mathcal{H}$. We used this method to construct codes based on extended BCH codes as $\bar{\mathcal{C}}$ and codes obtained from the tables in [2], [3] as $\bar{\mathcal{C}}_w$. Note that the full knowledge of the binary codes used in the construction is not required to compute the lower bound: given n and d, we only require to know the weight enumerator A_h of $\bar{\mathcal{C}}$, which can be found in [4]. On the other hand, for codes $\bar{\mathcal{C}}_w$, only the knowledge of the size k_h is required. The results are shown in Table I. For given n and $d_{\mathrm{B,min}}$, we report in the table the value $\log_2 M$. The upper bound on the size of codes over $\mathcal{H}$ of length n and minimum d_{B}-distance $d_{\mathrm{B,min}}$ is also given in the table in brackets (also in the form $\log_2 M$).

VI. CONCLUSION

In this paper, coding for a particular non-symmetric ternary channel where some transitions are not allowed, was addressed. We derived the maximum likelihood decoding rule for this channel and showed that it is too complex, since it depends on the error transition probability p. We then proposed an alternative decoding rule, called d_{A}-decoding, based on a more suitable distance measure. We showed that d_{A}-decoding and ML decoding are equivalent for values of p of interest. Further, we analyzed error correction capabilities of ternary codes over this particular channel under d_{A}-decoding. We gave an upper bound and a constructive lower bound on the code size, showing the existence of good codes. Following the proposed constructive method, we found good codes for several values of n and $d_{\mathrm{B,min}}$.

In the paper we did not make any considerations regarding the complexity of encoding and decoding. Further research includes finding binary codes better adapted to the construction method, and a simple method to map a binary input to the ternary codewords to be effectively used in memory devices.

REFERENCES

[1] T. J. Ting, T. Chang, T. Lin, C. S. Jenq, and K. K. C. Naiff, "A 50-ns CMOS 256k EEPROM," *IEEE J. Solid-St. Circ.*, vol. 23, pp. 1164–1170, Oct. 1988.

[2] M. Grassl, "Searching for linear codes with large minimum distance," in *Discovering Mathematics with Magma — Reducing the Abstract to the Concrete* (W. Bosma and J. Cannon, eds.), vol. 19 of *Algorithms and Computation in Mathematics*, pp. 287–313, Heidelberg: Springer, 2006.

[3] M. Grassl, "Bounds on the minimum distance of linear codes and quantum codes." Online available at http://www.codetables.de, 2007.

[4] R. H. Morelos-Zaragoza, *The Art of Error Correcting Coding*. John Wiley & Sons, 2006.