Maximizing the sum rate in symmetric networks of interfering links

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***Abstract*—We consider the power optimization problem of maximizing the sum rate of a symmetric network of interfering links in Gaussian noise. All transmitters have an average transmit power constraint, the same for all transmitters. We solve this nonconvex problem by indentifying some underlying convex structure. In particular, we characterize the maximum sum rate of the network, and show that there are essentially two possible states at the optimal solution depending on the cross-gain between the links, and/or the average power constraint: the first is a wideband (WB) state , in which all links interfere with each other, and the second is a frequency division multiplexing (FDM) state, in which all links operate in orthogonal frequency bands. The FDM state is optimal if the cross-gain between the links is above $1/\sqrt{2}$. If the cross-gain is below $1/\sqrt{2}$, then FDM is still optimal provided the average power of the links is sufficiently high. Assuming that $\epsilon < 1/\sqrt{2}$, we can say that the WB state occurs when the average power level is low (relative to the noise and the cross-gain factor between the links), but as we increase the average power level from low to high, there is a smooth transition from the WB state to the FDM state: For intermediate average power levels, the optimal configuration is a mixture, with some fraction of the bandwidth in the WB state, and the other fraction in the FDM state. This work has applications to DSL, as well as to wireless networks.**

I. Introduction

Wireless networks are plagued by two key problems not encountered in wireline networks: multipath fading, and interference between links. In the present paper, we focus primarily on the management of the second problem using optimized power allocation. The problem of interference also arises in DSL wireline access networks, and our results are applicable to this setting as well. We pose a power allocation problem in which the objective is to maximize the total rate achieved in the network. Each link has to choose a transmit power spectrum, but the choice impacts not only the rate achieveable on the desired link, but also the rates achieveable on the remaining links of the network.

Unlike traditional power control formulations, in which rate targets are constraints of the problem [1], the rate maximization formulation that we consider in this paper provides a more challenging nonlinear, nonconvex optimization problem. Recently, progress has been made, but under the assumption that the power allocation is time and frequency flat, with maximum power constraints on the links [2]–[4]. In particular, under these assumptions, a complete solution is provided for a network of N symmetric, interfering links in [4]. In the present paper, we provide a complete solution to the N link symmetric network problem when we remove the constraint that the power allocation be time and frequency flat, and we replace the peak per-link power constraints with average per-link power constraints.

Although the rate maximization problem is not itself convex, we exhibit an underlying convex structure to the problem, and this structure helps us identify the optimal solution. We show that the optimal power spectrum always consists of a relatively small number of modes, where a mode is a chunk of spectrum in which the power spectral density of all links is constant. Thus, the optimal total power spectrum is piecewise constant [5]. In this paper, we characterize the optimal solution precisely for the case of symmetric interfering links: We provide the bandwidths of the modes, and the power allocation for each link in each mode.

The general characteristic of the optimal solution is that it involves at most two states: a frequency division multiplexing (FDM) state, or a wideband (WB) state in which per-link power allocations are flat across the frequency band. In some scenarios, depending on the cross-gain factor and the signal to noise ratio, the optimal configuration is a mixture of these two states.

II. Problem formulation

We start with a base-band model of N communication links, each of bandwidth $W/2$ Hz, and each link is individually an additive white Gaussian noise channel (AWGN) with common noise power spectral density of σ^2 at the receivers. By re-scaling powers, we can assume without loss of generality that $\sigma^2 = 1$. The N links interfere with each other, as depicted in Figure 1, and we assume an additive model for the interference between links.

We assume each transmitter uses Gaussian signalling, and denote the stationary Gaussian process transmitted on link i by $X_i(t)$. The received signal on link i is $Y_i(t)$, where

$$Y_i(t) = X_i(t) + \sum_{j \neq i} \sqrt{\epsilon} X_j(t) + Z_i(t), \qquad (1)$$

$Z_i(t)$ is white Gaussian noise of unit power spectral density, and $\sqrt{\epsilon}$ is the cross-gain between the links of the network.

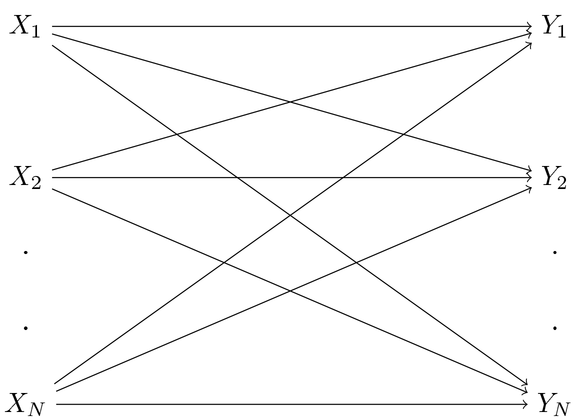

Fig. 1. A network of interfering links

If process $X_n(t)$ has power spectral density $\mathcal{P}_n(f)$ then an achievable rate on link i is given by [6]

$$R_i = \int_{-W/2}^{W/2} \log\left(1 + \frac{\mathcal{P}_i(f)}{1 + \epsilon \sum_{j \neq i} \mathcal{P}_j(f)}\right) df.$$

We impose the power constraint that for all i,

$$\int_{-W/2}^{W/2} \mathcal{P}_i(f) \leq P_{ave}.$$

The problem we address is that of computing the sum capacity of this network, under the above assumptions, which reduces to finding the optimal input spectra for the links.

