Polymatroids with Network Coding

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Abstract—The problem of network coding for multicasting a single source to multiple sinks has first been studied by Ahlswede, Cai, Li and Yeung in 2000, in which they have established the celebrated max-flow mini-cut theorem on non-physical information flow over a network of independent channels. On the other hand, in 1980, Han had studied the case with correlated multiple sources and a single sink from the viewpoint of polymatroidal functions in which a necessary and sufficient condition has been demonstrated for reliable transmission over the network. This paper presents an attempt to unify both cases, which leads to establish a necessary and sufficient condition for reliable transmission over a noisy network for multicasting correlated multiple sources altogether to every multiple sinks. Furthermore, we address also the problem of transmitting "independent" sources over a multiple-access-type of network as well as over a broadcast-type of network, which reveals that the (co-) polymatroidal structures are intrinsically involved in these types of network coding.

1. INTRODUCTION

The problem of network coding for multicasting a single source to multiple sinks has first been studied by Ahlswede, Cai, Li and Yeung [1] in 2000, in which they have established the celebrated max-flow mini-cut theorem on non-physical information flow over a network of independent channels. On the other hand, in 1980, Han [3] had studied the case with correlated multiple sources and a single sink from the viewpoint of polymatroidal functions in which a necessary and sufficient condition has been demonstrated for reliable transmission over a network.

This paper presents an attempt to unify both cases and to generalize it to quite a general case with stationary ergodic correlated sources and noisy channels (with arbitrary nonnegative real values of capacity that are not necessarily integers) satisfying the strong converse property (cf. Verdú and Han [7], Han [5]), which leads to establish a necessary and sufficient condition for reliable transmission over a noisy network for multicasting correlated multiple sources altogether to every multiple sinks.

It should be noted here that in such a situation with correlated multiple sources, the central issue turns out to be how to construct the matching condition between source and channel (i.e., joint source-channel coding), instead of of the traditional concept of capacity region (i.e., channel coding), although in the special case with non-correlated independent multiple sources the problem reduces again to how to describe the capacity region.

Several network models with correlated multiple sources have been studied by some people, e.g., by Barros and Servetto [10], Ho, Médard, Effros and Koetter [13], Ho, Médard, Koetter, Karger, Effros, Shi and Leong [14], Ramamoorthy, Jain, Chou and Effros [15]. Among others, [13], [14] and [15] consider (without attention to the converse part) a very restrictive case of error-free network coding for two stationary memoryless correlated sources with a single sink to study the error exponent problem, where we notice that all the arguments in [13], [14] and [15] can be validated only within the narrow class of stationary memoryless sources of integer bit rates and error-free channels (i.e., the identity mappings) all with one bit (or integer bits) capacity (these restrictions are needed solely to invoke “Menger’s theorem” in graph theory). The main result in the present paper is quite free from such severe restrictions, because we can dispense with the use of Menger’s theorem.

On the other hand, [10] revisits the same model as in Han [3], while [15] focuses on the network
with two correlated sources and two sinks to discuss the separation problem of distributed source coding (based on Slepian-Wolf theorem) and network coding. It should be noted that, in the case of networks with correlated multiple sources, such a separation problem is another central issue, although it is yet far from fully solved. In this paper, we mention a sufficient condition for separability in the case with multiple sources and multiple sinks. (cf. Remark 5.2).

The present paper consists of six sections. In Section 2 notations and preliminaries are described, and in Section 3 we state the main result as well as its proof. In Section 4 two examples are shown. Section 5 provides another type of necessary and sufficient condition for transmissibility. Also, some detailed comments on the previous papers are given. Finally, in Section 6 we address the routing capacity problems with polymatroids and co-polymatroids.

2. Preliminaries and Notations

A. Communication networks

Let us consider an acyclic directed graph $G = (V, E)$ where $V = \{1, 2, \cdots, |V|\}$ ($|V| < +\infty$), $E \subset V \times V$, but $(i, i) \not\in E$ for all $i \in V$. Here, elements of $V$ are called nodes, and elements $(i, j)$ of $E$ are called edges or channels from $i$ to $j$. Each edge $(i, j)$ is assigned the capacity $c_{ij} \geq 0$, which specifies the maximum amount of information flow passing through the channel $(i, j)$. If we want to emphasize the graph thus capacitated, we write it as $G = (V, E, C)$ where $C = (c_{ij})_{(i, j) \in E}$. A graph $G = (V, E, C)$ is sometimes called a (communication) network, and indicated also by $N = (V, E, C)$. We consider two fixed subsets $\Phi, \Psi$ of $V$ such that $\Phi \cap \Psi = \emptyset$ (the empty set) with

$\Phi = \{s_1, s_2, \cdots, s_p\}$,

$\Psi = \{t_1, t_2, \cdots, t_q\}$,

where elements of $\Phi$ are called source nodes, while elements of $\Psi$ are called sink nodes. Here, to avoid subtle irregularities, we assume that there are no edges $(i, s)$ such that $s \in \Phi$.

