

# Diversity and Coding Gain Evolution in Graph Codes

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**Abstract**—This work is a first attempt to analyze and understand the coding gain of full-diversity (MDS) low-density parity-check (LDPC) codes. A diversity population evolution is derived. The steady-state distribution of energy coefficients is found. Based on this new analysis, we show new full-diversity LDPC codes with improved performance. Finally, we describe a family of graph codes where all variable nodes, including parity nodes, achieve full diversity after a large number of decoding iterations. Full-diversity MDS low-density codes are not only useful for digital data transmission over wireless channels, some other applications can be found in situations where coding for a finite number of states is required.

## I. INTRODUCTION

The Hamming weight distribution of error-correcting codes [1][2] plays an important role in determining the performance of block and convolutional codes. The performance of codes on graphs studied in the new old Modern Coding Theory [4][5] is determined by the evolution of inference messages in a graph structure. Under iterative decoding and for infinite acyclic graphs, the key parameter of a code ensemble is known as its *threshold*, which corresponds to the parameter of the worst channel condition where the decoder is still capable of retrieving information with a vanishing error probability. Low-density parity-check codes have been extensively studied in the literature (see [5] and references therein) and ensemble thresholds have been determined and optimized on different types of non-zero capacity channels.

In this paper, we consider digital channel models with zero Shannon capacity (also known as non-ergodic channels [14]). As a trivial special case, consider a block erasure channel where the transmitted codeword is divided into two blocks, each block being erased with probability  $\epsilon$ . For any code length, the word error probability does not vanish and it is lower bounded by  $P_{out}(\epsilon) = \epsilon^2$ , where  $P_{out}$  is the outage probability limit of the channel. This block binary erasure channel (block BEC) is a special case of the block fading channel described in the next section. Capacity achieving codes designed for ergodic channels mutated into near outage achieving codes for non-ergodic channels by the embedding (so it was a controlled mutation!) of a deterministic structure into the random ensemble [7][9][10]. We propose in the sequel a simple but original analysis to understand the diversity and coding gain of root-LDPC ensembles originally published in [9][10]. The notions of diversity and coding gain will be recalled in the next two sections.

Unless otherwise stated, we assume that the channel is memoryless with additive white gaussian noise and multiplicative

real fading. The fading coefficients are only known at the decoder side. The input-output channel model is

$$y = \alpha x + \eta, \quad (1)$$

where  $x = \pm 1$  is the transmitted binary element, the fading coefficient  $\alpha$  is Rayleigh distributed, i.e.,  $\alpha^2$  is  $\chi^2$ -distributed with degree 2 and normalized moment  $\mathcal{E}[\alpha^2] = 1$ , and the noise  $\eta$  is  $\mathcal{N}(0, \sigma^2)$ . The BEC channel is derived from (1) by restricting the fading to its two extremal values 0 and  $+\infty$ .

## II. ITERATIVE DECODING ON BLOCK FADING CHANNELS

Let us briefly recall the expression of log-ratio probabilistic messages on a block fading channel. The error-correcting code is  $C[N, K]_2$  built as an instance from an LDPC ensemble defined by a Tanner graph [22] and its degree distribution [12]. The coding rate is denoted  $R = K/N$ . In our model, a codeword  $(c_1 c_2 \dots c_N)$  is split into two equal length parts. The first half-codeword is transmitted over a channel defined as in (1) by  $y_\ell = \alpha_1 c_\ell + \eta_\ell$ ,  $\ell = 1 \dots \frac{N}{2}$ . Similarly, the second half-codeword is transmitted via  $y_\ell = \alpha_2 c_\ell + \eta_\ell$ ,  $\ell = \frac{N}{2} + 1 \dots N$ . It is assumed that  $\alpha_1$  and  $\alpha_2$  are independent within the same codeword and from one codeword to another.

Before decoding, the message associated to  $c_\ell$  when  $y_\ell$  is observed at the channel output is

$$\begin{aligned} \Lambda_\ell &= \log \left( \frac{p(y_\ell | x_\ell = +1, \alpha)}{p(y_\ell | x_\ell = -1, \alpha)} \right) \\ &= \frac{2\alpha y_\ell}{\sigma^2} = \frac{2}{\sigma^2} (\alpha^2 c_\ell + \alpha \eta_\ell) \end{aligned} \quad (2)$$

where  $\alpha = \alpha_1$  if  $\ell \leq \frac{N}{2}$ , otherwise  $\alpha = \alpha_2$ . For example, consider the case  $c_1 = +1$ . A decision based on the message  $\Lambda_1 \propto (\alpha_1^2 + \eta_\ell')$  is erroneous iff  $\Lambda_1 < 0$ . In this case, the average error probability decreases as  $\frac{1}{\gamma}$ , where the parameter  $\gamma = E_b/N_0 = 1/(2R\sigma^2)$  is the signal-to-noise ratio per bit [17],  $\gamma \gg 1$ .

In the sequel, assume that the all-zero codeword has been transmitted, i.e.,  $c_\ell = +1, \forall \ell$ . What are we expecting from an efficient LDPC code designed for a block fading channel with an intrinsic double diversity (since a codeword undergoes two independent fading instances)? Once the observed messages are fed to the decoder, and after some number of iterations where messages evolve in the code graph, we expect that any message  $\Lambda_\ell^m$  at iteration  $m$  would be

$$\Lambda_\ell^m \propto (a\alpha_1^2 + b\alpha_2^2 + \eta'), \quad (3)$$

where  $a > 0$  and  $b > 0$ . The reals  $a$  and  $b$  are called *energy coefficients*. A decision based on the above message

yields an average error probability that decreases as  $\frac{1}{ab\gamma^2}$  [17]. Therefore, we state two main objectives for efficient LDPC coding on block fading channels. After a finite or an infinite number of decoding iterations:

- Probabilistic information messages must behave as a  $4^{\text{th}}$ -order  $\chi^2$  distribution in order to attain a double diversity error probability, i.e.,  $P_e \propto \frac{1}{\gamma^2}$ . In this case, the code is referred to as a *full-diversity code*.
- When full diversity is satisfied, probabilistic information messages must behave as a  $4^{\text{th}}$ -order  $\chi^2$  distribution with the highest possible energy coefficients  $a$  and  $b$ .

Notice that under Maximum-Likelihood (ML) decoding, the product  $ab$ , or more precisely its square root, is known as the *coding gain*. The two points stated above aim at rendering a word-error probability for the LDPC code as close as possible to the information theoretical limit  $P_{out}(\gamma)$  as defined in the following section.