Problem 1: Find the input spectra that achieve the maximum of the following:

$$\max \sum_{i=1}^{N} \int_{-W/2}^{W/2} \log\left(1 + \frac{\mathcal{P}_i(f)}{1 + \epsilon \sum_{j \neq i} \mathcal{P}_j(f)}\right) df \tag{2}$$

$$\text{s.t.} \quad \int_{-W/2}^{W/2} \mathcal{P}_i(f) df \leq P_{ave} \tag{3}$$

It can be shown that the optimum is achievable with spectra that are piece-wise constant: There are at most $N+2$ disjoint intervals in $[-W/2, W/2]$ with each link having constant power spectral density within each interval [5]. This can be proven via Caratheodory's convexity theorem [7], and is a consequence of the dimensionality of the problem. Here, the dimension of the problem is $N+1$, since there are N links, each of which has to choose a power level, and the sum-rate provides an additional dimension. Thus, the following problem is equivalent to Problem 1.

Problem 2: Let $M \geq N+2$. Find the normalized bandwidths $(\alpha_0, \alpha_1, \ldots, \alpha_{M-1})$ and power levels $(P_i^{(m)})$ $i = 1, 2, \ldots, N, m = 0, 1, \ldots, M-1$ to solve:

$$\max \sum_{i=1}^{N} \sum_{m=0}^{M-1} \alpha_m W \log\left(1 + \frac{P_i^{(m)}}{1 + \epsilon \sum_{j \neq i} P_j^{(m)}}\right) \tag{4}$$

$$\text{s.t.} \sum_{m=0}^{M-1} \alpha_m P_i^{(m)} \leq P_{ave}, \quad P_i^{(m)} \geq 0, \tag{5}$$

$$\sum_{m=0}^{M-1} \alpha_m \leq 1, \quad 0 \leq \alpha_m \leq 1, \tag{6}$$

From now on, we will assume $W = 1$ to reduce the notational burden, and note that there is no loss of generality in doing so.

Note that Caratheodory's theorem only states that there are *at most* $N+2$ distinct sub-bands on which the power levels are distinct. We will show that for some values of ϵ and P_{ave}, all the P_i's above are the same, which implies that there is really only one mode of behaviour in these cases: The power spectra are flat across the whole band. In fact, we will show that the number of distinct modes required is at most $N+1$ in all cases, for the symmetric network model.

To describe the result of this paper, we first define the following functions:

$$C(P) = \log(1+P) \tag{7}$$

$$f_1(P) = NC\left(\frac{P}{1+\epsilon(N-1)P}\right) \tag{8}$$

$$f_2(P) = C(NP) \tag{9}$$

In (7) we use the natural logarithm, so $C(P)$ is the capacity of a discrete time Gaussian noise channel, with SNR = P, measured in nats per channel use, $f_2(P)$ is the capacity at SNR = NP, and $f_1(P)$ is the capacity of a discrete time Gaussian link that receives interference from $N-1$ other links, as in our symmetric Gaussian network model.

The main result of this paper is the following theorem, which we will prove in Section IV. In the theorem, the values P_l and P_u are uniquely defined in the case $\epsilon < 1/2$, but the definition of these two values is relegated to Section III, where we will show that $0 < P_l < P_u$. The term $C(\epsilon, N, P_{ave})$ in the theorem is defined by

$$C(\epsilon, N, P_{ave}) = \begin{cases} f_2(P_{ave}) & \epsilon > 1/2 \text{ or } P_{ave} > P_u \\ f_1(P_{ave}) & \epsilon < 1/2 \text{ and } P_{ave} < P_l \\ \beta f_1(P_l) + (1-\beta) f_2(P_u) & o.w. \end{cases} \tag{10}$$

where, in the last case, *i.e.* $\epsilon < 1/2$ and $P_l < P_{ave} < P_u$, we define β to be the unique number in $(0,1)$ such that $P_{ave} = \beta P_l + (1-\beta) P_u$.

Theorem 1: $C(\epsilon, N, P_{ave})$ is the optimal value in Problem 2. If $\epsilon > 1/2$ or $P_{ave} > P_u$, the optimal value is achievable with N modes, $\alpha_m = 1/N$ for $m = 0, 1, \ldots, N-1$, and $P_i^{(m)} = NP_{ave} 1_{\{i=m-1\}}$, $i = 1, 2, \ldots N$. If $\epsilon < 1/2$ and $P_{ave} < P_l$ then the optimal value is achieveable with 1 mode, $\alpha_0 = 1$, and $P_i^{(0)} = P_{ave}$, $i = 1, 2, \ldots, N$. Otherwise, $\epsilon <$

$1/2$ and $P_{ave} = \beta P_l + (1-\beta)P_u$ for some unique $\beta \in (0,1)$. In this case, $C(\epsilon, N, P_{ave})$ is achieveable with $N+1$ modes: $\alpha_0 = \beta$, $\alpha_m = (1-\beta)/N$, $m = 1,2,\ldots,N$, $P_i^{(0)} = P_l$, $i = 1,2,\ldots,N$, $P_i^{(m)} = NP_u 1_{\{m=i\}}$, $i = 1,2,\ldots,N$, $m = 1,2,\ldots,N$.

