On Weight Enumerators and MacWilliams Identity for Convolutional Codes

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***Abstract*—Convolutional codes are defined to be equivalent if their code symbols differ only in how they are ordered and two generator matrices are defined to be weakly equivalent (WE) if they encode equivalent convolutional codes. It is shown that tailbiting convolutional codes encoded by WE minimal-basic generator matrices have the same spectra. Shearer and McEliece showed that MacWilliams identity does not hold for convolutional codes. However, for the spectra of truncated convolutional codes and their duals, MacWilliams identity clearly holds. It is shown that the dual of a truncated convolutional code is not a truncation of a convolutional code but its reversal is. Finally, a recursion for the spectra of truncated convolutional codes is given and the spectral components are related to those for the corresponding dual codes.**

I. Introduction

We regard a rate $R = b/c$ binary convolutional code over the field $\mathbb{F}_2$ as the image set of the linear mapping represented as

$$\boldsymbol{v}(D) = \boldsymbol{u}(D)G(D)$$

where the code sequence $\boldsymbol{v}(D)$ and information sequence $\boldsymbol{u}(D)$ are c- and b-tuples of Laurent series, respectively, and the generator matrix $G(D)$ has full rank over the field of rational functions $\mathbb{F}_2(D)$.

The path weight enumerator of a convolutional encoder plays an important role in this paper. It was introduced by Viterbi [1] and is the generating function of the weights of the paths which diverge from the allzero path at the root in the code trellis and do not reach the zero state until their termini. We call the sequence of path weights the *free distance spectrum*, in order to distinguish it from the spectrum of block codes.

The problem of equivalent convolutional codes was discussed many times (see, for example, [2], [3] and the references therein). The idea to find convolutional codes with better performances by multiplying or dividing each column of the polynomial generator matrix by a monomial arises in different papers. Since the path weight enumerators of the corresponding convolutional encoders differ from each other it gives ground for speculations about possible improvements of the best known convolutional codes in such a manner.

It is well-known, starting with the paper by Shearer and McEliece [4], that the MacWilliams identity [5] does not hold for the free distance spectra of convolutional codes. In [6], a MacWilliams-type identity was derived, but only for a special class of convolutional codes, namely, those with encoders consisting of only one memory element. A MacWilliams-type identity for convolutional codes involving the weight adjacency matrices for the encoders of a convolutional code $\mathcal{C}$ and its dual $\mathcal{C}^\perp$ was formulated in [7] and proved in [8] by Gluesing-Luerssen and Schneider. Their work inspired Forney, and in [9] and [10] he proved their results in a different way and generalized them to various kinds of weight adjacency matrices and to group codes defined on graphs.

We got interested in the MacWilliams-type of identity problems via searching for high-rate convolutional generator matrices. In particular, we were interested in the effect of "pulling out" a factor of D from a column of a generator matrix. This led us to study the MacWilliams identity for truncated convolutional codes and their duals. Recently, Forney [11] extended our approach to include tailbiting as a terminating procedure.

In Section II, we discuss the free distance spectrum and various notions of equivalence. Zero-tail terminated and truncated convolutional codes and their spectra are considered in Section III. The MacWilliams identity is revisited in Section IV and it is evident that it holds for truncated convolutional codes and their dual codes. In Section V, a recursion for the spectra of truncated convolutional codes is given and the spectral components of truncated convolutional codes are related to those of the corresponding dual codes.

II. Free distance spectrum and equivalences

Viterbi used the free distance spectrum to upper-bound the burst (first event) error probability when using a convolutional code for communication over the binary symmetric channel (BSC) with crossover probability ε and maximum-likelihood (ML) decoding [1],

$$P_B < T(W)\Big|_{W=2\sqrt{\varepsilon(1-\varepsilon)}}$$

where $T(W)$ is the path weight enumerator.

The free distance spectrum is an encoder property, not a code property. This is readily seen from the following example.