### III. SOME SIMPLE FACTS ON THE OUTAGE PROBABILITY

Consider the time-shared channels described in the previous section. For simplicity, let us drop the time index  $\ell$ . Channel  $i$  is defined by  $y_i = \alpha_i x_i + \eta_i$ ,  $i = 1, 2$ . When the input  $x_i$  is gaussian distributed, the instantaneous (i.e., conditioned on the fading instance) capacity of channel  $i$  is  $C_i = \frac{1}{2} \log_2(1 + 2R\alpha_i^2\gamma)$ , see [3], section 9.1. The instantaneous capacity of the global channel including the two parallel time-shared sub-channels is

$$C_{gauss} = \frac{1}{4} \log_2 [(1 + 2R\alpha_1^2\gamma)(1 + 2R\alpha_2^2\gamma)]. \quad (4)$$

When the input  $x_i$  is constrained to a binary  $\{\pm 1\}$  alphabet (BPSK), the instantaneous average mutual information  $I(x_i; y_i | \alpha_i)$  can be easily written as  $1 - \mathcal{E}_X[\log_2(1 + e^{-2X})]$ , where  $X \sim \mathcal{N}(2R\gamma, 2R\gamma)$ . The instantaneous capacity of the global time-shared channel denoted by  $C_{bpsk} = C(\gamma | \alpha_1, \alpha_2)$  becomes

$$C_{bpsk} = 1 - \mathcal{E}_{X_1}[\log_2(1 + e^{-2X_1})] - \mathcal{E}_{X_2}[\log_2(1 + e^{-2X_2})],$$

where  $X_1$  and  $X_2$  are i.i.d. gaussian  $\mathcal{N}(2R\gamma, 2R\gamma)$ . An outage event occurs each time  $C_{bpsk} < R$ . The outage probability limit is defined as  $P_{out}(\gamma) = \Pr(C(\gamma | \alpha_1, \alpha_2) < R)$ . Unfortunately,  $P_{out}(\gamma)$  has no simple closed form expression. Nevertheless, a curve fitting at rate  $R = 1/2$  shows that a very accurate approximation (for  $\gamma \geq 15\text{dB}$ ) can be

$$P_{out}(\gamma) \cong \frac{4}{\gamma^2}. \quad (5)$$

In the fading plane, for a fixed SNR, an outage occurs when the fading point is below the outage boundary [8] defined by  $C(\gamma | \alpha_1, \alpha_2) = R$  as illustrated in Fig. 1. We would like to determine the outage probability due to unbalanced fading. The latter is denoted by  $P_{uf}(\gamma)$  and corresponds to the outage area tail beyond  $\alpha_e$ , where  $\alpha_e$  (e stands for ergodic) is determined by  $C(\gamma | \alpha_e, \alpha_e) = R$ . The outage probability associated to fading points inside the square  $\alpha_e \times \alpha_e$  is denoted

by  $P_{so}(\gamma)$ . Expressions for  $P_{so}$  and  $P_{uf}$  at rate  $1/2$  can be found after solving  $\alpha_e$ . Let  $\Delta = 0.187\text{dB}$  be the signal-to-ratio gap between the capacity limit of a gaussian channel (no fading) with gaussian input and the capacity limit of the same channel with antipodal binary input. The outage boundary  $C_{gauss} = R$  intersects the ergodic line at  $\alpha^2 = 1/\gamma$ . Hence, we have  $\alpha_e^2 = \frac{10^{\Delta/10}}{\gamma}$ .

The outage inside the square can now be determined as  $P_{so}(\gamma) = [\Pr(\alpha_1 \leq \alpha_e)]^2$ . For  $\gamma \gg 1$ , we get

$$P_{so}(\gamma) \approx \alpha_e^4 = \frac{10^{\Delta/5}}{\gamma^2}.$$

The unbalanced fading outage is simply

$$P_{uf}(\gamma) = P_{out}(\gamma) - P_{so}(\gamma) \approx \frac{4 - 10^{\Delta/5}}{\gamma^2}.$$

Finally, we evaluate the percentage of outage for unbalanced fading and its complementary ( $\gamma \gg 1$ )

$$\frac{P_{so}(\gamma)}{P_{out}(\gamma)} \approx 27\%, \quad \frac{P_{uf}(\gamma)}{P_{out}(\gamma)} \approx 73\%. \quad (6)$$

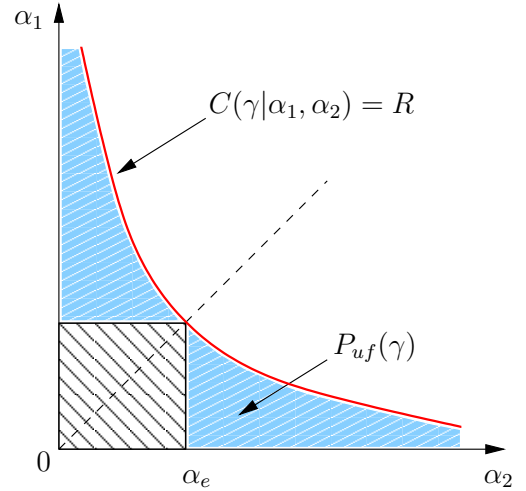


Fig. 1. Outage boundary in the fading plane for BPSK at rate  $R = 1/2$ . The unbalanced fading outage is the probability that an outage fading point be outside the square kissing the outage boundary on the ergodic line.

Numbers in (6) show that the outage probability on the block fading channel is mainly due to the critical unbalanced fading area where  $0 \leq \alpha_2^2 \leq \alpha_e^2 \leq \alpha_1^2 < +\infty$  and its symmetric region with respect to the ergodic line. An outage-achieving code should exhibit an outage boundary as close as possible to the curve  $C(\gamma | \alpha_1, \alpha_2) = R$  starting from the ergodic line and following it along the fading axes until infinity. A good ergodic threshold would definitely help in minimizing the outage probability in the area around the point  $(\alpha_e, \alpha_e)$  on the ergodic line. The numbers in (6) tell us that the major improvement can be obtained in critical cases such as  $\alpha_2^2 \ll \alpha_1^2$  which leads us to the block BEC.

Indeed, the exact performance of an LDPC ensemble (random or structured) can be numerically determined via density

evolution [5]. Unfortunately, such a numerical evaluation does not offer us any insight for understanding the behavior of the ensemble. The block BEC is a means to better understand the parameters influence of an LDPC ensemble and tune them for block fading channels. As derived from (1), the block BEC is the special case where fading takes only two extremal values 0 and  $+\infty$ . The probability of this special case is 0! Why then the block BEC is decisive for the design and analysis of full-diversity LDPC codes? Let us recall that the diversity order is defined by [17]

$$d = - \lim_{\gamma \rightarrow +\infty} \log(P_e) / \log(\gamma), \quad (7)$$

where  $P_e$  is the error probability at the decoder's output. The diversity  $d$  is identical whether  $P_e$  involves bit error probability or word error probability. Furthermore,  $d$  may depend on the decoding method and may vary for one code instance to another at finite length. In this study, a full diversity is equivalent to  $d = 2$  which is the diversity order of the outage limit for both gaussian and discrete input alphabets [19][18]. The reason for the BEC significance comes from the fact that *full diversity on the block BEC is a necessary and sufficient condition for full diversity on the block fading channel*. The proof may rely on sub-optimal min-sum decoding [21][20] as applied to block fading channels in [10].

In the sequel, the block BEC approach will be used to develop the concepts of diversity population evolution and coding gain evolution. Before describing these two concepts, we examine random LDPC ensembles in section IV and recall the structure of MDS ensembles in section V.

