Independent Source Coding for Control over Noiseless Channels

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***Abstract*—By focusing on a class of source coding schemes built around entropy coded dithered quantizers, we develop a framework to deal with average data-rate constraints in a tractable manner that combines ideas from both information and control theories. We focus on a situation where a noisy linear system has been designed assuming transparent feedback and, due to implementation constraints, a source coding scheme has to be deployed in the feedback path. We give a closed form expression for the minimal average data-rate required to achieve a given performance level, and also study the interplay between stability and average data-rates for the considered architecture.**

I. INTRODUCTION

Consider the networked control system (NCS) of Figure 1, where P is a given proper real rational transfer function, d is an exogenous signal, e is a signal related to closed loop performance, y can be measured, u is a manipulable input, and the channel is a noiseless digital channel. Partition P as

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad \begin{bmatrix} e \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} d \\ u \end{bmatrix},$$

where P_{22} is SISO and strictly proper, and both P_{12} and P_{21} are non-zero. The initial state x_o of P is a second order random variable and d is a second order wss process with PSD $S_d > 0$.

If no constraints but causality are imposed on the source encoder and decoder, then mild conditions guarantee that the NCS of Figure 1 is mean square stable (see definition in Section III) if and only if the average data-rate $\mathcal{R}$ of the coding scheme (i.e., across the channel) satisfies [1]

$$\mathcal{R} > \mathcal{R}_{\text{NE}} \triangleq \sum_{i=1}^{n_p} \ln |p_i| \,, \tag{1}$$

where p_i is the i^{th} unstable pole of P. When performance guarantees are sought, then much less is known. A causal rate-distortion inspired approach was followed in [2], but no achievable rate regions where established there.

In this work we assume that the feedback system of Figure 1 is internally stable and well-posed when $u = y$. We study the minimal average data-rate that allows one to attain a given performance level (as measured by the stationary variance σ_e^2 of e) in the considered NCS. (Details can be found in [3], [4].)

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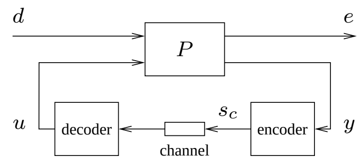

Fig. 1. NCS closed over a noiseless digital channel.

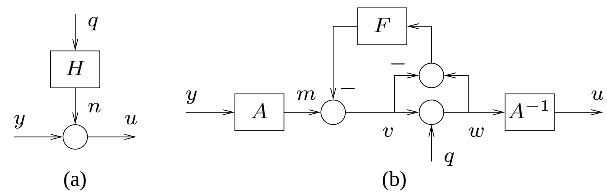

Fig. 2. (a) Independent source coding scheme and (b) equivalent rewriting.

II. THE SOURCE CODING SCHEME

The average data-rate of the source coding scheme in Figure 1 is defined via

$$\mathcal{R} \triangleq \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i),$$

where $R(i)$ is the expected length (in nats) of $s_c(i)$.

Theorem 1: Consider a causal source coding scheme inside a feedback loop, as in Figure 1. Under suitable assumptions (see [3]), $\mathcal{R} \geq I_\infty(y \to u)$, where $I_\infty(y \to u)$ denotes the directed mutual information rate defined by Massey [5]. ■

A. Independent source coding schemes

We focus on a class of source coding schemes:

Definition 1: The source coding scheme of Figure 1 is said to be independent iff the assumptions of Theorem 1 hold and the noise sequence $n \triangleq u - y$ obeys $n = \Omega q$, where q is a second order zero-mean i.i.d. sequence, q is independent of (d, x_o), and Ω is a stable and stably and causally invertible filter with deterministic initial state (see Figure 2(a)). ■

Any independent source coding scheme can be written as shown in Figure 2(b), where v and w are auxiliary signals, q is as in Definition 1, and A and F are auxiliary stable filters, and both A and $1-F$ are stably and causally invertible. Moreover:

Theorem 2: If an independent coding schemes is written as in Figure 2(b), then $I_\infty(y \to u) = I_\infty(v \to w)$. ■

The previous fact allows one to focus on $I_\infty(v \to w)$ instead of $I_\infty(y \to u)$. It is possible to establish upper bounds on $I_\infty(v \to w)$ that depend only on second order properties of w and v:

Theorem 3: If in the NCS of Figure 1 the source coding scheme is independent, then[1]

$$I_\infty(v \to w) \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + D(q(k)||q_G(k))$$
$$\leq \frac{1}{2}\ln(1+\gamma) + D(q(k)||q_G(k)), \quad \gamma \triangleq \frac{\sigma_v^2}{\sigma_q^2},$$

where S_w is the stationary PSD of w, σ_v^2 is the stationary variance of v, and σ_q^2 is the variance of q. Equality holds in the first inequality iff (d, x_o, q) is Gaussian, whereas equality holds in the second inequality of iff S_w/σ_q^2 is constant a.e. ■

An additional key fact of independent source coding schemes is stated next (see also [7]):

Theorem 4: For any given independent source coding scheme, there exists another independent source coding scheme, with the same noise color Ω and the same directed mutual information rate across it, such that the gap between the left and right hand sides of the second inequality of Theorem 3 can be made arbitrarily small. ■

B. Entropy Coded Dithered Quantizers (ECDQ)

We consider ECDQs as defined in [8] with dither d_h, input v and output w.