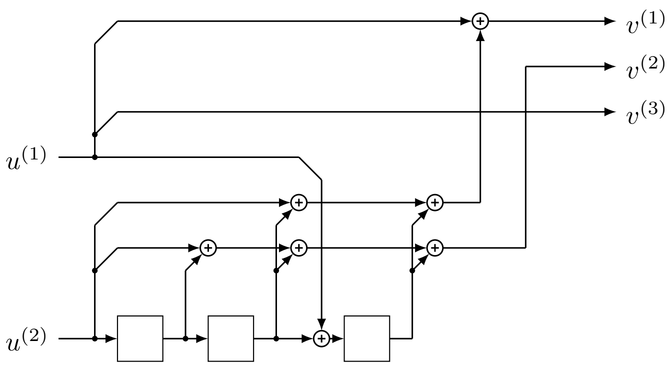

Fig. 1. A minimal encoder for the minimal generator matrix $G_2(D)$.

Example 1: The systematic generator matrix

$$G_1(D) = \begin{pmatrix} 1 & 0 & 1+D \\ 0 & 1 & D \end{pmatrix}$$

when realized in observer canonical form (ocf) has the path weight enumerator

$$\begin{aligned} T_1^{(\text{ocf})}(W) &= \frac{W^2 + W^3 - W^5}{1 - W - W^2} \\ &= W^2 + 4W^3 + 5W^4 + 8W^5 + 13W^6 + \cdots \end{aligned}$$

while when realized in controller canonical form (ccf) it has the path weight enumerator

$$\begin{aligned} T_2^{(\text{ccf})}(W) &= \frac{W^2 + 3W^3 - W^5}{1 - W - W^2 - 2W^3 + W^5} \\ &= W^2 + 4W^3 + 5W^4 + 10W^5 + 23W^6 + \cdots. \end{aligned}$$

We notice that the ocf realization yields the best Viterbi bound for the burst error probability, which is a code property. This is not surprising since this realization is a minimal encoder for the code encoded by $G_1(D)$. Moreover, consider only paths of weights up to $2d_{\text{free}} - 1$, that is, paths that consist of one complete (merging with the allzero path) detour plus possibly an incomplete detour starting from a zero state. The weights of such paths are a code property, so we conclude that equivalent generator matrices have the identical path weight enumerators up to weight $2d_{\text{free}} - 1$.

Example 2: The minimal generator matrix [12]

$$G_2(D) = \begin{pmatrix} 1+D & D & 1 \\ 1+D^2+D^3 & 1+D+D^2+D^3 & 0 \end{pmatrix}$$

has a minimal realization neither in controller canonical form nor in observer canonical form. However, the realization shown in Fig. 1 is a minimal encoder. Somewhat surprisingly the path weight enumerators are the same for all three realizations, namely,

$$\begin{aligned} T_2^{(\text{min})}(W) &= T_2^{(\text{ccf})}(W) = T_2^{(\text{ocf})}(W) \\ &= W^4 + 5W^5 + 24W^6 + 71W^7 + 238W^8 + \cdots. \end{aligned}$$

Example 3: Consider the following two generator matrices

$$\begin{aligned} G_3(D) &= \begin{pmatrix} 1 & 1+D \end{pmatrix} \\ G_4(D) &= \begin{pmatrix} D^2 & 1+D \end{pmatrix} \end{aligned}$$

and their path weight enumerators

$$\begin{aligned} T_3(W) &= \frac{W^3}{1-W} \\ &= W^3 + W^4 + W^5 + W^6 + \cdots \\ T_4(W) &= \frac{W^3}{1 - W - W^3} \\ &= W^3 + W^4 + W^5 + 2W^6 + \cdots. \end{aligned}$$

The two path weight enumerators are different, so the Viterbi bound on the burst error probability will be different.

Moreover, if we use sequential decoding, the computational performance using $G_4(D)$ will be much worse since its distance profile is not optimum while it is for $G_3(D)$. On the other hand, if we consider the convolutional code $\mathcal{C}_4$, encoded by $G_4(D)$, and shift the first code symbol in each tuple two steps to the left, we obviously obtain the first convolutional code. Hence, we expect that these convolutional codes have some common properties.

Definition 1: Two convolutional codes $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if the codewords $\boldsymbol{v} \in \mathcal{C}$ are finite-memory permutations of the codewords $\boldsymbol{v}' \in \mathcal{C}'$.