This theorem states that if $\epsilon > 1/2$ or $P_{ave} > P_u$ then the optimal configuration is that of frequency division multiplexing (FDM). If $\epsilon < 1/2$ and $P_{ave} < P_l$, then the optimal configuration is a single mode, with all links sharing the entire band $[-W/2, W/2]$. The optimal spectra are flat, and we describe this solution as being "wide-band" (WB) in nature. If $\epsilon < 1/2$ and $P_l < P_{ave} < P_u$, then the optimal configuration is a mixture of FDM and WB, requiring $N+1$ distinct modes (or sub-bands). Thus, there are essentially two distinct states for the system, FDM or WB. Which one is optimal depends on ϵ and P_{ave}, and, in an intermediate scenario, the optimal configuration is a mixture of the two.

Before we prove Theorem 1, we provide some preliminary results, including the definition of P_l and P_u, as required to complete the definition (10) as well as the statement of Theorem 1.

III. Preliminary results

Our first preliminary results concern the functions $f_1(P)$ and $f_2(P)$ defined in (8) and (9), respectively, and the corresponding curves defined by the graphs, which we denote by C_1 and C_2.

Lemma 1: If $\epsilon > 1/2$ then $f_2(P) > f_1(P)$ for all $P > 0$, *i.e.* the curve C_2 lies entirely above curve C_1. However, if $\epsilon < 1/2$, then there exists a unique $\tilde{P} > 0$ such that

$$f_1(\tilde{P}) = f_2(\tilde{P}) \quad (11)$$
$$f_2(P) < f_1(P) \quad \forall P < \tilde{P} \quad (12)$$
$$f_1(P) < f_2(P) \quad \forall P > \tilde{P} \quad (13)$$

i.e. C_1 is above C_2 for $P < \tilde{P}$, below C_2 for $P > \tilde{P}$, and $\tilde{P}$ is the point where they cross.

Proof: See Appendix A.

The case $\epsilon < 1/2$ is depicted in Figure 2.

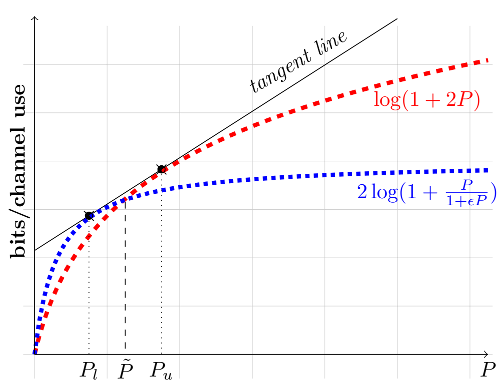

Fig. 2. The curves C_1 and C_2 in the case $\epsilon < 1/2$.

Lemma 2: If $\epsilon < 1/2$ then there is a unique tangent curve that touches both C_1 and C_2 at two points, namely $(P_l, f_1(P_l))$ and $(P_u, f_2(P_u))$, with $P_l < \tilde{P} < P_u$ (see Figure 2). Since both f_1 and f_2 are strictly concave, it follows that for all $P > 0$,

$$\max\{f_1(P), f_2(P)\} \le f_1(P_l) + f_1'(P_l)(P - P_l) \quad (14)$$

with strict inequality for $P \neq P_l, P_u$, and, since the tangent touches C_2 at $(P_u, f_2(P_u))$, we have that

$$f_2(P_u) = f_1(P_l) + f_1'(P_l)(P_u - P_l). \quad (15)$$

Thus, both curves lie below this unique tangent line. For $P^* < P_l$, there is a unique supporting tangent to the curve C_1 at the point $(P^*, f_1(P^*))$, and both C_1 and C_2 lie below this line *i.e.* for all $P > 0$,

$$\max\{f_1(P), f_2(P)\} \le f_1(P^*) + f_1'(P^*)(P - P^*) \quad (16)$$

Similarly, for $P^* > P_u$, there is a unique supporting tangent to the curve C_2 at the point $(P^*, f_2(P^*))$, both C_1 and C_2 lie below this line, *i.e.* for all $P > 0$,

$$\max\{f_1(P), f_2(P)\} \le f_2(P^*) + f_2'(P^*)(P - P^*) \quad (17)$$

Proof: See Appendix B.

The following results concern the function to be maximized in Problem 2, which we observe is neither concave, nor convex, and it possesses local maxima, making standard numerical approaches problematic. Nevertheless, we will provide an algorithmic approach that is always guaranteed to improve the objective function. We will do this by weakening the constraints, and in so doing, we will *not* be *guaranteed* to obtain a feasible allocation from this approach. However, we will be able to obtain useful upper bounds, which will end up being achieveable after all.

The key idea is to focus on any one of the modes, choose two links, and trade power from one link to the other without changing the sum of the powers, in such a way that the objective function always increases. By repeating this technique in an iterative manner, we will obtain a final upper bound, and then show that this upper bound is in fact $C(\epsilon, N, P_{ave})$ and hence achieveable.

Consider the following function, that provides the sum rate in the two link case: for $a > 0, \epsilon > 0$,

$$g(\epsilon, a, P_1, P_2) = C\left(\frac{P_1}{a + \epsilon P_2}\right) + C\left(\frac{P_2}{a + \epsilon P_1}\right). \quad (18)$$

where a represents the background noise level.

The following lemma considers the function restricted to the segment

$$\mathcal{P} = \{(P_1, P_2) : P_1 + P_2 = \hat{P}, P_1 \ge 0, P_2 \ge 0\}, \quad (19)$$

for some fixed total sum power on the two links, $\hat{P}$.

Lemma 3: For fixed ϵ, a, the function $g(\epsilon, a, \cdot, \cdot)$ is

- Schur-concave [8] on $\mathcal{P}$ if $\epsilon \le \epsilon^*(a, \hat{P})$.
- Schur-convex [8] on $\mathcal{P}$ if $\epsilon \ge \epsilon^*(a, \hat{P})$.

where $\epsilon^*(a, \hat{P}) = \sqrt{a}\frac{\sqrt{a+\hat{P}} - \sqrt{a}}{\hat{P}}$.

