Lattice Paths: Coding and Complexity

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***Abstract*— We shall study several sequences of numbers arising in the array $\beta(n,k) = 3 \cdot \binom{n+k}{k} - 2 \cdot \binom{n+k}{k-1}$. Some of them relate to an old problem of Berlekamp, which arose in the construction of certain convolutional burst–error detecting codes. The other sequences are of recent interest in combinatorial theory.**

I. Introduction

Consider the following array of numbers $\beta(n,k) = 3 \cdot \binom{n+k}{k} - 2 \cdot \binom{n+k}{k-1}$

$$\begin{array}{r|rrrrrr}
-2 & -13 & -45 & -105 & -165 & -99 & 429 \\
-2 & -11 & -32 & -60 & -60 & 66 & 528 \\
-2 & -9 & -21 & -28 & 0 & 126 & 462 \\
-2 & -7 & -12 & -7 & 28 & 126 & 336 \\
-2 & -5 & -5 & 5 & 35 & 98 & 210 \\
-2 & -3 & 0 & 10 & 30 & 63 & 112 \\
-2 & -1 & 3 & 10 & 20 & 33 & 49 \\
-2 & 1 & 4 & 7 & 10 & 13 & 16 \\
 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}$$

The numbers around the 0s in this array can be expressed via several closed formulae

$\frac{1}{5n+1}\binom{5n+1}{2n} : 2, 5, 60, \ldots$
$\frac{1}{5n+2}\binom{5n+2}{2n} : 1, 5, 66, \ldots$
$\frac{1}{5n+3}\binom{5n+3}{2n+1} : 1, 7, 99, \ldots$
$\frac{3}{5n+1}\binom{5n+1}{2n+1} : 3, 10, 126, \ldots$
$\frac{2}{5n-1}\binom{5n-1}{2n} : 3, 28, 429$

The first three sequences come into play in the solution to an old problem of Berlekamp [15], [3], which arose in his study of burst–error detecting convolutional codes [2]. The last two sequences are of recent interest in combinatorial theory motivated by the so called tennis ball problem, e. g. [4].

The analysis of both problems can be reduced to the enumeration of lattice paths not touching a specified boundary. The boundary relevant for Berlekamp's problem is obtained from a line through the origin, which may not be touched, the boundary for the tennis ball problem is of staircase type.

After presenting Berlekamp's problem and the corresponding lattice path model we shall sketch the method based on generating functions which leads to a an expression involving the above numbers. Further, we shall demonstarte that this generating function method also allows to derive the identities from [6], which were obtained by a bijective argument. Finally, some open problems concerning further generalizations of the above numbers and th complexity to obtain these numbers or their reduction modulo 2 will be discussed.

II. Berlekamp's Problem

At the 3rd Waterloo Conference on Combinatorics ([15], pp. 341 – 342), Berlekamp presented the following combinatorial problem. The problem will be illustrated with the following example also due to Berlekamp in [15].

$$\begin{array}{r|rrrrrr}
8 & & & & & & 1 \\
7 & & & & & 1 & 1 \\
6 & & & & & 1 & 2 \\
5 & & & & 1 & 1 & 3 \\
4 & & & 1 & 1 & 2 & 7 \\
3 & & & 1 & 2 & 5 & 19 \\
2 & & 1 & 1 & 3 & 9 & 37 \\
1 & 1 & 1 & 2 & 7 & 23 & 99 \\
0 & 1 & 2 & 5 & 19 & 66 & 293 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array} \qquad (1)$$

Berlekamp defines an array to be unitary if any square submatrix whose upper left corner falls on the boundary of the array has a determinant equal to 1. For instance, in the array above

$$\det \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 9 \end{pmatrix} = 1$$

The problem then he states as follows: "... A periodic quasilinear boundary represents the best staircase approximation to a straight line of rational slope. ... Exact formulas are known for the values of the numbers in the unitary arrays generated by periodic quasilinear boundaries of slopes $1/n$ or n, but no such formulas are known (to me) for the values in the arrays with boundaries of slopes m/n where $1 < m < n$. The simplest such case is slope $2/3$" - this is shown above in (1), actually this array was presented in [15] by flipping rows and columns.

This problem arose already in Berlekamp's paper [2], where the numbers in the array above reduced modulo 2 were suggested as generating sequence of a convolutional code detecting burst errors.