#### IV. RANDOM LDPC ENSEMBLES: IRREGULARITY IS NOT SUFFICIENT FOR ACHIEVING FULL DIVERSITY

We study in this section the diversity tunnel of irregular LDPC codes, i.e., how the fraction of full-diversity bits evolve with the number of decoding iterations. Consider the random ensemble of rate-1/2 binary  $(\lambda(x), \rho(x))$  LDPC codes, where  $\lambda(x)$  defines the left degree distribution and  $\rho(x)$  defines the right degree distribution, both from an edge perspective [12]. Assume that half of bits are white and half are red. This random coloring is equivalent to transmitting a codeword on a block BEC, where a fading is named 'color'. Assume that white bits are perfectly known (i.e.,  $\alpha_1 = +\infty$ ) and erase all red bits (i.e.,  $\alpha_2 = 0$ ). Denote by  $p_m$  the fraction of red bits having full diversity at iteration  $m$ , with  $p_0 = 1/2$  at the first iteration. Filling the erased value of a bit means that the bit attained full diversity. Is there a degree distribution  $(\lambda(x), \rho(x))$  such that  $p_\infty = 1$ ?

The above question can be translated into: Do random rate-1/2 full-diversity LDPC ensembles exist? For an infinite acyclic graph, it is easy to prove the following proposition via standard tree techniques:

$$\text{Proposition 1: } p_{m+1} = 1 - \lambda(1 - \rho(\frac{1+p_m}{2})).$$

Define  $\epsilon_m = (1 - p_m)/2$ , then

$$\epsilon_{m+1} = \frac{1}{2} \lambda(1 - \rho(1 - \epsilon_m)).$$

The above recurrence is the one associated to an iid BEC with  $\epsilon_0 = 1/2$ . Since the code rate is  $R = 1/2 = C(\epsilon_0)$ , the fraction  $p_m$  of non-erased red bits wont converge to 1 when  $m \rightarrow +\infty$ . Thus, the following proposition can be stated for asymptotically large acyclic graph ensembles.

*Proposition 2: A random rate-1/2  $(\lambda(x), \rho(x))$  LDPC ensemble cannot achieve full diversity.*

Some numerical examples can be found in the fifth column of Table I. The 5 given examples are by far too weak to approach full-diversity, i.e.,  $p_\infty = 100\%$ . For any integer  $L$ ,  $L \geq 3$ , the above proposition is still valid for rate-1/ $L$  LDPC ensembles on  $L$ -order diversity channels where a codeword undergoes  $L$  fading instances ( $L$  channel states). A trivial solution for full diversity is to slightly decrease the coding rate. A better solution is to build root-LDPC codes [9][10] and enhance their coding gain as described in the following sections.

#### V. MDS CODES: ROOT-LDPC ENSEMBLES

Let us first enlighten the subtle difference between the MDS property and the property of full diversity. As we already mentioned in sections II and III for diversity-2 channels, an LDPC ensemble is full diversity if the diversity order  $d$  is equal to  $L$ , where  $d$  is defined by (7) and  $L$  is the number of fading instances per codeword, i.e., the intrinsic channel diversity. Under ML decoding, the diversity order  $d$  is upper-bounded by a Singleton-like bound written as  $d \leq \lfloor L(1 - R) \rfloor + 1$  [15][16]. A code is referred to as *MDS code* when equality occurs in the Singleton bound [1]. Since  $d \leq \lfloor L(1 - R) \rfloor + 1 \leq L$ , a full-diversity code is necessarily MDS with coding rate  $R \leq 1/L$ , but an MDS code is not necessarily full diversity.

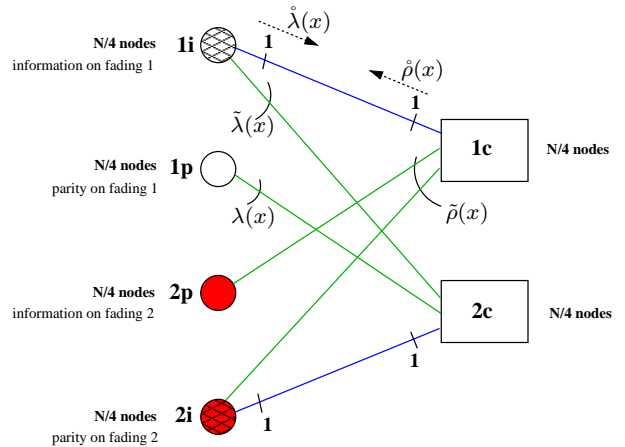


Fig. 2. Compact Tanner graph representation for a rate 1/2 irregular root-LDPC. You can erase one color out of 2, erased information bits are always recovered. This ensemble is full-diversity on a 2-state block fading channel (hence it is MDS).

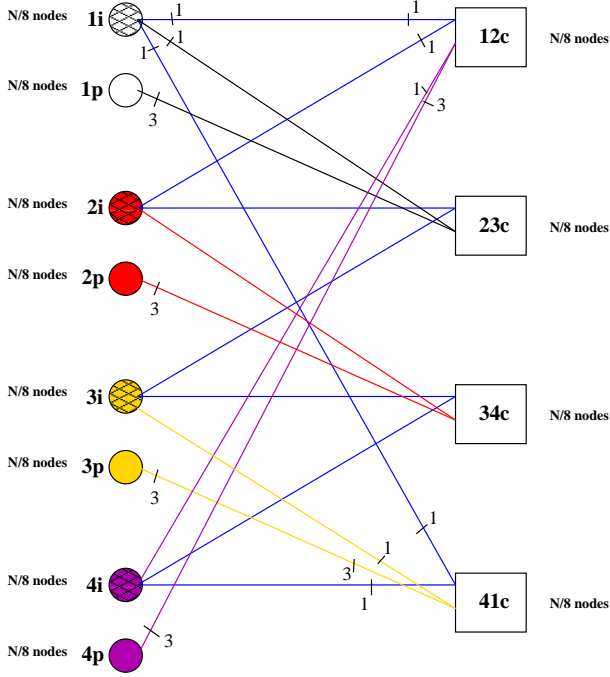


Fig. 3. Compact Tanner graph representation for a rate-1/2 regular (3, 6) root-LDPC. You can erase any 2 colors out of 4, information bits are always recovered. This ensemble attains diversity 3 on a 4-state block fading channel (it is MDS but not full-diversity).

It is not clear if the Singleton bound on  $d$  holds under iterative probabilistic decoding. From the previous section, we know at least that  $R < 1/L$  is a necessary condition for random ensembles to achieve full diversity. Root-LDPC ensembles briefly described in the current section are MDS under iterative decoding and they are full-diversity when the coding rate is tuned to  $1/L$ .

Firstly, binary elements within a codeword are divided into two half-codewords to be transmitted on fading  $\alpha_1$  and  $\alpha_2$  respectively, as explained in section II. Inside each half-codeword, we separate information digits from parity digits. We finally obtain 4 classes of digits, 1i, 1p, 2i, and 2p, each class of size  $N/4$  as illustrated in Fig. 2. Similarly, parity check equations are separated into 2 classes 1c and 2c. The key idea is to guarantee full diversity for information bits only by connecting every information bit to a rootcheck. A *rootcheck* with respect to its root is a checknode with all its leaves red but its root bit being white [9][10]. The second type of rootchecks is obtained by switching white and red.