Proof: See Appendix C. See also [4]

Corollary 1: For fixed ϵ, a, the maximization of the function $g(\epsilon, a, \cdot, \cdot)$ over $\mathcal{P}$ occurs at

- $(\hat{P}/2, \hat{P}/2)$ if $\epsilon \leq \epsilon^*(a, \hat{P})$.
- $(0, \hat{P})$ or $(\hat{P}, 0)$ if $\epsilon \geq \epsilon^*(a, \hat{P})$

Lemma 3 states that if we are trying to maximize the sum rate of two links, with the sum of the powers of the two links held fixed, then the solution is either to allocate all of the power to one of the links (if the function is Schur-convex) or distribute it equally between the two links (if it is Schur-concave). This underlying structure can be exploited in an iterative manner, for an arbitrary number of links, N, to solve the following optimization problem:

Problem 3:

$$\max_{\boldsymbol{P}} \sum_{i=1}^{N} C\left(\frac{P_i}{1+\epsilon\sum_{j\neq i} P_j}\right) \tag{20}$$

$$\text{s.}t. \quad P_i \geq 0 \ \forall i, \qquad \sum_{i=1}^{N} P_i = \hat{P}$$

where $\boldsymbol{P} = (P_1, P_2, \ldots P_N)$ is a vector of power levels.

Lemma 4: The solution, $U(\epsilon, N\hat{P})$, to Problem 3 is given by:

$$\begin{aligned} U(\epsilon, N, \hat{P}) &= \max\{NC\left(\frac{\hat{P}/N}{1+\epsilon(N-1)\hat{P}/N}\right), C(\hat{P})\} \\ &= \max\{f_1(\hat{P}/N), f_2(\hat{P}/N)\} \end{aligned} \tag{21}$$

Proof: See Appendix D.

IV. Proof of Theorem 1

We are now in a position to prove Theorem 1. First, the achievability of $C(\epsilon, N, P_{ave})$ can be immediately checked; the issue we address here is the converse, namely that there is no other strategy that can beat $C(\epsilon, N, P_{ave})$. This is established in the following lemma.

Lemma 5: Let $\boldsymbol{\alpha}, \boldsymbol{P}$ be a feasible allocation of normalized bandwidths, and power levels, respectively, for Problem 2, and let $C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P})$ denote the corresponding sum-rate. Then

$$C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P}) \leq C(\epsilon, N, P_{ave}) \tag{22}$$

Proof: Let $\boldsymbol{\alpha}, \boldsymbol{P}$ be a feasible allocation of normalized bandwidths, and power levels, respectively, and let $C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P})$ denote the corresponding sum-rate:

$$C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P}) = \sum_{m=0}^{M-1} \sum_{i=1}^{N} \alpha_m C\left(\frac{P_i^{(m)}}{1+\epsilon(N-1)\sum_{j\neq i} P_j^{(m)}}\right).$$

Note that $\boldsymbol{\alpha}$ is of dimension M, and $\boldsymbol{P} = \left(P_i^{(m)}\right)$ is of dimension NM. Let $\hat{P}^{(m)} = \sum_{i=1}^{N} P_i^{(m)}$. Then the upper bound

$$C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P}) \leq \sum_{m=0}^{M-1} \alpha_m U(\epsilon, N, \hat{P}^{(m)}) \tag{23}$$

clearly holds, where $U(\epsilon, N, \hat{P}^{(m)})$ is defined in (20). By Lemma 4, we can write the RHS of (23) as

$$\sum_{m=0}^{M-1} \alpha_m \max\{f_1\left(\frac{\hat{P}^{(m)}}{N}\right), f_2\left(\frac{\hat{P}^{(m)}}{N}\right)\} \tag{24}$$

Now re-order the modes so that for modes $m = 0, 1, \ldots, k$, the maximum in (24) is $f_1(\hat{P}^{(m)}/N)$ (if there are no such modes, let $k = -1$) and for modes $k+1, k+2, \ldots, M-1$, the maximum in (24) is $f_2(\hat{P}^{(m)}/N)$ (if there are no such modes, k will equal $M-1$). But the functions f_1 and f_2 are both concave functions, so if we define:

$$\alpha = \sum_{m=0}^{k} \alpha_m \tag{25}$$

$$P^{(a)} = \sum_{m=0}^{k} (\alpha_m/\alpha)(\hat{P}^{(m)}/N) \tag{26}$$

$$P^{(b)} = \sum_{m=k+1}^{M-1} (\alpha_m/(1-\alpha))(\hat{P}^{(m)}/N) \tag{27}$$

then the following upper bound must also be true:

$$C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P}) \leq \alpha f_1(P^{(a)}) + (1-\alpha) f_2(P^{(b)}) \tag{28}$$

Since the initial mode and power allocations $(\boldsymbol{P}, \boldsymbol{\alpha})$ are feasible for Problem 2, it must also be true that

$$\alpha P^{(a)} + (1-\alpha) P^{(b)} \leq P_{ave}$$

We conclude that the solution to the following optimization problem must also upper bound $C(\epsilon, N, \boldsymbol{\alpha}, \boldsymbol{P})$:

Problem 4:

$$\max_{\alpha, P^{(a)}, P^{(b)}} \alpha f_1(P^{(a)}) + (1-\alpha) f_2(P^{(b)}) \tag{29}$$

$$\text{s.t.} \quad 0 \leq \alpha \leq 1, \quad P^{(a)} \geq 0, \quad P^{(b)} \geq 0$$

$$\alpha P^{(a)} + (1-\alpha) P^{(b)} \leq P_{ave} \tag{30}$$

The proof is complete once we establish that the maximum value achieved in Problem 4 is $C(\epsilon, N, P_{ave})$. This is established in Lemma 6 below.