The examples above suggest the following notion of equivalence.

Definition 2: Two generator matrices $G(D)$ and $G'(D)$ are *weakly equivalent* (WE) if they encode equivalent convolutional codes.

Consider a permutation matrix Π and replace all 1s with monomials D^i, where i can be different integers for different entries; then we obtain a *generalized permutation* matrix $\Pi(D)$. Let $G(D)$ be a generator matrix of a rate $R = b/c$ convolutional code $\mathcal{C}$, then by multiplication of $G(D)$ from the right by a generalized permutation matrix $\Pi(D)$ of size $c \times c$ we obtain a generator matrix $G'(D)$ of an equivalent convolutional code $\mathcal{C}'$.

It follows from Definition 2 that all generator matrices $G^{'}(D)$ of the form

$$G'(D) = T(D)G(D)\Pi(D)$$

are WE to $G(D)$, where $T(D)$ is a nonsingular $b \times b$ matrix and $\Pi(D)$ is a $c \times c$ generalized permutation matrix. It is easy to see that by factoring out D^i, $i > 0$, from a column of $G(D)$ we obtain a WE generator matrix.

In particular,

$$G_4(D) = G_3(D) \begin{pmatrix} D^2 & 0 \\ 0 & 1 \end{pmatrix}$$

which means that these convolutional generator matrices are WE to each other.

Next we consider a less trivial example.

Example 4: The following matrices

$$G_5(D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1+D & 1+D & 0 & 1 \\ 0 & D & 1 & 1+D & D \end{pmatrix}$$

$$G_6(D) = \begin{pmatrix} 1+D & 1+D & 0 & 1 & 0 \\ D & 1 & 1+D & D & 0 \\ D & D & D & 0 & 1 \end{pmatrix}$$

$$G_7(D) = \begin{pmatrix} 1 & 1+D & D & 0 & 1 \\ D & D & 0 & 1 & 1 \\ D^2+D & 0 & D & 0 & 1+D \end{pmatrix}$$

are considered "equivalent" in [3] since they can be obtained from each other by cyclic shifting of the trellis module. In our terminology these matrices are WE to each other because

$$G_6(D) = T(D)G_5(D)\Pi(D)$$

and

$$G_7(D) = T(D)G_6(D)\Pi(D)$$

where

$$T(D) = \begin{pmatrix} 0 & D^{-1} & 0 \\ 0 & 0 & D^{-1} \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\Pi(D) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ D & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 \\ 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & D & 0 \end{pmatrix}.$$

If we tailbite convolutional codes encoded by the WE generator matrices $G_5(D)$, $G_6(D)$, and $G_7(D)$, we obtain the same (block code) spectrum for all these codes, although they have different overall constraint lengths. For tailbiting length $t = 1000$ we have

$$B_{t=1000}(W) = 1 + 1000W^4 + 12000W^5 + 32000W^6 + \cdots.$$

This observation motivated the introduction of the *weakly equivalence* concept.

III. Spectra for zero-tail terminated and truncated convolutional codes

Consider a rate $R = b/c$ convolutional code with memory m that is zero-tail terminated to the length of $t+m$ c-tuples. If we compute the (block code) spectrum for the two zero-tail terminated convolutional codes with the two weakly equivalent generator matrices $G_3(D)$ and $G_4(D)$ using $t = 1000$ we obtain identical spectra

$$B^{(\text{zt})}_{3,t=1000}(W) = B^{(\text{zt})}_{4,t=1000}(W) = 1 + 1000W^3 + 999W^4 + 998W^5 + 499498W^6 + \cdots.$$

Notice that we have to choose the block length for $\mathcal{C}^{(\text{zt})}_4$ to be one c-tuple longer than that for $\mathcal{C}^{(\text{zt})}_3$ since $m_4 = m_3 + 1 = 2$.