Informally, our problem is how to simultaneously transmit the information generated at the source nodes in $\Phi$ altogether to all the sink nodes in $\Psi$. More formally, this problem is described as in the following subsection.

B. Sources and channels

Each source node $s \in \Phi$ generates a stationary and ergodic source process

$$X_s = (X_s^{(1)}, X_s^{(2)}, \cdots),$$

where $X_s^{(i)} (i = 1, 2, \cdots)$ takes values in finite source alphabet $\mathcal{X}_s$. Throughout in this paper we consider the case in which the whole joint process $X_\Phi \equiv (X_s)_{s \in \Phi}$ is stationary and ergodic. It is then evident that the joint process $X_T \equiv (X_s)_{s \in T}$ is also stationary and ergodic for any $T$ such that $\emptyset \neq T \subset \Phi$. The component processes $X_s (s \in \Phi)$ may be correlated. We write $X_T$ as

$$X_T = (X_T^{(1)}, X_T^{(2)}, \cdots)$$

and put

$$X_T^n = (X_T^{(1)}, X_T^{(2)}, \cdots, X_T^{(n)}),$$

where $X_T^{(i)} (i = 1, 2, \cdots)$ takes values in $\mathcal{X}_T = \prod_{s \in T} \mathcal{X}_s$.

On the other hand, it is assumed that all the channels $(i, j) \in E$, specified by the transition probabilities $w_{ij} : A_{ij}^n \rightarrow B_{ij}^n$ with finite input alphabet $A_{ij}$ and finite output alphabet $B_{ij}$, are statistically independent and satisfy the strong converse property (see Verdú and Han [7]). It should be noted here that stationary and memoryless (noisy or noiseless) channels with finite input/output alphabets satisfy, as very special cases, this property (cf. Gallager [8], Han [5]). Barros and Servetto [10] have considered the case of stationary and memoryless sources/channels with finite alphabets. The following lemma plays a crucial role in establishing the relevant converse of the main result:
Lemma 2.1: (Verdú and Han [7]) The channel capacity \( c_{ij} \) of a channel \( w_{ij} \) satisfying the strong converse property with finite input/output alphabets is given by

\[
c_{ij} = \lim_{n \to \infty} \frac{1}{n} \max_{X^n} I(X^n; Y^n),
\]

where \( X^n, Y^n \) are the input and the output of the channel \( w_{ij} \), respectively, and \( I(X^n; Y^n) \) is the mutual information (cf. Cover and Thomas [9]).

C. Encoding and decoding

In this section let us state the necessary operation of encoding and decoding for network coding with correlated multiple sources to be multicast to multiple sinks.

With arbitrarily small \( \delta > 0 \) and \( \varepsilon > 0 \), we introduce an \((n, (R_{ij})(i,j) \in E, \delta, \varepsilon)\) code as the one as specified by (2.4) ~ (2.9) below, where we use the notation \([1, M]\) to indicate \(\{1, 2, \ldots, M\}\). How to construct a “good” \((n, (R_{ij})(i,j) \in E, \delta, \varepsilon)\) code will be shown in Direct part of the proof of Theorem 3.1.

1) For all \((s, j) \ (s \in \Phi)\), the encoding function is

\[
f_{sj} : \mathcal{X}_{s}^{n} \to [1, 2^{n(R_{sj}-\delta)}],
\]

where the output of \(f_{sj}\) is carried over to the encoder \(\varphi_{sj}\) of channel \(w_{sj}\), while the decoder \(\psi_{sj}\) of \(w_{sj}\) outputs an estimate of the output of \(f_{sj}\), which is specified by the stochastic composite function:

\[
h_{sj} \equiv \psi_{sj} \circ w_{sj} \circ \varphi_{sj} \circ f_{sj} : \mathcal{X}_{s}^{n} \to [1, 2^{n(R_{sj}-\delta)}];
\]

2) For all \((i, j) \ (i \not\in \Phi)\), the encoding function is

\[
f_{ij} : \prod_{k: (k,i) \in E} [1, 2^{n(R_{ki}-\delta)}] \to [1, 2^{n(R_{ij}-\delta)}],
\]

where the output of \(f_{ij}\) is carried over to the encoder \(\varphi_{ij}\) of channel \(w_{ij}\), while the decoder \(\psi_{ij}\) of \(w_{ij}\) outputs an estimate of the output of \(f_{ij}\), which is specified by the stochastic composite function:

\[
h_{ij} \equiv \psi_{ij} \circ w_{ij} \circ \varphi_{ij} \circ f_{ij} : \prod_{k: (k,i) \in E} [1, 2^{n(R_{ki}-\delta)}] \to [1, 2^{n(R_{ij}-\delta)}].
\]