■

Lemma 6: The maximum value achieved in Problem 4 is $C(\epsilon, N, P_{ave})$.

Proof: Consider the case that $P_{ave} < P_l$, and let $\alpha, P^{(a)}, P^{(b)}$ be feasible for Problem 4. Then

$$\begin{aligned} & \alpha f_1(P^{(a)}) + (1-\alpha) f_2(P^{(b)}) \\ \leq\ & \alpha f_1(P^{(a)}) + \alpha f_1'(P_{ave})(P^{(a)} - P_{ave}) \\ +\ & (1-\alpha) f_1(P^{(a)}) + (1-\alpha) f_1'(P_{ave})(P^{(b)} - P_{ave}) \\ \leq\ & f_1(P_{ave}) \\ =\ & C(\epsilon, N, P_{ave}) \end{aligned}$$

where the first inequality follows from (16), and the second inequality from (30). But clearly $f_1(P_{ave})$ is achieveable if we set $\alpha = 1$ and $P^{(a)} = P_{ave}$. The case $P_{ave} > P_u$ follows in the analogous way, using (17) in place of (16).

The remaining case is $P_{ave} = \beta P_l + (1-\beta)P_u$, for $0 < \beta < 1$, with $\alpha, P^{(a)}, P^{(b)}$ feasible for Problem 4. Then

$$\begin{aligned}
& \alpha f_1(P^{(a)}) + (1-\alpha) f_2(P^{(b)}) \\
\leq\ & \alpha f_1(P_l) + \alpha f_1'(P_l)(P^{(a)} - P_l) \\
+\ & (1-\alpha) f_1(P_l) + (1-\alpha) f_1'(P_l)(P^{(b)} - P_l) \\
\leq\ & f_1(P_l) + f_1'(P_l)(P_{ave} - P_l) \\
\leq\ & \beta f_1(P_l) + (1-\beta) f_1(P_l) + (1-\beta) f_1'(P_l)(P_u - P_l) \\
=\ & \beta f_1(P_l) + (1-\beta) f_2(P_u) \\
=\ & C(\epsilon, N, P_{ave})
\end{aligned}$$

where the first inequality follows from (14), the second inequality from (30), and the first equality from (15). ■

V. Conclusion

In this paper, we have solved the sum-rate maximization problem for a symmetric network of an arbitrary number of links. Each link has an average power constraint, which is the same for all links. We have shown that the critical value of the cross-gain parameter, ϵ, is $\epsilon = 1/2$, above which the optimal spectra consist of N modes, with only one link active in each mode, providing a frequency division multiplexing (FDM) characteristic to the solution.

When $\epsilon < 1/2$, the FDM configuration is still optimal, provided the average power is high enough. However, if the average power is sufficiently low, the optimal power spectra will consist of one mode, giving a wideband (WB) characteristic to the solution. For intermediate values of the average power, the optimal spectra is a mixture of these two states; in these cases there are $N+1$ modes, with N modes of FDM, and one mode in which the links interfere with each other.

Although the symmetric network is a very special case of the general problem of interfering links, our solution provides a very clean characterization of the optimal behaviour in this particular case, and it may provide insight into more general network problems. The paper [5] shows that the piecewise constant form of the optimal input spectra holds for general networks. However, the problem of finding the optimal modes and power levels to use in each mode is left completely open. In general, it appears to be a very difficult problem.

The fact that we need to find the common tangent line to the two curves C_1 and C_2 is a manifestation of the convexity that is a characteristic of all capacity regions. It is well known that capacity regions are always convex: Usually, time-sharing arguments are invoked, but in the present paper, the convexification is obtained in the frequency domain: Our solutions are time-invariant.

An early paper that considered the impact of interference on the capacity of a cellular network is [9]. This paper showed that in some scenarios, pure TDM partitioning of cells into disjoint time-slots, or pure WB strategies, can be beaten by a mixed strategy that they called "fractional inter-cell time sharing". This is consistent with our findings in the present paper, in which the mixed state is shown to be optimal in some cases.

Prior work on rate maximization has been done for CDMA, or UWB networks, in which the rate function is modeled as a linear function of the SIR [10], [11], [12]. More recent work has modeled the rate function as we do in the present paper (logarithmic in the SIR) but under the assumption that the power allocation is time and frequency flat: [2], [13], [4]. The latter approaches can be combined with higher-layer scheduling algorithms [10], [14] providing time-varying approaches to resource allocation.

Finally, we note that the assumption that each link treats the other links as sources of Gaussian noise can be relaxed. A link is instead allowed to know about the codebooks used on the other links. One then enters the difficult territory of the interference channel, although important recent progress has been made [15].