Theorem 5: Consider the system of Figure 2(b) with A and F as before. If an ECDQ is used as the link between v and w, $\Delta < \infty$ and the dither d_h is i.i.d., independent of (x_o, d), and uniformly distributed on $(-\Delta/2, \Delta/2)$, then the system of Figure 2(b) becomes an independent source coding scheme and the entropy coder inside the ECDQ can be chosen so that

$$\mathcal{R} < \frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \frac{S_w}{\sigma_q^2} d\omega + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$$

If the entropy coder inside the ECDQ is memoryless, then it can be chosen so that

$$\mathcal{R} < \frac{1}{2}\ln(1+\gamma) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2.$$

■

III. Mean Square Stability

***Definition** 2:* A system $x(k+1) = M_x x(k) + M_w w(k)$, $k \in \mathbb{N}_0$, $x(k) \in \mathbb{R}^n$, $x(0) = x_o$, where x_o is a second order random variable, M_x and M_w are constant matrices, and w is a second order wss process is said to be mean square stable (MSS) iff there exist finite $\mu \in \mathbb{R}^n$ and a finite and positive semi-definite $M \in \mathbb{R}^{n\times n}$ such that $\lim_{k\to\infty} \mathcal{E}\{x(k)\} = \mu$, $\lim_{k\to\infty} \mathcal{E}\{x(k)x(k)^T\} = M$, regardless of the initial state x_o. ■

The previous definition applies to the NCS of Figure 1, when the source coding scheme is independent.

Theorem 6: Consider the setup and assumptions of Theorem 5. Then, irrespective of whether the entropy coder inside the ECDQ has memory or not, the minimal average data-rate compatible with MSS, say $\mathcal{R}_{\text{MSS}}$, satisfies (cf. (1))

$$\mathcal{R}_{\text{NE}} < \mathcal{R}_{\text{MSS}} < \sum_{i=1}^{n_p} \ln|p_i| + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2.$$

■

The bound on $\mathcal{R}_{\text{MSS}}$ provided by Theorem 6 is, by construction, achievable.

IV. Performance Guarantees

Define $S \triangleq (1-P_{22})^{-1}$, $T_{de} \triangleq P_{11} + P_{12}SP_{21}$, $Y \triangleq |S|^2 \sqrt{P_{12}^H P_{12} P_{21} S_d P_{21}^H}$, and $D_{\inf} \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} T_{de} S_d T_{de}^H d\omega$.

Theorem 7: Consider the setup and assumptions of Theorem 5. If $D_{\inf} < D < \infty$, then, irrespective of whether the entropy coder inside the ECDQ has memory or not, the minimal average data-rate that allows one to attain a performance level $\sigma_e^2 \leq D$, say $\mathcal{R}_D$, obeys

$$\mathcal{R}_D < \frac{1}{2}\ln(1+\gamma_D) + \frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2, \qquad (2)$$

where

$$\gamma_D = \exp\left(\frac{1}{\pi}\int_{-\pi}^{\pi} \ln\left(\sqrt{\frac{Y^2}{\lambda_D} + |S|^2} + \frac{Y}{\sqrt{\lambda_D}}\right) d\omega\right) - 1,$$

and λ_D is the unique positive real satisfying

$$D = D_{\inf} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda_D Y}{2\left(\sqrt{Y^2 + \lambda_D |S|^2} + Y\right)} d\omega.$$

If $D < D_{\inf}$ then $\mathcal{R}_D$ does not exist, whereas achieving $D = D_{\inf}$ requires an infinite average data-rate. ■

The bound on $\mathcal{R}_D$ provided by Theorem 6 is, by construction, achievable. To our knowledge, our result corresponds to the first closed form bound on the achievable average data-rate needed to attain a given performance level.

We note that, if (x_o, d) is Gaussian, then the bound on $\mathcal{R}_D$ given in (2) is tight up to $\frac{1}{2}\ln\left(\frac{2\pi e}{12}\right) + \ln 2$ nats/sample.

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[1] $D(x||y)$ denotes relative entropy [6].