Here, if \(\{k : (k, i) \in E\}\) is empty, we use the convention that \(f_{ij}\) is an arbitrary constant function taking a value in \([1, 2^{n(R_{ij}-\delta)}]\);

3) For all \(t \in \Psi\), the decoding function is

\[
g_{t} : \prod_{k: (k,t) \in E} [1, 2^{n(R_{kt}-\delta)}] \to \mathcal{X}_{\Phi, t}^{n}.
\]

4) Error probability

All sink nodes \(t \in \Psi\) are required to reproduce a “good” estimate \(\hat{X}_{\Phi, t}^{n}\) \((\equiv \text{the output of the decoder } g_{t})\) of \(X_{\Phi, t}^{n}\), through the network \(N = (V, E, C)\), so that the error probability \(\Pr\{\hat{X}_{\Phi, t}^{n} \neq X_{\Phi, t}^{n}\}\) be as small as possible. Formally, for all \(t \in \Psi\), the probability \(\lambda_{n,t}\) of decoding error committed at sink \(t\) is required to satisfy

\[
\lambda_{n,t} \equiv \Pr\{\hat{X}_{\Phi, t}^{n} \neq X_{\Phi, t}^{n}\} \leq \varepsilon
\]

for all sufficiently large \(n\). Clearly, \(\hat{X}_{\Phi, t}^{n}\) are the random variables induced by \(X_{\Phi, t}^{n}\) that were generated at all source nodes \(s \in \Phi\).

We now need the following definitions.

Definition 2.1 (rate achievability): If there exists an \((n, (R_{ij})(i,j) \in E, \delta, \varepsilon)\) code for any arbitrarily small \(\varepsilon > 0\) as well as any sufficiently small \(\delta > 0\), and for all sufficiently large \(n\), then we say that the rate \((R_{ij})(i,j) \in E\) is achievable for the network \(G = (V, E)\).

Definition 2.2 (transmissibility): If, for any small \(\tau > 0\), the augmented capacity rate \((R_{ij} = c_{ij} + \tau)(i,j) \in E\) is achievable, then we say that the source \(X_{\Phi}\) is transmissible over the network \(N = (V, E, C)\), where \(c_{ij} + \tau\) is called the \(\tau\)-capacity of channel \((i, j)\).
The proof of Theorem 3.1 (both of the converse part and the direct part) are based on these definitions.

D. Capacity functions

Let \( \mathcal{N} = (V, E, C) \) be a network. For any subset \( M \subset V \) we say that \((M, \overline{M})\) (or simply, \(M\)) is a cut and
\[
E_M \equiv \{(i, j) \in E | i \in M, j \in \overline{M}\}
\]
the cutset of \((M, \overline{M})\) (or simply, of \(M\)), where \(\overline{M}\) denotes the complement of \(M\) in \(V\). Also, we call
\[
c(M, \overline{M}) \equiv \sum_{(i, j) \in E, i \in M, j \in \overline{M}} c_{ij}
\]
the value of the cut \((M, \overline{M})\). Moreover, for any subset \(S\) such that \(\emptyset \neq S \subset \Phi\) (the source node set) and for any \(t \in \Psi\) (the sink node sets), define
\[
\rho_t(S) = \min_{M: S \subset M, t \in \overline{M}} c(M, \overline{M});
\]
(2.11)
\[
\rho_\mathcal{N}(S) = \min_{t \in \Psi} \rho_t(S).
\]
(2.12)
We call this \(\rho_\mathcal{N}(S)\) the capacity function of \(S \subset V\) for the network \(\mathcal{N} = (V, E, C)\).

**Remark 2.1:** A set function \(\sigma(S)\) on \(\Phi\) is called a co-polymatroid \(^*\) (function) if it holds that
1) \(\sigma(\emptyset) = 0\),
2) \(\sigma(S) \leq \sigma(T)\) (\(S \subset T\)),
3) \(\sigma(S \cap T) + \sigma(S \cup T) \geq \sigma(S) + \sigma(T)\).

It is not difficult to check that \(\sigma(S) = H(X_S | X_\Phi)\) is a co-polymatroid (see, Han [3]). On the other hand, a set function \(\rho(S)\) on \(\Phi\) is called a polymatroid if it holds that
1') \(\rho(\emptyset) = 0\),
2') \(\rho(S) \leq \rho(T)\) (\(S \subset T\)),
3') \(\rho(S \cap T) + \rho(S \cup T) \leq \rho(S) + \rho(T)\).

\(^*\)In Zhang, Chen, Wicker and Berger [18], the co-polymatroid here is called the contra-polymatroid.

It is also not difficult to check that for each \(t \in \Psi\) the function \(\rho_t(S)\) in (2.11) is a polymatroid (cf. Han [3], Meggido [22]), but \(\rho_\mathcal{N}(S)\) in (2.12)) is not necessarily a polymatroid. These properties have been fully invoked in establishing the matching condition between source and channel for the special case of \(|\Psi| = 1\) (cf. Han [3]).