Acknowledgment

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Appendix

A. Proof of Lemma 1

Proof: To account for the dependence of f_1 on ϵ, let us redefine f_1 as a function of two variables:

$$f_1(\epsilon, P) = N \log\left(1 + \frac{P}{1 + \epsilon(N-1)P}\right)$$

As before, we have

$$f_2(P) = \log(1 + NP).$$

Clearly, $f_1(\cdot, P)$ is a decreasing function (for any fixed P), $f_1(0, P) > f_2(P)$ and $f_1(\infty, P) = 0 < f_2(P)$, so there exists a unique $\epsilon^\star(P)$ for which

$$f_1(\epsilon^\star(P), P) = f_2(P),$$

which can be explicitly computed:

$$\epsilon^\star(P) = \frac{1 + P - (1+NP)^{1/N}}{(N-1)P((1+NP)^{1/N} - 1)},$$

which is a decreasing function of P. By the monotonicity of $f_1(\cdot, P)$ (for fixed P), we have that

$$f_1(\epsilon, P) > f_1(\epsilon^\star(P), P) = f_2(P), \text{ for } \epsilon < \epsilon^\star(P) \tag{31}$$

$$f_1(\epsilon, P) < f_1(\epsilon^\star(P), P) = f_2(P), \text{ for } \epsilon > \epsilon^\star(P) \tag{32}$$

Since $\epsilon^\star(\cdot)$ is decreasing, we can define $\epsilon^\star(0)$ and $\epsilon^\star(\infty)$ by:

$$\begin{aligned}
\epsilon^\star(0) &= \lim_{P \downarrow 0} \epsilon^\star(P) = 1/2 \\
\epsilon^\star(\infty) &= \lim_{P \uparrow \infty} \epsilon^\star(P) = 0.
\end{aligned}$$

It follows that if $\epsilon > 1/2$ then $f_1(\epsilon, P) < f_2(P)$ for all $P > 0$, but if $\epsilon < 1/2$, then there exists a unique $\tilde{P}(\epsilon) > 0$ such that $\epsilon^\star(\tilde{P}(\epsilon)) = \epsilon$. For $P < \tilde{P}(\epsilon)$, $\epsilon^\star(P) > \epsilon$, and so $f_1(\epsilon, P) > f_2(P)$ by (31). For $P > \tilde{P}(\epsilon)$, $\epsilon^\star(P) < \epsilon$, and so $f_1(\epsilon, P) < f_2(P)$ by (32). ■

B. Proof of Lemma 2

Proof: If the tangent to C_1 at the point $(P_1, f_1(P))$ is to intersect C_2 at $(P, f_2(P))$ then P must solve the equation:

$$h(P) = J(P_1) \tag{33}$$

where

$$\begin{aligned} h(P) &:= f_2(P) - f_1'(P_1)P \\ J(P) &:= f_1(P) - f_1'(P)P. \end{aligned}$$

Since $h''(P) = \frac{-N^2}{(1+NP)^2} < 0$, it follows that $h(P)$ is a concave function, that increases to its maximum value $M(P_1)$, where $M(P)$ is given by:

$$M(P) = \log N - \log f_1'(P) - 1 + f_1'(P)/N$$

which is achieved at $P = \frac{1}{f_1'(P_1)} - \frac{1}{N}$, and $h(P)$ decreases on $(\frac{1}{f_1'(P_1)} - \frac{1}{N}, \infty)$. Thus, the following statements of equivalence hold: there are exactly two solutions to (33) iff $M(P_1) > J(P_1)$, there are no solutions to (33) iff $M(P_1) < J(P_1)$, and there is exactly one solution to (33) iff $M(P_1) = J(P_1)$. But

$$M'(P) - J'(P) = \frac{N-1}{N} P f_1''(P) k(\epsilon, N, P)$$

where

$$k(\epsilon, N, P) = 1 - 2\epsilon - \epsilon(1 + \epsilon(N-1))P$$

so if $\epsilon < 1/2$ then $M(P) - J(P)$ is decreasing on $(0, \frac{1-2\epsilon}{\epsilon(1+\epsilon(N-1))})$, and increasing on $(\frac{1-2\epsilon}{\epsilon(1+\epsilon(N-1))}, \infty)$. In the following, we assume that $\epsilon < 1/2$, as in the statement of the result. But since $\epsilon < 1/2$, $M(0+) - J(0+) < 0$ and $M(\infty) - J(\infty) > 0$, so there is a unique P_l such that $M(P_l) = J(P_l)$, which implies that the tangent line to C_1 at $(P_l f_1(P_l))$ touches C_2 at a unique point $(P_u, f_2(P_u))$. For $P < P_l$, $M(P) < J(P)$, and hence the tangent line to C_2 at $(P, f_1(P))$ does not intersect C_2 at all. For $P > P_l$, $M(P) > J(P)$, and hence the tangent line to C_2 at $(P, f_1(P))$ intersects C_2 at two points.

Equation (15) follows from fact that the tangent to C_1 at $(P_l f_1(P_l))$ touches C_2 at $(P_u, f_2(P_u))$, as does the equation:

$$f_1(P_l) = f_2(P_u) + f_2'(P_u)(P_l - P_u). \tag{34}$$

If $P_u < \tilde{P}$ then $f_2(P_u) < f_1(P_u)$, by (12), but then

$$f_1(P_u) > f_1(P_l) + f_1'(P_l)(P_u - P_l),$$

by (15), which contradicts the concavity of f_1. Similarly, if $\tilde{P} < P_l$ then $f_1(P_l) < f_2(P_l)$, by (13), but then

$$f_2(P_l) > f_2(P_u) + f_2'(P_u)(P_l - P_u),$$

by (34), which contradicts the concavity of f_2. Hence $P_l < \tilde{P} < P_u$. The remaining statements of the lemma are either straightforward consequences of the strict concavity of the functions f_1 and f_2, or of results proven above. ■