The root-LDPC structure shown in Fig. 2 is full-diversity ( $d = L = 2$ ). The root-LDPC structure shown in Fig. 3 is MDS ( $R = 1/2$  and  $d = \lfloor L(1 - R) \rfloor + 1 = 3$ ) but not full-diversity ( $d < L = 4$ ). The compact Tanner representation is sufficient to indicate the multiedge-type nature of root-LDPC codes since more than one type of message density will propagate on the graph edges [13]. On the other hand, the bipartite graph representation does

not put in evidence how the edge permutations are defined. Indeed, let us focus on the first family of rate-1/2 root-LDPC ensembles as illustrated in Fig. 2. Consider the set of edges linking 1i and 1p to 2c and the set of edges linking 2i and 2p to 1c. Depending on how these two sets of edges are built, we distinguish between two families of root-LDPC codes. In both families the edge sets connecting 1i and 2i to their rootchecks 1c and 2c respectively are defined by an identity matrix (or equivalently any  $N/4 \times N/4$  permutation matrix). The two root-LDPC families will be introduced after defining some polynomials involved in the degree distribution.

Let  $d_b$  and  $d_c$  denote the maximum degree of bitnodes and checknodes respectively. The left and right degree distributions of a random LDPC ensemble are given by the standard polynomials [5]:

$$\lambda(x) = \sum_{i=2}^{d_b} \lambda_i x^{i-1}, \quad \rho(x) = \sum_{j=2}^{d_c} \rho_j x^{j-1}.$$

The average degree of bitnodes is  $\bar{d}_b = 1/\sum_{i=2}^{d_b} \lambda_i/i$  and  $\bar{d}_c = 1/\sum_{j=2}^{d_c} \rho_j/j$  is the average degree of checknodes. From a node perspective, the distribution coefficients should be divided by the degree and then normalized to get

$$\tilde{\lambda}(x) = \bar{d}_b \sum_{i=2}^{d_b} \lambda_i/i x^{i-1}, \quad \tilde{\rho}(x) = \bar{d}_c \sum_{j=2}^{d_c} \rho_j/j x^{j-1}.$$

Consider a set of bitnodes (e.g., class 1i) with degree distribution  $\lambda(x)$ . Put aside one edge per node, what is the new degree distribution  $\tilde{\lambda}(x)$ ? The new average degree becomes  $\bar{d}_b - 1$  and  $\tilde{\lambda}(x)$  becomes  $\lambda(x)/x$ . Hence, we have  $\bar{d}_b \sum_{i=2}^{d_b} \lambda_i/i x^{i-2} = (\bar{d}_b - 1) \sum_{i=1}^{d_b-1} \tilde{\lambda}_i/i x^{i-1}$  leading to the new definitions

$$\tilde{\lambda}(x) = \sum_{i=1}^{d_b-1} \tilde{\lambda}_i x^{i-1} = \frac{\bar{d}_b}{\bar{d}_b - 1} \sum_{i=1}^{d_b-1} i \lambda_{i+1}/(i+1) x^{i-1},$$

$$\tilde{\rho}(x) = \frac{\bar{d}_c}{\bar{d}_c - 1} \sum_{j=1}^{d_c-1} j \rho_{j+1}/(j+1) x^{j-1}.$$

A simple development of  $\tilde{\lambda}(x)$  expression yields the relation  $\bar{d}_b \lambda(x) = (\bar{d}_b - 1)x\tilde{\lambda}(x) + \lambda(x)$ . A similar relation exists for the polynomial  $\rho(x)$ .

Now, let us carefully examine the set  $S_E$  of edges connecting 1i and 1p to 2c as in Fig. 2. We have  $S_E = S_{1i} \cup S_{1p}$ , with  $|S_{1p}|$  is the number of 1p-2c edges and  $|S_{1i}| = \sum_i (\lambda_i |S_{1p}|/i)(i-1)$  is the number of 1i-2c edges. Next, define the multiedge fraction  $f_e$  as

$$f_e = \frac{|S_{1i}|}{|S_{1i}| + |S_{1p}|} = \frac{\sum_i (i-1) \frac{\lambda_i}{i}}{\sum_i (i-1) \frac{\lambda_i}{i} + 1} = \frac{\bar{d}_b - 1}{2\bar{d}_b - 1}. \quad (8)$$

Finally, the two root-LDPC families are:

- **Root-LDPC( $4\pi$ ) family:** Consider a checknode of a given degree  $j$ . One edge is reserved for the rootbit and  $j - 1$  edges belong to  $S_E$  (i.e., connected to bitnodes

of the opposite color). The checknode will be forced to have  $\ell$  edges from  $S_{1i}$  and  $j - 1 - \ell$  from  $S_{1p}$ , such that  $\lfloor (j - 1)f_e \rfloor \leq \ell \leq \lceil (j - 1)f_e \rceil$  and  $\mathcal{E}[\ell] = (j - 1)f_e$ . This constraint is applied to all checknodes  $\forall j \leq d_c$ . By symmetry, a similar construction is performed on  $S_{2i} \cup S_{2p}$ .

- **Root-LDPC( $2\pi$ ) family:** Consider a checknode of a given degree  $j$ . The  $j - 1$  edges belonging to  $S_E$  are randomly selected. Hence, an edge belongs to  $S_{1i}$  with probability  $f_e$  and it belongs to  $S_{1p}$  with probability  $g_e = 1 - f_e$ . This random selection is applied to all checknodes  $\forall j \leq d_c$ . By symmetry, a similar construction is performed on  $S_{2i} \cup S_{2p}$ .

All 1i-1c and 2i-2c links are assumed to be ordered in a natural order, i.e., two  $N/4 \times N/4$  identity matrices are included in the LDPC parity-check matrix. Then, the Root-LDPC( $4\pi$ ) family is defined by 4 graph permutations whereas Root-LDPC( $2\pi$ ) requires 2 graph permutations only. Density evolution analysis of Root-LDPC( $4\pi$ ) would be then slightly more complicated than density evolution for Root-LDPC( $2\pi$ ). But the main reason for preferring Root-LDPC( $2\pi$ ) will be given in the next section. Furthermore, by default, the name Root-LDPC stands for Root-LDPC( $2\pi$ ). The complete list of density evolution equations for a root-LDPC ensemble is given in (9). Six bitnode $\rightarrow$ checknode densities are updated by 6 recurrent equations, where  $m$  is the decoding iteration number. Functions  $\mu_i(x)$  are the probability density functions of  $\Lambda$  given in (2) with  $\alpha = \alpha_i$ ,  $c_\ell = +1$ , and  $i = 1, 2$ . Update equations in (9) are directly established from the propagation trees depicted in Figures 9-14. In the special case of a regular LDPC, we have  $\tilde{\lambda}(x) = \lambda(x)$ ,  $\tilde{\lambda}(x) = \lambda(x)/x$ ,  $g_1^m(x) = q_1^m(x) \forall m$ , and  $g_2^m(x) = q_2^m(x) \forall m$ . Density evolution equations in [9][10] are meant for regular LDPC and should be modified as in (9) to cope with an irregular distribution.

After this not-so-long introduction to the MDS root-LDPC family and its notations, we proceed in the next section to the study of how diversity propagates under iterative decoding.

## VI. DIVERSITY POPULATION EVOLUTION

Following the same arguments and notations as in previous sections, let us study how full diversity propagates after a given number of decoding iterations in a rate-1/2 root-LDPC ensemble. The extremal unbalanced fading case  $\alpha_2^2 \ll 1 \ll \alpha_1^2$  is assumed in this section, i.e., block BEC channel where white bits are perfectly known and red bits are erased by the channel.