With these preparations we will demonstrate the main result in the next section.

3. Main Result

The problem that we deal with here is not that of establishing the “capacity region” as usual, because the concept of “capacity region” does not make sense for the general network with correlated sources. Instead, we are interested in the matching problem between the correlated source \(X_\Phi\) and the network \(\mathcal{N} = (V, E, C)\) (transmissibility: cf. Definition 2.2). Under what condition is such a matching possible? This is the key problem here. An answer to this question is just our main result to be stated here.

**Theorem 3.1:** The source \(X_\Phi\) is transmissible over the network \(\mathcal{N} = (V, E, C)\) if and only if
\[
H(X_S | X_\Phi) \leq \rho_\mathcal{N}(S) \quad (\emptyset \neq \forall S \subset \Phi)
\]
(3.13)
holds.

**Remark 3.1:** The case of \(|\Psi| = 1\) was investigated by Han [3], and subsequently revisited by Barros and Servetto [10], while the case of \(|\Phi| = 1\) was investigated by Ahlswede, Cai, Li and Yeung [1].

**Remark 3.2:** If the sources are mutually independent, (3.13) reduces to
\[
\sum_{i \in S} H(X_i) \leq \rho_\mathcal{N}(S) \quad (\emptyset \neq \forall S \subset \Phi).
\]
Then, setting the rates as \(R_i = H(X_i)\) we have another equivalent form:
\[
\sum_{i \in S} R_i \leq \rho_\mathcal{N}(S) \quad (\emptyset \neq \forall S \subset \Phi).
\]
(3.14)
This specifies the capacity region of independent message rates in the traditional sense. In other
words, in case the sources are independent, the concept of capacity region makes sense. In this case too, channel coding looks like for non-physical flows (as for the case of $|\Phi| = 1$, see Ahlswede, Cai, Li and Yeung [1]; and as for the case of $|\Phi| > 1$ see, e.g., Koetter and Medard [16], Li and Yeung [17]). It should be noted that formula (3.14) is not derivable by a naive extension of the arguments as used in the case of single-source ($|\Phi| = 1$), irrespective of the comment in [1].

Proof of Theorem 3.1: The proof is based on joint typicality, strong converse property of the channel, acyclicity of the network, ergodicity of the source, random coding arguments, Fano’s inequality, subtle classification of the error patterns, and so on. As for the details, see Han [4]).

4. Examples

In this section we show two examples of Theorem 3.1 with $\Phi = \{s_1,s_2\}$ and $\Psi = \{t_1,t_2\}$.

Example 1. Consider the network as in Fig.1(called the butterfly) where all the solid edges have capacity 1 and the independent sources $X_1, X_2$ are binary and uniformly distributed. The capacity function of this network is computed as follows:

\[
\rho_1(\{s_2\}) = \rho_2(\{s_1\}) = 1,
\rho_1(\{s_1\}) = \rho_2(\{s_2\}) = 2,
\rho_1(\{s_1,s_2\}) = \rho_2(\{s_1,s_2\}) = 2;
\]

\[
\rho_N(\{s_1\}) = \min(\rho_1(\{s_1\}), \rho_2(\{s_1\})) = 1,
\rho_N(\{s_2\}) = \min(\rho_1(\{s_2\}), \rho_2(\{s_2\})) = 1,
\rho_N(\{s_1,s_2\}) = \min(\rho_1(\{s_1,s_2\}), \rho_2(\{s_1,s_2\})) = 2.
\]

On the other hand,

\[
H(X_1|X_2) = H(X_1) = 1,
H(X_2|X_1) = H(X_2) = 1,
H(X_1X_2) = H(X_1) + H(X_2) = 2.
\]

Therefore, condition (3.13) in Theorem 3.1 is satisfied with equality, so that the source is transmissible over the network. Then, how to attain this transmissibility? That is depicted in Fig.2 where $\oplus$ denotes the exclusive OR. Fig. 3 depicts the corresponding capacity region, which is within the framework of the previous work (e.g., see Ahlswede et al. [1]).