C. Proof of Lemma 3

We consider only the case where $a = 1$, otherwise we can re-scale the powers. With ϵ and a fixed, and under the constraint (19), we can write (18) as a function of a single variable

$$g(P_1) = g(\epsilon, a, P_1, \hat{P} - P_1) \tag{35}$$

Writing $c = \frac{\hat{P}}{2}$, and employing the change of variables: $P_1 = b + c, P_2 = c - b$, (35) becomes:

$$g(b) = C\left(\frac{b+c}{1+\epsilon(c-b)}\right) + C\left(\frac{c-b}{1+\epsilon(b+c)}\right). \tag{36}$$

and the constraint (19) becomes: $-c \le b \le c$. Let $d_1 = 1 + \epsilon c$, $d_2 = \epsilon b$, $d_3 = d_1 + c$ and $d_4 = b - d_2$. Then we can write the derivative of $g(\cdot)$ as

$$g'(b) = 2b \frac{(\epsilon d_3 - (1-\epsilon)d_1)(\epsilon d_3 + (1-\epsilon)d_1)}{(d_1^2 - d_2^2)(d_3^2 - d_4^2)} \tag{37}$$

Since d_1 and d_3 are independent of b, if $(\epsilon d_3 - (1-\epsilon)d_1) \neq 0$, the only root of $g'(b)$ happens at $b = 0$. The only positive ϵ for which $(\epsilon d_3 - (1-\epsilon)d_1)$ becomes zero is the $\epsilon^*(\cdot, \cdot)$ given in the lemma. This proves that the function $g(\cdot)$ in (35) increases in the interval $(0, \hat{P}/2)$ and then decreases to $\log(1 + \hat{P})$ at $P_1 = \hat{P}$, if $\epsilon \le \epsilon^*$, and vice versa otherwise. The fact that $g(\cdot)$ is strictly increasing and then strictly decreasing, along with its symmetric property around $\hat{P}/2$ implies that $g(\epsilon, a, \cdot, \cdot)$ in (18) is Schur concave [8] on $\mathcal{P}$ when $\epsilon \le \epsilon^*$, and Schur convex otherwise.

D. Proof of Lemma 4

In the following, we will denote the objective function value of Problem 3 by

$$C_s(\boldsymbol{P}) = \sum_{i=1}^{N} C\left(\frac{P_i}{1 + \epsilon \sum_{j \neq i} P_j}\right)$$

for a feasible power vector $\boldsymbol{P}$. Consider an arbitrary feasible power vector $\boldsymbol{P}^{(1)} = (P_1^{(1)}, P_2^{(1)}, \ldots, P_N^{(1)})$, satisfying $\sum_{j=1}^{N} P_j^{(1)} = \hat{P}$. Without loss of generality, we assume the components of all our power vectors are sorted in decreasing order, so that

$$P_1^{(1)} \ge P_2^{(1)} \ge \ldots \ge P_N^{(1)}.$$

Define

$$\begin{aligned} \bar{P}^{(1)} &= \sum_{j=3}^{N} P_j^{(1)} \\ a_1 &= 1 + \epsilon \bar{P}^{(1)} \\ \hat{P}^{(1)} &= P_1^{(1)} + P_2^{(1)} \end{aligned}$$

and consider the function $g^{(1)}(\cdot) = g(\epsilon, a_1, \cdot)$ (see (18) for the definition of g) restricted to the domain

$$\mathcal{P}^{(1)} = \{(P_1, P_2) : P_1 + P_2 = \hat{P}^{(1)}\}.$$

Lemma 3 implies that if $\epsilon \le \epsilon^*(a_1, \hat{P}^{(1)})$ then $g^{(1)}$ is Schur-concave on $\mathcal{P}^{(1)}$, but if if $\epsilon > \epsilon^*(a_1, \hat{P}^{(1)})$ then $g^{(1)}$ is Schur-convex on $\mathcal{P}^{(1)}$. In either case, we can construct a sequence of power vectors that cannot decrease the achieveable sum-rate, as we now show.

Case 1: $\epsilon \leq \epsilon^*(a_1, \hat{P}^{(1)})$.

First, consider another arbitrary, feasible power vector $\boldsymbol{Q}$, ordered in decreasing order as above. For any component i, we can define a new vector $\boldsymbol{Q}'$ by decreasing Q_i and increasing Q_{i+1} by the same amount. Provided the amount swapped between the two vectors is no more than $Q_i - Q_{i+1}$, the vector $\boldsymbol{Q}'$ will also be ordered in decreasing order, and $\boldsymbol{Q} \succ \boldsymbol{Q}'$. Such a transfer is known as a Pigou-Dalton transfer, and it is given the name "elementary Robin Hood operation" in [16]. It is well known [16] that if $\boldsymbol{Q} \succ \boldsymbol{R}$ then one can generate $\boldsymbol{R}$ from $\boldsymbol{Q}$ via a countable sequence of elementary Robin Hood operations.