Based on the tree-like neighborhood drawn in Fig. 12, it is obvious that information bits  $2i$  reach full diversity after one decoding iteration only, thanks to checknodes of type  $2c$  (the associated rootchecks). This immediate convergence of information bits toward full diversity is true for both root-LDPC( $4\pi$ ) and root-LDPC( $2\pi$ ) families. Unfortunately, the highest slope  $d = 2$  of  $P_e(\gamma)$  is not sufficient to close the gap to the outage capacity limit as shown in Fig. 5 for code A in its root-LDPC version. On the contrary, code E that

performs 0.6dB from capacity limit has the second advantage of a high coding gain or equivalently high energy coefficients as defined in (3). The link between diversity evolution and coding gain will come to light below.

How can we improve the energy coefficients of information bits? Assume that an erased bitnode  $\vartheta$  from class  $2i$  is connected to  $\delta$  checknodes. On the BEC, the message produced by the rootcheck filling the value of  $\vartheta$  is sufficient. On the block fading channel, it is beneficial to let other checknodes from class  $1c$  solve the value of  $\vartheta$ . Assume that a total of  $\xi$  out of  $\delta$  checknodes are capable of delivering the erased value of  $\vartheta$ . Hence, each of the  $\xi$  messages is greater than or equal to  $\alpha_1^2$ . Then, the total message received by the bitnode  $\vartheta$  is at least  $\xi\alpha_1^2$ . Consequently, the number of checknodes solving the erased information value is a lower bound for the energy coefficient.

Consider a checknode  $\Phi$  of type  $1c$  connected to  $\vartheta$ . All its leaves must be known in order for  $\Phi$  to deliver a message greater than or equal to  $\alpha_1^2$ . But the leaves of  $\Phi$  are all selected in the classes  $2i$  and  $2p$ , which means that some parity bits of class  $2p$  are non-erased after a number of decoding iterations. Thus, energy coefficients of information bits are related to the fraction of full-diversity parity bits. *Diversity Population Evolution* (DPE) studies the evolution of the fraction  $p_m$  of full-diversity parity bits in a root-LDPC versus the decoding iteration number  $m$ , with  $p_0 = 0$  since all bits in class  $2p$  are erased by the channel.

- **DPE for root-LDPC( $4\pi$ ).** The local tree neighborhood of a bitnode  $2p$  is similar to that illustrated in Fig. 14, but the leaves of checknodes  $1c$  must be selected as defined in section V for root-LDPC( $4\pi$ ). Thus, a checknode  $1c$  of degree  $j$  has at least among its leaves  $\lfloor (j - 1)g_e \rfloor - 1$  parity bits from class  $2p$ ,  $g_e = 1 - f_e$ , where  $f_e$  has been established in (8). Assume that  $\rho_i = 0$  for  $i \leq 4$ , then  $\lfloor (j - 1)g_e \rfloor - 1 > 1$ . After one decoding iteration, no parity bit  $2p$  can be solved because it is connected via  $1c$  to other erased  $2p$  parity bits! After two or more iteration, the situation will not improve. Hence, we can announce the following dead-end DPE for this type of root-LDPC.

*Proposition 3:* For root-LDPC( $4\pi$ ), we have  $p_m = 0, \forall m$ .

Root-LDPC( $4\pi$ ) ensembles suffer from their weak coding gain despite their full diversity, a full diversity restricted to information bits.

- **DPE for root-LDPC( $2\pi$ ).** The local tree neighborhood of a bitnode  $2p$  is illustrated in Fig. 14. A leaf of a checknode  $1c$  is randomly selected as  $2i$  with probability  $f_e$  and  $2p$  with probability  $g_e$ . A checknode  $1c$  of degree  $j$  may have no parity bits among its leaves, this event has probability  $f_e^{j-2}$ . So, after one decoding iteration, the fraction of non-erased (i.e. full-diversity) parity bits will be non-negative. Based on the tree graph of Fig. 14, it is easy to prove

*Proposition 4:* For root-LDPC( $2\pi$ ), we have

$$p_{m+1} = 1 - \lambda(1 - \tilde{\rho}(f_e + (1 - f_e)p_m)).$$

The DPE function is defined by

$$DPE(x) = 1 - \lambda \left[ 1 - \tilde{\rho} \left( \frac{x + \sum_i \frac{(i-1)\lambda_i}{i}}{1 + \sum_i \frac{(i-1)\lambda_i}{i}} \right) \right].$$

The proof is similar to standard DE on BEC, one should take into account the leaf selection with fractions  $f_e, g_e$  and take into account the polynomial  $\tilde{\rho}(x)$  for averaging at the checknode output. Some numerical examples for  $p_\infty$  are indicated in the sixth column of Table I. Assuming  $p_\infty$  is known, we would like to get an estimate for the energy coefficients. This is the subject of the next section.

## VII. CODING GAIN EVOLUTION

A diversity of order 2 is associated to a 4th order  $\chi^2$  distribution or any other probability density function which is linear around the origin. In our study, the 4th order  $\chi^2$  distribution is encountered when combining the squared fadings as in (3). An optimal probabilistic LDPC decoder takes the channel messages from (2) and renders new messages whose distribution can be accurately estimated via (9) at a given iteration  $m$ . It is not guaranteed that those output messages can be written in the form  $a\alpha_1^2 + b\alpha_2^2$  added to some noise (but we are sure from exact DE results shown in Fig. 5 that it involves densities that are linear around the origin). The analytical intractability of optimal iterative decoding on gaussian channels with or without fading is a well known fact. On AWGN channels, the evolution procedure is dramatically simplified by assuming that the output distribution is gaussian or a mixture of gaussian, i.e., the output distribution has the same type as the input distribution. This *Gaussian Approximation* technique has been successfully used to study transfer functions of iterative decoding, e.g., as in [11].

Using the block BEC approach as in previous sections and under a  $\chi^2$  *Approximation*, we assume that a log-likelihood-ratio message is always proportional to

$$\xi\alpha_1^2 + \zeta\alpha_2^2,$$

at very high signal-to-noise ratio, where  $\xi, \zeta \in \mathbb{N}$ . Furthermore, we restrict the analysis to the unbalanced fading case where  $\alpha_2^2 \ll 1 \ll \alpha_1^2$ . Thus, we would like to estimate the probability mass function of  $\xi$ , the latter being the energy coefficient of messages  $\xi\alpha_1^2$  delivered from  $2i$  to  $1c$  as in the tree graph drawn in Fig. 12. Messages  $2i \rightarrow 1c$  achieves full diversity after the first decoding iteration and can be used to track the coding gain in further iterations.

The energy coefficient PMF is written as a set of pairs,

$$L = \left\{ (i_k, p_{i_k}) \right\}_{k=1}^{n(L)}$$

where  $\Pr(\xi = i_k) = p_{i_k}$ ,  $i_k \in \mathbb{N}$ , and  $n(L) = |L|$ . If  $L$  contains a pair  $(0, p)$ , it implies that full diversity is not

attained. A high coding gain should translate into a highly spread energy distribution for values of  $\xi$  greater than 1. Three different operations apply on messages during *Coding Gain Evolution* (CGE):

- *OR operation* at the input of checknodes. Given two PMFs  $L_1$  and  $L_2$ , the average PMF  $L = f_e L_1 \text{ OR } g_e L_2$  is

$$L = \left\{ (i_{k1}, p_{i_{k1}}) \right\}_{k1=1}^{n(L_1)} \cup \left\{ (i_{k2}, p_{i_{k2}}) \right\}_{k2=1}^{n(L_2)}$$

The symbol  $\cup$  is the union operation in set theory, except for  $\{(i, p_1)\} \cup \{(i, p_2)\} = \{(i, p_1 + p_2)\}$ .