Example 2. Consider the network with noisy channels as in Fig.4 where the solid edges have capacity 1 and the broken edges have capacity $h(p) < 1$. Here, $h(p)$ ($0 < p < \frac{1}{2}$) is the binary entropy defined by $h(p) = -p \log_2 p - (1-p) \log_2 (1-p)$. The source $(X_1, X_2)$ generated at the nodes $s_1, s_2$ is the binary symmetric source with crossover probability $p$, i.e.,

\[
\Pr\{X_1 = 1\} = \Pr\{X_1 = 0\} = \Pr\{X_2 = 1\}
\]
Notice that the entropy of $X_1$, $X_2$ is

to recover

\[ \rho_{t_1}(\{s_2\}) = \rho_{t_2}(\{s_1\}) = h(p), \]
\[ \rho_{t_1}(\{s_1, s_2\}) = \rho_{t_2}(\{s_1, s_2\}) = 2; \]
\[ \rho_N(\{s_2\}) = \min(\rho_{t_1}(\{s_2\}), \rho_{t_2}(\{s_2\})) = h(p), \]
\[ \rho_N(\{s_1\}) = \min(\rho_{t_1}(\{s_1\}), \rho_{t_2}(\{s_1\})) = h(p), \]
\[ \rho_N(\{s_1, s_2\}) = \min(\rho_{t_1}(\{s_1, s_2\}), \rho_{t_2}(\{s_1, s_2\})) = 2. \]

On the other hand,
\[ H(X_1|X_2) = h(p), \]
\[ H(X_2|X_1) = h(p), \]
\[ H(X_1X_2) = 1 + h(p). \]

Therefore, condition (3.13) in Theorem 3.1 is satisfied with strict inequality, so that the source is transmissible over the network. Then, how to attain this transmissibility? That is depicted in Fig. 5 where $X_1, X_2$ are $n$ independent copies of $X_1, X_2$, respectively, and $A$ is an $m \times n$ matrix ($m = nh(p) < n$). Notice that the entropy of $X_1 \oplus X_2$ (component-wise exclusive OR) is $nh(p)$ bits and hence it is possible to recover $X_1 \oplus X_2$ from $A(x_1 \oplus x_2)$ (of length $m = nh(p)$) with asymptotically negligible probability of decoding error, provided that $A$ is appropriately chosen (see Körner and Marton [20]).

It should be remarked that this example cannot be justified by the previous works such as Ho et al. [13], Ho et al. [14], and Ramamoorthy et al. [15], because all of them assume noiseless channels with capacity of one bit, i.e., this example is outside the previous framework.

5. ALTERNATIVE TRANSMISSIBILITY CONDITION

In this section we demonstrate an alternative transmissibility condition equivalent to the necessary and sufficient condition (3.13) given in Theorem 3.1.

To do so, for each $t \in \Psi$ we define the polyhedron
\(C\) as the set of all nonnegative rates \((R_s; s \in \Phi)\) such that
\[
\sum_{i \in S} R_i \leq \rho_i(S) \quad (\emptyset \neq \forall S \subset \Phi),
\] (5.15)
where \(\rho_i(S)\) is the capacity function as defined in (2.11) of Section 2. Moreover, define the polyhedron \(R_{SW}\) as the set of all nonnegative rates \((R_s; s \in \Phi)\) such that
\[
H(X_S|X_{\overline{S}}) \leq \sum_{i \in S} R_i \quad (\emptyset \neq \forall S \subset \Phi),
\] (5.16)
where \(H(X_S|X_{\overline{S}})\) is the conditional entropy rate as defined in Section 2. Then, we have the following theorem on the transmissibility over the network \(\mathcal{N} = (V, E, C)\).

**Theorem 5.1:** The following two statements are equivalent:

1) \[
H(X_S|X_{\overline{S}}) \leq \rho_N(S) \quad (\emptyset \neq \forall S \subset \Phi),
\] (5.17)

2) \[
R_{SW} \cap C_t \neq \emptyset \quad (\forall t \in \Psi).
\] (5.18)

In order to prove Theorem 5.1 we need the following lemma:

**Lemma 5.1** (Han [3]): Let \(\sigma(S), \rho(S)\) be a co-polymatroid and a polymatroid, respectively, as defined in Remark 2.1. Then, the necessary and sufficient condition for the existence of some nonnegative rates \((R_s; s \in \Phi)\) such that
\[
\sigma(S) \leq \sum_{i \in S} R_i \leq \rho(S) \quad (\emptyset \neq \forall S \subset \Phi)
\] (5.19)
is that
\[
\sigma(S) \leq \rho(S) \quad (\emptyset \neq \forall S \subset \Phi).
\] (5.20)

**Proof of Theorem 5.1:** It is easy to see this.

**Remark 5.1:** The necessary and sufficient condition of the form (5.18) appears (without the proof) in Ramamoorthy, Jain, Chou and Effros [15] with \(|\Phi| = 2, |\Psi| = 2\), which they call the feasibility. They attribute the sufficiency part simply to Ho, Médard, Effros and Koetter [13] with \(|\Phi| = 2, |\Psi| = 1\) (also, cf. Ho, Médard, Koetter, Karger, Effros, Shi, and Leong [14] with \(|\Phi| = 2, |\Psi| = 1\), while attributing the necessity part to Han [3], Barros and Servetto [10]. However, notice that all the arguments in [13], [14] ([13] is included in [14]) can be validated only within the class of stationary memoryless sources of integer bit rates and error-free channels (i.e., the identity mappings) all with one bit capacity (this restriction is needed to invoke “Menger’s theorem” in graph theory); while the present paper, without such severe restrictions, treats “general” acyclic networks, allowing for general correlated stationary ergodic sources as well as general statistically independent channels with each satisfying the strong converse property (cf. Lemma 2.1). Moreover, as long as we are concerned also with noisy channels, the way of approaching the problem as in [13], [14] does not work as well, because in this noisy case we have to cope with two kinds of error probabilities, one due to error probabilities for source coding and the other due to error probabilities for network coding (i.e., channel coding); thus in the noisy channel case or in the noiseless channel case with non-integer capacities and/or i.i.d. sources of non-integer bit rates, [15] cannot attribute the sufficiency part of (5.18) to [13], [14].