Next we consider the generator matrix

$$G_8(D) = \begin{pmatrix} 1+D+D^2+D^3 & 1+D^2+D^3 \end{pmatrix}.$$

Weight	d_{free} spectrum	Normalized zt spectrum $t = 1000$
0	0	0.001
1-5	0	0
6	1	0.999
7	3	2.995
8	5	4.981
9	11	10.940
10	25	24.829
11	55	54.541
12	121	614.319
13	267	3227.073

TABLE I
NORMALIZED ZERO-TAIL TERMINATED SPECTRUM FOR $G_8(D)$

The corresponding convolutional code has free distance $d_{\text{free}} = 6$ and memory $m = 3$. The normalized (by t) spectrum for a zero-tail terminated convolutional code is a good approximation up to $2d_{\text{free}} - 1$ of the free distance spectrum for the corresponding minimal encoder as shown in Table I.

Notice that we do not get integers when we compute the normalized spectrum for the zero-tail terminated convolutional code. The reason is that for this generator matrix, the length of the detour of weight d_{free} is $m+2$, not $m+1$, c-tuples. Had it been $m+1$, we would have had perfect agreement at least for d_{free}.

Zero-tail termination causes a rate loss. For a rate R convolutional code we obtain the rate

$$R^{(\text{zt})} = \frac{t}{t+m}R$$

for the zero-tail terminated convolutional code. If we simply truncate the convolutional code after t c-tuples we do not have this rate loss, but this truncation method has other disadvantages as we will see.

Let $\boldsymbol{G}$ denote the semi-infinite generator matrix corresponding to

$$G(D) = G_0 + G_1D + G_2D^2 + \cdots + G_mD^m$$

that is,

$$\boldsymbol{G} = \begin{pmatrix} G_0 & G_1 & G_2 & \dots & G_m & & \\ & G_0 & G_1 & G_2 & \dots & G_m & \\ & & \ddots & \ddots & \ddots & & \ddots \end{pmatrix}.$$

If we truncate $\boldsymbol{G}$ after t c-tuples, we obtain the $t \times t$ matrix

$$\boldsymbol{G}_t = \begin{pmatrix} G_0 & G_1 & G_2 & \cdots & G_m & & & & \\ & G_0 & G_1 & G_2 & \cdots & G_m & & & \\ & & \ddots & \ddots & \ddots & & \ddots & & \\ & & & G_0 & G_1 & G_2 & \cdots & G_m \\ & & & & G_0 & G_1 & \cdots & G_{m-1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & G_0 & G_1 \\ & & & & & & & G_0 \end{pmatrix}.$$

The normalized spectrum for the truncated convolutional code with generator matrix $G_8(D)$ is given in Table II.

Weight	d_{free} spectrum	Normalized zt spectrum $t=1000$	Normalized truncated spectrum $t=200$	$t=1000$
0	0	0.001	0.005	0.001
1	0	0	0	0
2	0	0	0.005	0.001
3	0	0	0.015	0.003
4	0	0	0.035	0.007
5	0	0	0.080	0.016
6	1	0.999	1.150	1.030
7	3	2.995	3.305	3.061
8	5	4.981	6.635	6.127
9	11	10.940	18.185	17.237
10	25	24.829	48.015	46.403
11	55	54.541	120.910	118.582
12	121	614.319	390.185	788.037
13	267	3227.073	1272.990	3675.400

TABLE II
NORMALIZED TRUNCATED SPECTRUM FOR $G_8(D)$

Notice that only for the spectral components for weights d_{free} and $d_{\text{free}}+1$ (since $c=2$ in this case), we obtain a good approximation of the corresponding components in the free distance spectrum. For higher weights, a significant number of codewords with weights $d_{\text{free}}+c$, $d_{\text{free}}+c+1, \ldots$ that will not merge with the allzero state contribute to the normalized spectrum. This is a drawback with truncated convolutional codes.

IV. MacWilliams identity for truncated Convolutional Codes

Next we shall consider dual convolutional codes. We have [12]:

Definition 3: A *dual* code $\mathcal{C}^\perp$ to a rate $R=b/c$ convolutional code $\mathcal{C}$ is the set of all sequences $\boldsymbol{v}^\perp \in \mathcal{C}^\perp$ such that the inner product

$$(\boldsymbol{v},\boldsymbol{v}^\perp) = \boldsymbol{v}(\boldsymbol{v}^\perp)^T = 0$$

that is, $\boldsymbol{v}$ and $\boldsymbol{v}^\perp$ are orthogonal, for all finite sequences $\boldsymbol{v}$ in $\mathcal{C}$.