- *MIN operation* at the output of checknodes. Given two PMFs  $L_1$  and  $L_2$ , the Minimum PMF  $L = \text{Min}(L_1, L_2)$  is

$$L = \left\{ (\text{Min}(i_{k1}, i_{k2}), p_{i_{k1}} \times p_{i_{k2}}) \right\}_{k1=1, k2=1}^{n(L_1), n(L_2)}$$

- *ADD operation* at the output of a bitnode. Given two PMFs  $L_1$  and  $L_2$ , the sum PMF  $L = \text{Add}(L_1, L_2)$  is

$$L = \left\{ (i_{k1} + i_{k2}, p_{i_{k1}} \times p_{i_{k2}}) \right\}_{k1=1, k2=1}^{n(L_1), n(L_2)}$$

The CGE described above can be seen as a simplified sub-optimal version of DE. CGE is suitable for the prediction of the marginal distribution of  $\xi$  on unbalanced block fading channels. The CGE is initialized by two observations  $L_{obs1} = \{(1, 1)\}$  and  $L_{obs2} = \{(0, 1)\}$ , then PMFs evolve through propagation trees as for DE in Figures 9-14.

As a nice simple example, let us apply CGE on the (3,6)-regular root-LDPC ensemble. We use the notations as in (9) for exact DE, but upper cases replace lower cases.

-Decoding Iteration 1:

$$Q_1^1 = G_1^1 = \{(1, 1.0)\} \text{ for } 1i \rightarrow 1c \text{ and } 1p \rightarrow 2c.$$

$$F_1^1 = \{(1, 1.0)\} \text{ for } 1i \rightarrow 2c \text{ messages.}$$

$$Q_2^1 = G_2^1 = \{(0, 1.0)\} \text{ for } 2i \rightarrow 2c \text{ and } 2p \rightarrow 1c.$$

$$F_2^1 = \{(1, 1.0)\} \text{ for } 2i \rightarrow 1c \text{ messages.}$$

-Decoding Iteration 2:

$$Q_1^2 = \{(1, 1.0)\}$$

$$F_1^2 = \{(1, 1 - f_e^5), (2, f_e^5)\}$$

$$Q_2^2 = \{(0, (1 - f_e^4)^2), (0, 2f_e^4(1 - f_e)^4), (0, f_e^8)\}$$

$$F_2^2 = \{(1, 1 - f_e^4), (2, f_e^4)\}.$$

-Decoding Iteration 3:

At this iteration, the reader will notice that finding  $F_2^3$  is already cumbersome by hand, unless OR, MIN, and ADD operations are embedded into a CGE computer program where the 4 evolution trees are taken into account to update the 4 PMFs in parallel. The PMF of  $\chi$  will be tracked via  $F_2^m$ . For many different types of root-LDPC codes, we noticed that  $F_2^m$  reaches its steady state after a small number of iterations, typically around  $m = 20 - 25$  iterations.

Fig. 4 shows the distribution of  $\xi$  in its steady state for 3 different ensembles. Code D has an excellent threshold on BIAWGN and it is taken from [12]. Code E has been selected in a database of 250 ensembles [6]: -In a first step, we evaluated  $p_\infty$  and we kept 5 codes with the highest  $p_\infty$ , -In a second step, we kept code E which had the best distribution of energy coefficients.

As expected, code E outperforms all other codes. Its performance at 0.6dB from the outage capacity limit is the best known in the current literature on LDPC codes for block fading channels. The increase in coding gain due to the full diversity of parity bits has been measured but not controlled. Results illustrated in Table I and Figures 4&5 involves *Uncontrolled Doping* of coding gain. While terminating this preliminary work, the uncontrolled doping found in DPE and CGE lead us to a new construction with *Controlled Doping* described at the end of this section.

The numerical implementation of CGE does offer an interesting insight to better understand why a given ensemble is weak or strong in presence of block fading. Unfortunately, similarly to standard DE, it does not explicitly relates  $p_\infty$  to the distribution of  $\xi$ . We establish now a lower bound on the energy coefficient  $\xi$  with the corresponding PMF as a function of  $p_\infty$  and other code parameters. The lower bound is determined from the tree graph in Fig 12 used to update  $2i \rightarrow 1c$  message densities. After a large number of iterations, the situation becomes well known for all leaves in that tree: information bits  $2i$  are all solved, information bits  $1i$  and parity bits  $1p$  are not erased (same unbalanced fading condition as before), parity bits  $2p$  are solved with probability  $p_\infty$  and erased with probability  $1 - p_\infty$ . Bitnode  $2i$  at the top of the tree has degree  $\delta + 2$ ,  $0 \leq \delta \leq d_b - 2$ . Hence, the energy coefficient (in its lower bound) satisfies  $1 \leq \xi \leq d_b - 1$ . Of course,  $\Pr(\xi = 0) = 0$  because the rootcheck  $2c$  is delivering its  $\alpha_1^2$  message to  $2i$ , this explains why the range of  $\xi$  has been shifted by 1. Focus on checknodes  $1c$  in that tree. Let  $j$  be the checknode degree and  $\ell$  the number of solved parity bits  $2p$  within the  $j - 2$  incoming edges. The checknode of type  $1c$  and fixed degree  $j$  produces an  $\alpha_1^2$  message with probability

$$\sum_{\ell=0}^{j-2} \binom{j-2}{\ell} f_e^{j-2-\ell} (1-f_e)^\ell p_\infty^\ell = (f_e + (1-f_e)p_\infty)^{j-2}$$

Let  $\pi_e$  be the average probability of a checknode  $1c$  generating a non-erased message to  $2i$ . After  $\tilde{\rho}(x)$  expectation, we get

$$\pi_e = \sum_{j=2}^{d_c} \tilde{\rho}_{j-1} (f_e + (1-f_e)p_\infty)^{j-2} = \tilde{\rho}(f_e + (1-f_e)p_\infty).$$

Define the quantity  $F_\infty(\xi, \delta)$ , the probability of energy coefficient  $\xi$  when the bitnode  $2i$  has degree  $\delta + 2$ . We have

$$F_\infty(\xi, \delta) = \binom{\delta}{\xi-1} \pi_e^{\xi-1} (1-\pi_e)^{\delta-\xi+1}$$

After  $\tilde{\lambda}(x)$  expectation, we get

$$\begin{aligned} F_\infty(\xi) &= \sum_{\delta=0}^{d_b-2} \tilde{\lambda}_{\delta+1} \binom{\delta}{\xi-1} \pi_e^{\xi-1} (1-\pi_e)^{\delta-\xi+1} \\ &= \left( \frac{\pi_e}{1-\pi_e} \right)^{\xi-1} \sum_{i=1}^{d_b-1} \binom{i-1}{\xi-1} \tilde{\lambda}_i (1-\pi_e)^{i-1} \end{aligned}$$

The above expression is written in a compressed form and the result is stated as follows:

*Proposition 5:* The energy coefficient admits a lower bound whose PMF is given by

$$F_\infty(\xi = n + 1) = \frac{1}{n!} \left( \frac{\pi_e}{1-\pi_e} \right)^n \left[ x^n \frac{\partial^n \tilde{\lambda}(x)}{\partial x^n} \right]_{x=(1-\pi_e)}$$

where  $1 \leq \xi \leq d_b - 1$  and  $\pi_e = \tilde{\rho}(f_e + (1-f_e)p_\infty)$ . The above analytical lower bound is an alternative to the numerical implementation of CGE. Finally, we terminate this paper by explaining the principle of controlled doping.