Now let us denote the vector $(\hat{P}/N, \hat{P}/N, \ldots, \hat{P}/N)$ by $\boldsymbol{P}^*$. Since $\boldsymbol{P}^{(1)} \succ \boldsymbol{P}^*$ it follows that there is a sequence $\boldsymbol{P}^{(n)}$ of feasible power vectors, starting at $\boldsymbol{P}^{(1)}$, converging to $\boldsymbol{P}^*$, where $\boldsymbol{P}^{(n+1)}$ is obtained from $\boldsymbol{P}^{(n)}$ by an elementary Robin Hood operation. Let $i_n, i_n + 1$ denote the components where the transfer takes place at step n of this sequence, and without loss of generality, let $i_1 = 1$. For each $n \in \mathbb{Z}_+$, define

$$\begin{aligned} \bar{P}^{(n)} &= \sum_{j=1}^{N} P_j^{(n)} I_{\{j \neq i_n, i_n+1\}} \qquad (38) \\ a_n &= 1 + \epsilon \bar{P}^{(n)} \\ \hat{P}^{(n)} &= P_{i_n}^{(n)} + P_{i_n+1}^{(n)} \end{aligned}$$

It is trivial to show, by induction, that for any $i \in \{1, 2, \ldots, N-1\}$, and any $n \in \mathbb{Z}_+$, we have that

$$P_i^{(n)} + P_{i+1}^{(n)} \leq P_1^{(1)} + P_2^{(1)}$$

so in particular,

$$P_{i_n}^{(n)} + P_{i_n+1}^{(n)} \leq P_1^{(1)} + P_2^{(1)}$$

It follows from (38) that

$$\begin{aligned} \bar{P}^{(n)} &\geq \bar{P}^{(1)} \\ a_n &\geq a_1 \\ \hat{P}^{(n)} &\leq \hat{P}^{(1)} \end{aligned}$$

Define the function $g^{(n)}(\cdot) = g(\epsilon, a_n, \cdot)$ restricted to the domain

$$\mathcal{P}^{(n)} = \{(P_1, P_2) : P_1 + P_2 = \hat{P}^{(n)}\}.$$

Now, since $\epsilon \leq \epsilon^*(a_1, \hat{P}^{(1)})$, it follows that $\epsilon \leq \epsilon^*(a_n, \hat{P}^{(n)})$, for all $n \in \mathbb{Z}_+$, and hence $g^{(n)}$ is Schur-concave on $\mathcal{P}^{(n)}$ for all such n. Since the power vectors $\boldsymbol{P}^{(n)}$ decrease in order of majorization, it follows that $C_s(\boldsymbol{P}^{(1)}) < C_s(\boldsymbol{P}^*)$, unless $\boldsymbol{P}^{(1)} = \boldsymbol{P}^*$.

Case 2: $\epsilon > \epsilon^*(a_1, \hat{P}^{(1)})$.

For $n = 2, 3, \ldots, N$, define the power vector $\boldsymbol{P}^{(n)}$ by:

$$P_j^{(n)} = \begin{cases} \sum_{i=1}^{n} P_i^{(1)} & j = 1 \\ P_{j+n-1}^{(1)} & j = 2, 3, \ldots, N-n+1 \\ 0 & j = N-n+2, N-n+3, \ldots, N \end{cases}$$

which gives us a sequence of power vectors, all satisfying the feasibility constraint that $\sum_{j=1}^{N} P_j^{(n)} = \hat{P}$, and all with the property of decreasing order for the components of each power vector. Further, as a sequence of power vectors, the vectors are increasing in order of majorization, with the final element in the sequence being $\boldsymbol{P}^{(N)} = (\hat{P}, 0, 0, \ldots, 0)$. For each n, define

$$\begin{aligned} \bar{P}^{(n)} &= \sum_{j=2}^{N-n+1} P_j^{(n)} = \sum_{j=n+2}^{N} P_j^{(1)} \\ a_n &= 1 + \epsilon \bar{P}^{(n)} \\ \hat{P}^{(n)} &= \sum_{i=1}^{n+1} P_i^{(1)} \end{aligned}$$

Note that $\bar{P}^{(n)}$ (and hence a_n) decrease with n, but $\hat{P}^{(n)}$ increases with n. Define also the function $g^{(n)}(\cdot) = g(\epsilon, a_n, \cdot)$ restricted to the domain

$$\mathcal{P}^{(n)} = \{(P_1, P_2) : P_1 + P_2 = \hat{P}^{(n)}\}.$$

Now, since a_n decreases, and $\hat{P}^{(n)}$ increases, it follows that $\epsilon^*(a_n, \hat{P}^{(n)})$ decreases with n. Since $\epsilon > \epsilon^*(a_1, \hat{P}^{(1)})$, it follows that $\epsilon > \epsilon^*(a_n, \hat{P}^{(n)})$, for all $n = 2, 3, \ldots, N$, and hence $g^{(n)}$ is Schur-convex on $\mathcal{P}^{(n)}$ for all such n. Since the power vectors $\boldsymbol{P}^{(n)}$ increase in order of majorization, it follows that $C_s(\boldsymbol{P}^{(1)}) < C_s(P^{(N)})$, unless $\boldsymbol{P}^{(1)} = \boldsymbol{P}^{(N)}$.

Now suppose that the vector $\boldsymbol{P}^{(1)}$ is optimal for Problem 3. Then either $\boldsymbol{P}^{(1)} = (\hat{P}/N, \hat{P}/N, \ldots, \hat{P}/N))$ (if $\epsilon \leq \epsilon^*(a_1, \hat{P}^{(1)})$) or $\boldsymbol{P}^{(1)} = (\hat{P}, 0, 0, \ldots, 0)$ (if $\epsilon > \epsilon^*(a_1, \hat{P}^{(1)})$), for otherwise, we can improve the objective function, as described above in the two separate cases. We conclude that

$$U(\epsilon, N, \hat{P}) = \max\{f_1(\hat{P}/N), f_2(\hat{P}/N)\}$$

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