It should be noted here also that [13] and [14], though demonstrating relevant error exponents (the direct part), do not have the converse part.

**Remark 5.2 (Separation):** Here, the term of separation is used to mean separation of distributed source coding and network coding with independent sources. Theorem 3.1 does not immediately guarantee separation in this sense. However, when \(\rho_N(S)\) is, for example, a polymatroid, separation in this sense is ensured, because in this case it is guaranteed by Lemma 5.1 that there exist some nonnegative rates \(R_i (i \in \Phi)\) such that
\[
H(X_S|X_{\overline{S}}) \leq \sum_{i \in S} R_i \leq \rho_N(S) \quad (\emptyset \neq \forall S \subset \Phi)
\] (5.21)
Then, the first inequality ensures reliable distributed source coding by virtue of the theorem of Slepian and Wolf (cf. Cover [6]), while the second inequality ensures reliable network coding, that looks like
for non-physical flows, with independent distributed sources of rates $R_i$ ($i \in \Phi$; see Remark 3.2). Condition (5.21) is equivalently written as

$$\mathcal{R}_{SW} \cap \left( \bigcap_{i \in \Psi} C_i \right) \neq \emptyset.$$  

(5.22)

for any general network $\mathcal{N}$. We notice here that the existence of rates $R_i$‘s satisfying condition (5.21) is actually sufficient for separability despite the polymatroid property of $\rho_N(S)$. Incidentally, it is concluded that, in general, condition (5.22) is not only sufficient but also necessary for separability.

6. Routing Capacity Regions

So far we have considered the problem of multicasting multiple sources to multiple sinks over a noisy network in which all the sources may be mutually correlated. The fundamental tool for this kind of reliable transmission are mainly routing and coding at each node of the network. Along this line we have established Theorem 3.1 to give a necessary and sufficient condition for reliable transmission. On the other hand, several class of network coding may not need the operation of coding but only that of routing. However, Theorem 3.1 does not provide any explicit suggestions or answers in this respect.

In this section, we address this problem with mutually independent sources specified only by their rates $R_i$’s. The set of all such achievable rates will be called the routing capacity of the network. In what follows, as illustrative cases, we take three types of network routings, i.e., multiple-access-type of routing, broadcast-type of routing, and interference-type of routing. In doing so, for each edge $(i, j) \in E$ we impose not only upper capacity $c_{ij}$ restriction but also lower capacity $d_{ij}$ restriction ($0 \leq d_{ij} \leq c_{ij}$), which means that information flows $g_{ij}$ passing the channel $(i, j) \in E$ are restricted so that $d_{ij} \leq g_{ij} \leq c_{ij}$ for all $(i, j) \in E$. A motivation for the introduction of such lower capacities $d_{ij}$ is that in some situations “informational outage” over a network is to be avoided, for example.

Let us first state “Hoffman’s theorem” in a graph theory which is needed to discuss the routing capacities. For simplicity we put $D = (d_{ij})_{(i,j) \in E}$ like $C = (c_{ij})_{(i,j) \in E}$, and also write as $[d_{ij}, c_{ij}]$.

**Definition 6.1:** Given a graph $G = (V, E, C, D)$, a flow $g_{ij}$ is said to be circular if $d_{ij} \leq g_{ij} \leq c_{ij}$ for all $(i, j) \in E$, and $\sum_{j \in V} g_{ij} = \sum_{j \in V} g_{ji} = 0$ for all $i \in V$ (the conservation law).