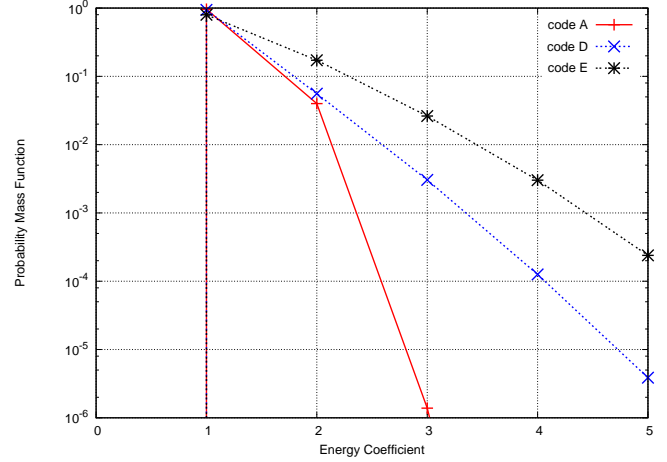


Fig. 4. Distribution of energy coefficients  $\xi$  for three root-LDPC ensembles.

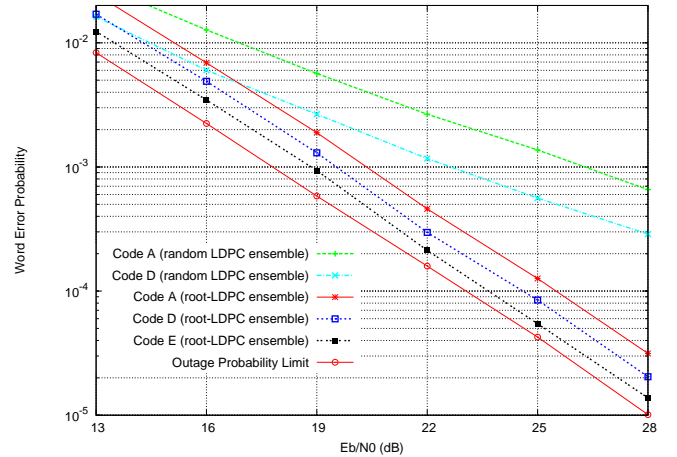


Fig. 5. Word error probability versus signal-to-noise ratio of some rate-1/2 LDPC ensembles on a 2-state block fading channel. The information theoretical limit (outage probability limit) is also indicated.

Controlled doping is done by forcing a fraction  $\theta$  of parity bits to acquire full diversity. This can happen only after 2 decoding iterations or more. As illustrated in Fig. 6, a smaller identity matrix is added to the parity-check matrix  $H$  of the LDPC code. Thus, a fraction  $\theta$  of  $1p$  and  $2p$  has rootchecks of second order. Is there an upper limit for  $\theta$ ? To answer this question, consider the simple case of (3,6)-regular ensemble and its Tanner graph with doping as in Fig. 7. Let  $\delta_a$  denote the number of edges outgoing from a checknode  $2c$  toward the

set of bitnodes  $1i$ . Similarly, let  $\delta_a$  denote the number of edges outgoing from a checknode  $2c$  toward the set of bitnodes  $1\tilde{p}$ . From the graph properties, it can be shown that

$$\delta_a = \frac{2(1-2\theta)}{1-\theta}, \quad \delta_b = \frac{2\theta}{1-\theta}.$$

Because  $\delta_a \geq 0$ , we get  $0 \leq \theta \leq \frac{1}{2}$ . Therefore, let us push  $\theta$  to its maximal value and repeat the same procedure on  $1/4$  of the parity and then  $1/8$  of the parity, etc, as shown in the parity-check matrix at Fig. 8. This new construction yield a full-diversity LDPC code where all variables, both information and parity, attain full diversity after a large number of iterations.

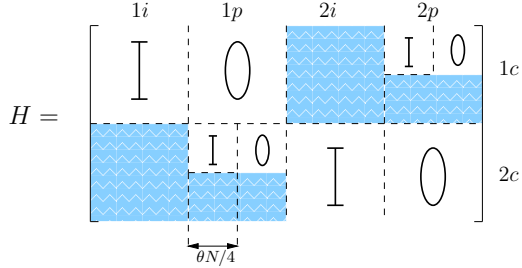


Fig. 6. Root-LDPC with controlled doping. A fraction  $\theta$  of parity bits will acquire full diversity after 2 decoding iterations.

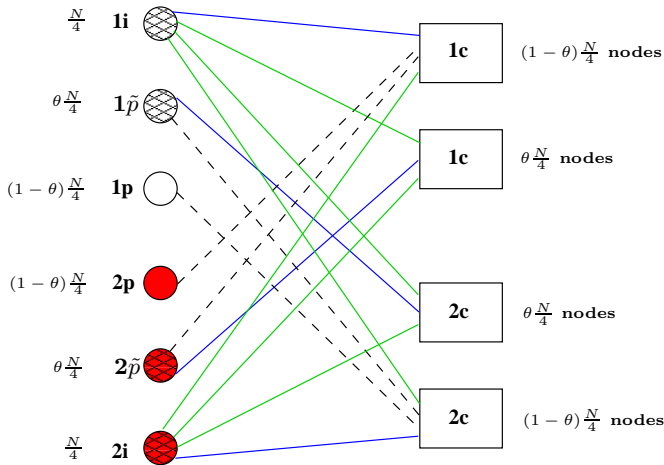


Fig. 7. Compact Tanner graph for a regular-(3,6) root-LDPC where doping is applied to a fraction  $\theta$  of parity bits.

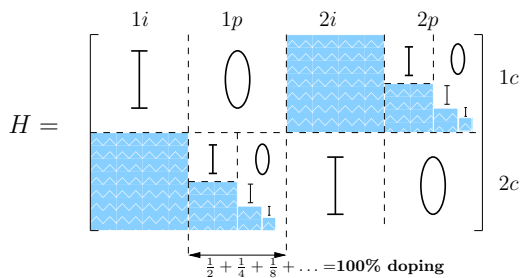


Fig. 8. Root-LDPC with full controlled doping. All parity bits will acquire full diversity after a large number of decoding iterations.

My first research results on coding for block-fading channels [7] was submitted to Allerton's Conference in Illinois, in September 2004. One day before the presentation at Allerton's House, I completely lost my voice because of a strong cold. The same evening, Ralf kindly took a copy of my 10-page paper, he carefully read it late at night. The next day, he made a quick look at my slides before the session. Finally, he gave the talk while I was standing besides the screen, following his lecture, and waving with my hands as to confirm his statements. It was a wonderful souvenir. The audience witnessed a serious but a funny lecture. Today, Ralf passed away while I was writing this manuscript for ITA'2009.