The case slope $2/3$ and $3/2$, which yield codes of rates $2/5$ and $3/5$, respectively, was studied in further detail by Berlekamp in [3], where he derived some formulas for special parameters and then stated: “The patterns are clear but I know no explanation. Why does the formula apply to an individual entry, then to sums of pairs of entries from different rows, and then to the *negative* of an entry?”. This question had been answered in our papers [11] (without being aware of the reference [3] at that time) and [13].

Carlitz, Roselle, and Scoville [5] later presented a fast algorithm for the computation of the number of such lattice paths by getting rid of the determinant calculation. They showed that the entries in this array enumerate the lattice paths from the beginning of the row to the top of the column which determine the respective entry, where these paths are not allowed to cross the boundary given by the 1's. For instance, in the array above the positions of the 1's are below the boundary determined by $u_0 = 2$, $u_1 = 3$, $u_2 = 5$, $u_3 = 6$, $u_4 = 8$, $u_5 = 9$, etc.

A path here is a sequence of pairs (s_i, t_i), $i = 0, 1, \ldots$ of nonnegative integers where (s_i, t_i) is either $(s_{i-1} + 1, t_{i-1})$ or $(s_{i-1}, t_{i-1} + 1)$. So, a particle following such a path can move either one step to the right, i. e. $s_i = s_{i-1} + 1$, or one step upwards, i. e. $t_i = t_{i-1} + 1$ in each time unit i. We shall assume that a path starts in the origin $(0, 0)$.

Observe that the numbers from (1) occur under the boundary of the three arrays below whose entries obey the recursion $a(x, y) = a(x - 1, y) + a(x, y - 1)$

					293
				66	293
				66	227
			19	66	161
		5	19	47	95
		5	14	28	48
	2	5	9	14	20
1	2	3	4	5	6
1	1	1	1	1	1

					136
					136
				37	136
			9	37	99
			9	28	62
		3	9	19	34
	1	3	6	10	15
	1	2	3	4	5
1	1	1	1	1	1

					99
				23	99
				23	76
			7	23	53
		2	7	16	30
		2	5	9	14
	1	2	3	4	5
1	1	1	1	1	1

III. Lattice Path Enumeration

We shall consider paths in an integer lattice from the origin $(0, 0)$ to the point (n, u_n), which never touch any of the points (i, u_i), $i = 0, 1, \ldots, n-1$. In [7] Gessel introduced a general probabilistic method to determine the number of such paths, denoted by f_n , which he studied for the case that the subsequence $(u_i)_{i=1,2\ldots}$ is periodic.

For period length 2 the elements of the sequence $(u_i)_{m=0,1,2,\ldots}$ are on the 2 lines (for $i = 0, 1, 2, \ldots$)

$$u_{2i} = s + ci \text{ and } u_{2i+1} = s + \mu + ci,$$

Gessel's probabilistic method is as follows. A particle starts at the origin $(0, 0)$ and successively moves with probability p one unit to the right and with probability $q = 1 - p$ one unit up. The particle stops if it touches one of the points (i, u_i).

The probability that the particle stops at (n, u_n) is $p^n q^{u_n} \cdot f_n$.

Setting

$$f(t) = \sum_{n=0}^{\infty} f_n t^n = \sum_{n=0}^{\infty} f_{2n} t^{2n} + \sum_{n=0}^{\infty} f_{2n+1} t^{2n+1} = g(t^2) + t \cdot h(t^2)$$

the probability that the particle eventually stops is

$$q^{u_0} g(p^2 q^c) + p q^{u_1} h(p^2 q^c)$$

If p is sufficiently small, the particle will touch the boundary $(i, u_i)_{i=0,1,\ldots}$ with probability 1. So for small p and with $t = pq^{c/2}$ we have

$$q(t)^{u_0} g(t^2) + p(t) q(t)^{u_1} h(t^2) = 1$$

For p sufficiently small one may invert $t = p(1-p)^{c/2}$ to express p as a power series in t, namely $p = p(t)$. Then changing t to $-t$ and denoting $p(-t)$ by $\overline{p}(t)$ and similarly $q(-t)$ by $\overline{q}(t)$ yields the system of equations

$$q^s \cdot g(t^2) + p \cdot q^{s+\mu} \cdot h(t^2) = 1,$$
$$\overline{q}^s \cdot g(t^2) + \overline{p} \cdot \overline{q}^{s+\mu} \cdot h(t^2) = 1$$