**Theorem 6.1:** [Hoffman’s theorem] There exists a circular flow $(g_{ij} : (i,j) \in E)$ if and only if
c\(c(M, \overline{M}) \geq d(M, M)\) for all subset $M \subset V,$

(6.23)

where $c(M, \overline{M})$ was specified in (2.10) and $d(M, M)$ is similarly defined by replacing $c, M, \overline{M}$ by $d, \overline{M}, M,$ respectively, where $\overline{M}$ denotes the complement of $M$ in $V$.■

**E. Multiple-access-type of network routing**

Suppose that we are given a network $\mathcal{N} = (V, E, C, D)$ as in Fig.6:

![Fig. 6. Multiple-access-type of network](image)

We modify this network by adding $p$ fictitious edges as in Fig.7:

![Fig. 7. Modified network $\mathcal{N}^*$](image)

to obtain the modified network $\mathcal{N}^* = (V, E^*, C, D, [R_1, R_1], \ldots, [R_p, R_p])$. We notice here that a rate
(R₁, · · · , Rₚ) is achievable over the original network N if and only if the network N has a circular flow. Thus, by writing down all the conditions in Hoffman’s theorem for N*, we obtain the following two kinds of inequalities:

1. \( M → M : \)
   \[
   \sum_{i ∈ A} R_i ≥ d(M, M) − c(M, M) ≡ d^*(M, M)
   \]
   \( (A ⊂ M, M ⊃ A; M ⊃ \overline{A}) \)

2. \( M → M : \)
   \[
   \sum_{i ∈ A} R_i ≤ c(M, M) − d(M, M) ≡ c^*(M, M)
   \]
   \( (A ⊂ M, M ⊃ A; M ⊃ \overline{A}) \)

\[ \text{can be omitted} \]

**Proof:** It suffices only to use Lemma 5.1. ■

**Remark 6.1:** It is easy to see that \( σ_m(A) = 0 \) for all \( A ⊂ M \) if \( d_{ij} = 0 \) (for all \( i, j \) ∈ E), and hence Theorem 6.3 turns out to be a special case of Theorem 3.1 with independent sources. This implies that in this case the capacity region given by Thoorem 3.1 can actually be attained only with network routing but without network coding.

**F. Broadcast-type of network routing**

Suppose that we are given a network \( N = (V, E, C, D) \) as in Fig.9:

![Fig. 9. Broadcast-type of network](image)

We modify this network by adding \( q \) fictitious edges as in Fig.10:

![Fig. 10. Modified network \( N^* \)](image)

In order to express these conditions in a compact form, define the “capacity function” as follows:

**Definition 6.2:** For each subset \( A \) of \( \Phi \), let
\[
ρ_m(A) = \min \{ c^*(M, M) | M ⊃ A ; M ⊃ \overline{A} \},
\]
\[
σ_m(A) = \max \{ d^*(M, M) | M ⊃ A ; M ⊃ \overline{A} \}.
\]

**Theorem 6.2:** \( ρ_m(A) \), \( σ_m(A) \) are a polymatroid and a co-polymatroid, respectively.

**Theorem 6.3:** There exists a circular flow if and only if
\[
σ_m(A) ≤ ρ_m(A) \quad \text{for all } A ⊂ \Phi,
\]
and the routing capacity region of multiple-access-type of network is given by
\[
σ_m(A) ≤ \sum_{i ∈ A} R_i ≤ ρ_m(A) \quad \text{for all } A ⊂ \Phi.
\]

**Proof:** It suffices only to use Lemma 5.1. ■

**Remark 6.1:** It is easy to see that \( σ_m(A) = 0 \) for all \( A ⊂ \Phi \) if \( d_{ij} = 0 \) (for all \( i, j \) ∈ E), and hence Theorem 6.3 turns out to be a special case of Thoorem 3.1 with independent sources. This implies that in this case the capacity region given by Thoorem 3.1 can actually be attained only with network routing but without network coding.

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\]

**Proof:** It suffices only to use Lemma 5.1. ■

**Remark 6.1:** It is easy to see that \( σ_m(A) = 0 \) for all \( A ⊂ \Phi \) if \( d_{ij} = 0 \) (for all \( i, j \) ∈ E), and hence Theorem 6.3 turns out to be a special case of Thoorem 3.1 with independent sources. This implies that in this case the capacity region given by Thoorem 3.1 can actually be attained only with network routing but without network coding.
Accordingly, the “capacity function” is defined as:

**Definition 6.3:** For each subset $A$ of $\Psi$, let

\[ \rho_b(A) \equiv \min \{ c^*(M, M) | M \ni s_1, M \supset A \}, \]
\[ \sigma_b(A) \equiv \max \{ d^*(M, M) | M \ni s_1, M \supset A \}. \]

**Theorem 6.4:** $\rho_b(A)$, $\sigma_b(A)$ are a polymatroid and a co-polymatroid, respectively.

**Theorem 6.5:** There exists a circular flow if and only if $\sigma_b(A) \leq \rho_b(A)$ for all $A \subset \Psi$, (6.26)

and the routing capacity region of broadcast-type of network is given by

\[ \sigma_b(A) \leq \sum_{i \in A} R_i \leq \rho_b(A) \text{ for all } A \subset \Psi. \quad (6.27) \]

**Proof:** It suffices only to use Lemma 5.1. 

**Remark 6.2:** The capacity region (6.27) remains unchanged if network coding is allowed in addition to network routing.