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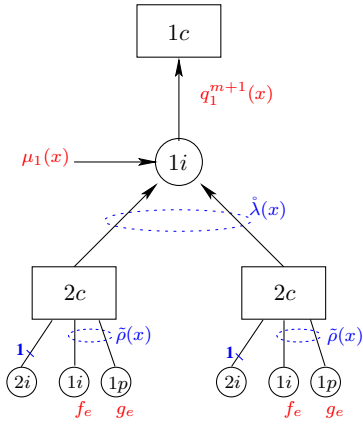


Fig. 9. Local tree neighborhood of bitnode  $1i$ . This tree is used to determine the evolution of messages  $1i \rightarrow 1c$  denoted by  $q_1^m(x)$  at iteration  $m$ .

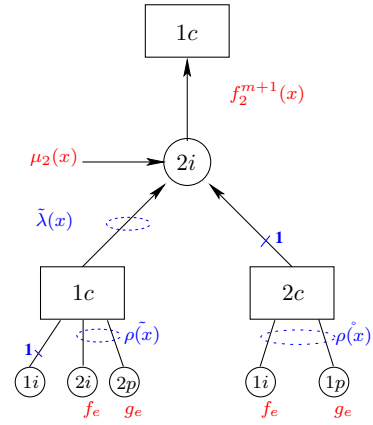


Fig. 12. Local tree neighborhood of bitnode  $2i$ . This tree is used to determine the evolution of messages  $2i \rightarrow 1c$  denoted by  $f_2^m(x)$ .

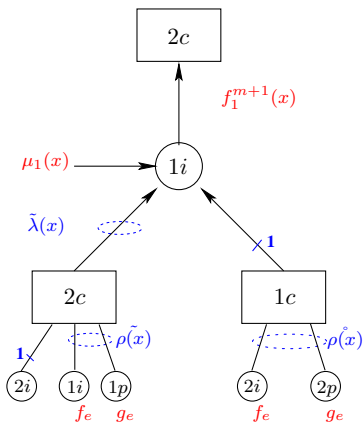


Fig. 10. Local tree neighborhood of bitnode  $1i$ . This tree is used to determine the evolution of messages  $1i \rightarrow 2c$  denoted by  $f_1^m(x)$ .

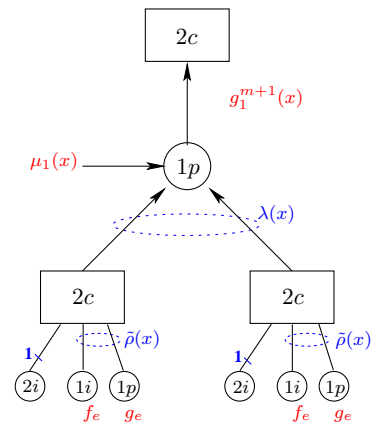


Fig. 13. Local tree neighborhood of bitnode  $1p$ . This tree is used to determine the evolution of messages  $1p \rightarrow 2c$  denoted by  $g_1^m(x)$ .

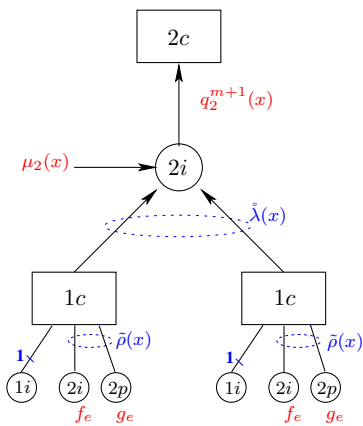


Fig. 11. Local tree neighborhood of bitnode  $2i$ . This tree is used to determine the evolution of messages  $2i \rightarrow 2c$  denoted by  $q_2^m(x)$ .

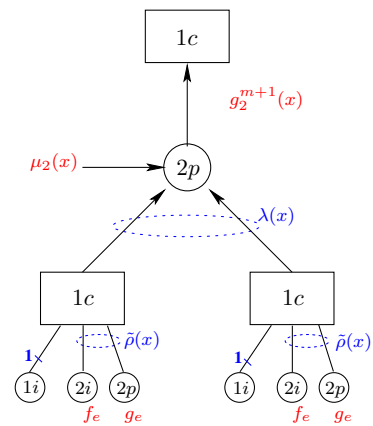


Fig. 14. Local tree neighborhood of bitnode  $2p$ . This tree is used to determine the evolution of messages  $2p \rightarrow 1c$  denoted by  $g_2^m(x)$ .

$$\begin{aligned}
q_1^{m+1}(x) &= \mu_1(x) \otimes \mathring{\lambda}(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
f_1^{m+1}(x) &= \mu_1(x) \otimes \tilde{\lambda}(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \otimes \mathring{\rho}(f_e f_2^m(x) + g_e g_2^m(x)) \\
q_2^{m+1}(x) &= \mu_2(x) \otimes \mathring{\lambda}(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))) \\
f_2^{m+1}(x) &= \mu_2(x) \otimes \tilde{\lambda}(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))) \otimes \mathring{\rho}(f_e f_1^m(x) + g_e g_1^m(x)) \\
g_1^{m+1}(x) &= \mu_1(x) \otimes \lambda(\tilde{\rho}(q_2^m(x), f_e f_1^m(x) + g_e g_1^m(x))) \\
g_2^{m+1}(x) &= \mu_2(x) \otimes \lambda(\tilde{\rho}(q_1^m(x), f_e f_2^m(x) + g_e g_2^m(x))).
\end{aligned} \tag{9}$$


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Name	Left Degrees	Right Degrees	AWGN Threshold	$p_\infty$ (random)	$p_\infty$ (root)	Energy Distribution at $\infty$ (root)	Gap to Outage Limit $10^{-2\dots-3}$ $10^{-4\dots-5}$
code A	$\lambda_3 = 1$	$\rho_6 = 1$	1.10	09.67%	07.82%	0.960 0.039 1e-6 4e-13	2.45 2.38
code B	$\lambda_2 = 0.4$ $\lambda_3 = 0.6$	$\rho_5 = 1$	1.07	19.29%	17.24%	0.817 0.173 0.009 5e-5	1.81 1.58
code C	$\lambda_{2,3,4}$	$\rho_{5,6}$	0.77	15.68%	13.30%	0.855 0.136 0.008 2e-4	1.64 1.48
code D	$\lambda_{2,3}$ $\lambda_{6,7,8,9,10}$ $\lambda_{28}$ $\lambda_{30}$	$\rho_{8,9,10}$	0.27	06.77%	04.95%	0.924 0.073 0.003 2e-5	1.66 1.46
code E	$\lambda_{2,3,4,5}$ $\lambda_{10}$	$\rho_{6,7}$	1.50	21.45%	16.96%	0.798 0.172 0.026 0.003	0.94 0.61

TABLE I  
CHARACTERISTICS OF 5 CODE ENSEMBLES FOR RATE 1/2.

AWGN threshold and gap to outage capacity limit are expressed in decibels. The 3 most right columns are for root-LDPC( $2\pi$ ) ensembles. The limit  $p_\infty$  for random ensembles includes both erased parity and erased information bits. The limit  $p_\infty$  for root-LDPC involves erased parity bits only.