which for $g(t^2)$ and $h(t^2)$ yield the solutions

$$g(t^2) = \frac{p^{-1} q^{-s-\mu} - \overline{p}^{-1} \overline{q}^{-s-\mu}}{p^{-1} q^{-\mu} - \overline{p}^{-1} \overline{q}^{-\mu}} = \frac{q^{c/2-\mu-s} + \overline{q}^{c/2-\mu-s}}{q^{c/2-\mu} + \overline{q}^{c/2-\mu}} \tag{2}$$

and

$$h(t^2) = \frac{q^{-s} - \overline{q}^{-s}}{t \cdot (q^{\mu - c/2} + \overline{q}^{\mu - c/2})} \qquad (3)$$

By Lagrange inversion (cf. e.g. [10]) for any α we have

$$q^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha}{(c/2+1)n+\alpha} \binom{(c/2+1)n+\alpha}{n} \cdot t^n \qquad (4)$$

Now we are able to explain the entries from Berlekamp's example array for slope $\frac{2}{3}$. We have to inspect the parameter choices $(s = 1, \mu = 1)$, $(s = 1, \mu = 2)$, and $(s = 2, \mu = 1)$. The generating functions look as follows.

Theorem [13]:

$$\sum_{n=0}^{\infty} \frac{1}{5n+1} \binom{5n+1}{2n} x^n - \frac{x}{2} \cdot [h(x)]^2$$

$$= 1 + 2x + 23x^2 + 377x^3 + \dots,$$

$$\sum_{n=0}^{\infty} \frac{1}{5n+1} \binom{5n+1}{2n} x^n + \frac{x}{2} \cdot [h(x)]^2$$

$$= 1 + 3x + 37x^2 + 624x^3 + \dots,$$

$$h(x) = \sum_{n=0}^{\infty} \frac{1}{5n+2} \binom{5n+2}{2n+1} x^n$$

$$= 1 + 5x + 66x^2 + 1156x^3 + \dots,$$

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{5n+3} \binom{5n+3}{2n+1} x^n =$$

$$1 + 7x + 99x^2 + 1768x^3 + \dots,$$

$$\sum_{n=1}^{\infty} \frac{1}{5n-1} \binom{5n-1}{2n} x^{n-1} - \frac{1}{2} [g(x)]^2$$

$$= 1 + 9x + 136x^2 + \dots,$$

$$\sum_{n=1}^{\infty} \frac{1}{5n-1} \binom{5n-1}{2n} x^{n-1} + \frac{1}{2} \cdot [g(x)]^2$$

$$= 2 + 19x + 293x^2 + 5332x^3 + \dots.$$

IV. Staircase Boundaries

Recently, in [6], lattice paths not touching staircase boundaries have been considered, where the boundary is obtained by moving successively v steps north and w steps east, allowing maybe an additional first column in the beginning. The two arrays for $v = 3$ and $w = 2$ are as follows:

					126	659
					126	533
					126	407
			10	45	126	281
			10	35	81	155
			10	25	46	74
1	3	6	10	15	21	28
1	2	3	4	5	6	7
1	1	1	1	1	1	1

				81	429
				81	348
				81	267
		6	28	81	186
		6	22	53	105
		6	16	31	52
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

Observe that the sequences $3, 10, 126, \dots$ and $3, 28, 429, \dots$ are old friends from the introduction.

A general result was formulated in [6]. The authors define as $A_{w,v}$ the infinite staircase path that starts at $(0, v)$, then takes w steps east, v steps north, w steps east, v steps north, and so on. Then they derive by a bijective argument similar to the reflection principle the following two results

Theorem ([6])

(1) Let S_1 be the set of lattice paths from $(0, 0)$ to $(wn + 1, vn)$ that avoid $A_{w,v}$. There are $v\binom{wn+vn}{vn} - w\binom{wn+vn}{vn-1}$ lattice paths in S_1 that avoid Aw, v.

(2) Let S_2 be the set of lattice paths from $(1, 0)$ to $(wn, vn-1)$ that avoid $A_{w,v}$. There are $v\binom{wn+vn-2}{vn-1}$ - $w\binom{wn+vn-2}{vn-2}$ lattice paths in S_2 that avoid $A_{w,v}$.