**G. Interference-type of network routing**

Suppose that we are given a network $N = (V, E, C, D)$ as in Fig.12:

![Interference-type of network](image1)

We modify this network by adding $p$ fictitious edges as in Fig.13 below.

In this case too, again by virtue of Hoffman’s theorem, the achievability condition is summarized as

\[ \max_{(A,B)} d^*(M, M) \leq \sum_{i \in A_2} R_i - \sum_{i \in A_3} R_i \leq \min_{(A,B)} c^*(M, M), \quad (6.28) \]

where $\max_{(A,B)}$ (resp. $\min_{(A,B)}$) denotes max (resp. min) over all possible cuts $(M, \overline{M})$ and partitions (with $A_2, A_3$ fixed) as was shown in Fig.14:

![Modified network $N^*$](image2)

For simplicity, let us now consider the case with $|\Phi| = |\Psi| = 2$:

![Modified network $N^*$](image3)

Then, the achievability condition reduces to that as
in Fig.16 below, where, on the contrary to the case of multiple-access-type of network routing as well as to the case of broadcast-type of network routing, the term $R_1 - R_2$ appears in addition to the term $R_1 + R_2$, which suggests that the interference-type of network routing may be more complicated with a rather pathological behavior.

\[
0 \leq R_1 \leq \min_M c(M, \overline{M}) \text{ where } \begin{cases} [s_1 \in M, t_1 \in \overline{M}] \text{ and } [\{s_2 \in M, t_2 \in M\} \\
\text{or } \{s_2 \in \overline{M}, t_2 \in \overline{M}\}\end{cases}
\]

Case 1: $0 \leq R_1 \leq \min_M c(M, \overline{M})$ where two elementary directed cycles: one with $R_1 = 0, R_2 = 1$; the other with $R_1 = 1, R_2 = 1$. The first flow actually specifies a flow in $\mathcal{N}$, while the latter flow contains two fictitious edges and hence does not specify any flow in $\mathcal{N}$. Thus, in the case of the interference-type of network, we have to take another approach to establish the routing capacity region. An approach, though brute, would be to list up all the elementary paths in $\mathcal{N}$ connecting source $s_i$ and sink $t_i$ for every $i = 1, 2, \ldots, p$. Incidentally, it should also be remarked that any polymatroidal structure does not appear here.

The following Fig.17 shows a typical region of Fig.16, which looks like a rather untraditional shape.

\[
0 \leq R_1 \leq \min_M c(M, \overline{M}) \leq \min_M c(M, \overline{M})
\]

Case 2: $0 \leq R_2 \leq \min_M c(M, \overline{M})$ where two elementary directed cycles: one with $R_2 = 0, R_1 = 1$; the other with $R_1 = 1, R_2 = 1$. The first flow actually specifies a flow in $\mathcal{N}$, while the latter flow contains two fictitious edges and hence does not specify any flow in $\mathcal{N}$. Thus, in the case of the interference-type of network, we have to take another approach to establish the routing capacity region. An approach, though brute, would be to list up all the elementary paths in $\mathcal{N}$ connecting source $s_i$ and sink $t_i$ for every $i = 1, 2, \ldots, p$. Incidentally, it should also be remarked that any polymatroidal structure does not appear here.

For example, let us consider the network $\mathcal{N}$ as in Fig.18. It is easy to check that the achievability condition in Fig. 16 for this network reduces to $0 \leq R_1 \leq R_2 \leq 1$. Fig.19 shows that two elementary directed cycles: one with $R_1 = 0, R_2 = 1$; the other with $R_1 = 1, R_2 = 1$. The first flow actually specifies a flow in $\mathcal{N}$, while the latter flow contains two fictitious edges and hence does not specify any flow in $\mathcal{N}$. Thus, in the case of the interference-type of network, we have to take another approach to establish the routing capacity region. An approach, though brute, would be to list up all the elementary paths in $\mathcal{N}$ connecting source $s_i$ and sink $t_i$ for every $i = 1, 2, \ldots, p$. Incidentally, it should also be remarked that any polymatroidal structure does not appear here.

For example, let us consider the network $\mathcal{N}$ as in Fig.18. It is easy to check that the achievability condition in Fig. 16 for this network reduces to $0 \leq R_1 \leq R_2 \leq 1$. Fig.19 shows that two elementary directed cycles: one with $R_1 = 0, R_2 = 1$; the other with $R_1 = 1, R_2 = 1$. The first flow actually specifies a flow in $\mathcal{N}$, while the latter flow contains two fictitious edges and hence does not specify any flow in $\mathcal{N}$. Thus, in the case of the interference-type of network, we have to take another approach to establish the routing capacity region. An approach, though brute, would be to list up all the elementary paths in $\mathcal{N}$ connecting source $s_i$ and sink $t_i$ for every $i = 1, 2, \ldots, p$. Incidentally, it should also be remarked that any polymatroidal structure does not appear here.
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