Proof (via generating functions): For the special case $w = 2$ and $v = 3$ this just yields the fourth and fifth sequence from the introduction as pointed out above. The proof of the second result in this case is easy by simply chosing as parameters $c = 3$, $s = 3$ and $\mu = 0$ in the generating function h in (2).

In order to prove the first result, one has to take into account that the boundary is not periodic as required in the previous section. However, the paths beginning in the origin $(0, 0)$ will enter the axis with $x = 1$ in one of the coordinates $(1, i)$, $i = 0, \dots, 2$. From this on the paths are below a perodic boundary, such that one can sum up over all i all paths from $(1, i)$ to the destination $(wn + 1, vn)$.

So we have to add up the generating functions in (2) for $s = 1, 2, 3$ and fixed $c = 3$, $\mu = 0$ in order to obtain the generating function F, say, we are interested in. Thus

$$F(t^2) = \sum_{s=1}^{3} \frac{p^{-1}q^{-s-\mu} - \overline{p}^{-1}\overline{q}^{-s-\mu}}{p^{-1} - \overline{p}^{-1}}.$$

Now, $q^{-1} + q^{-2} + q^{-3} = q^{-1}\frac{q^{-3}-1}{q^{-1}-1} = \frac{1}{1-q}(q^{-3} - 1) = \frac{1}{p}(q^{-3} - 1)$.

Accordingly, $\sum_{s=1}^{3} \overline{q}^{-s} = \frac{1}{\overline{p}}(\overline{q}^{-3} - 1)$.

Hence, using $pq^{3/2} = t$ and $\overline{pq}^{3/2} = -t$,

$$F(t^2) = \frac{-p^{-2} + \overline{p}^{-2}}{p^{-1} - \overline{p}^{-1}} = -p^{-1} - \overline{p}^{-1} = \frac{\overline{q}^{3/2}}{t} - \frac{q^{3/2}}{t},$$

whose coefficients are $\frac{3}{5n+1}\binom{5n+1}{2n+1}$ by (4).

A generalization to arbitrary w and v is in progress [14], taking into account that for period length $d \geq 3$ one has to solve the larger system of equations

$$\left(p(\omega^i t)^j q(\omega^i t)^{u_j}\right)_{i,j=0,\dots,d-1} \cdot \begin{pmatrix} f^{(0)}(t^d) \\ \vdots \\ f^{(d-1)}(t^d) \end{pmatrix}$$

in order to obtain d generating functions $f^{(0)}, \dots f^{(d-1)}$. In our case $d = w$ and $u_o = u_1 = \dots u_{w-1} = v$, which simplifies to evaluating several Vandermonde determinants.

V. Concluding Remarks and Open Problems

1) The evaluation of determinants for the array (1) had been replaced in [5] by calculation of the recurrence $a(x, y) = a(x - 1, y) + a(x, y - 1)$ which is, of course, much more efficient. In fact, Berlekamp [2] was rather interested in the entries of the array (1) reduced modulo 2. In some cases these numbers can be computed via a copy-paste algorithm. For instance, the Catalan numbers modulo 2 are the sequence 101000100000001... Here the segment from a 1 until the 0 preceding the next 1 is simply obtained by copying the segment from the previous 1 until the 0 preceding the actual 1 and appending to this segment the all-zero sequence. Similar procedures can be given for sequences reduced modulo 2 in an array bounded by lines of integer slope, cf. [2].

Berlekamp in [2] asked if there would exist comparably efficient procedures to generate the numbers reduced modulo 2 in an array of noninteger slope. Even for the most simple example presented in this paper this question is still open. Even the patterns for the numbers in the odd or even positions, respectively, do not seem to be realizable by such a nice algorithm.

2) Computer observations strongly suggest that nice identities also exist for arrays with a periodic boundary of period length $d > 2$. Recently, Irving and Rattan [8] found identities similar to the Catalan numbers for the number of lattice paths summed up over all cyclic shifts of a periodic boundary.

3) In [13] two further applications were pointed out. There is a one–to–one correspondence between s–ary regular trees and ballot–type $\{0, 1\}$–sequences which can be obtained from the lattice paths under consideration. This correspondence can be exploited to store regular trees, by assigning to them as codewords the ballot – type sequence. The codes thus obtained form a prefix code, cf. [9]. Further, these numbers we studied also denote the size of downsets in the so–called pushing order studied, for instance, in [1].

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