The dual code $\mathcal{C}^\perp$ to a rate $R=b/c$ convolutional code $\mathcal{C}$ encoded by the semi-infinite generator matrix $\boldsymbol{G}$ is a rate $R=(c-b)/c$ convolutional code encoded by the semi-infinite generator matrix $\boldsymbol{G}^\perp$, where

$$\boldsymbol{G}^\perp = \begin{pmatrix} G_0^\perp & G_1^\perp & G_2^\perp & \ldots & G_{m^\perp}^\perp & & \\ & G_0^\perp & G_1^\perp & G_2^\perp & \cdots & G_{m^\perp}^\perp & \\ & & \ddots & \ddots & \ddots & & \ddots \end{pmatrix}$$

and $\boldsymbol{G}^\perp$ satisfies

$$\boldsymbol{G}\left(\boldsymbol{G}^\perp\right)^T = \boldsymbol{0}.$$

Truncation of $\boldsymbol{G}^\perp$ by t yields the $t\times t$ matrix

$$\boldsymbol{G}_t^\perp = \begin{pmatrix} G_0^\perp & G_1^\perp & G_2^\perp & \cdots & G_{m^\perp}^\perp & & & & \\ & G_0^\perp & G_1^\perp & G_2^\perp & \cdots & G_{m^\perp}^\perp & & & \\ & & \ddots & \ddots & \ddots & & \ddots & & \\ & & & G_0^\perp & G_1^\perp & G_2^\perp & \cdots & G_{m^\perp}^\perp \\ & & & & G_0^\perp & G_1^\perp & \cdots & G_{m^\perp-1}^\perp \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & G_0^\perp & G_1^\perp \\ & & & & & & & G_0^\perp \end{pmatrix}$$

but now

$$\boldsymbol{G}_t\left(\boldsymbol{G}_t^\perp\right)^T \neq \boldsymbol{0}.$$

However, if

$$\boldsymbol{v}_t^\perp = \boldsymbol{u}_t^\perp \widetilde{\boldsymbol{G}}_t^\perp$$

where

$$\widetilde{\boldsymbol{G}}_t^\perp = \begin{pmatrix} G_{m^\perp}^\perp & & & & & & \\ G_{m^\perp-1}^\perp & G_{m^\perp}^\perp & & & & & \\ \vdots & \vdots & \ddots & & & & \\ G_0^\perp & G_1^\perp & \cdots & G_{m^\perp}^\perp & & & \\ & G_0^\perp & G_1^\perp & \cdots & G_{m^\perp}^\perp & & \\ & & \ddots & \ddots & & \ddots & \\ & & & G_0^\perp & G_1^\perp & \ldots & G_{m^\perp}^\perp \end{pmatrix}$$

we obtain

$$\boldsymbol{v}_t\left(\boldsymbol{v}_t^\perp\right)^T = \boldsymbol{u}_t\boldsymbol{G}_t\left(\widetilde{\boldsymbol{G}}_t^\perp\right)^T\left(\boldsymbol{u}_t^\perp\right)^T = 0.$$

But $\widetilde{\boldsymbol{G}}_t^\perp$ is *not* a generator matrix of a truncated convolutional code. However, if we write both the rows and the columns in reversed order we obtain

$$\begin{pmatrix} G_{m^\perp}^\perp & \cdots & G_1^\perp & G_0^\perp & & & \\ & G_{m^\perp}^\perp & \cdots & G_1^\perp & G_0^\perp & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & G_{m^\perp}^\perp & \cdots & G_1^\perp & G_0^\perp \\ & & & & G_{m^\perp}^\perp & \cdots & G_1^\perp \\ & & & & & \ddots & \vdots \\ & & & & & & G_{m^\perp}^\perp \end{pmatrix}$$

which is equal to the reversal of $\boldsymbol{G}_t^\perp$ and denoted $\overleftarrow{\boldsymbol{G}}_t^\perp$.

The dual truncated code $\mathcal{C}_t^\perp$ is in general not a truncated convolutional code but it is the reversal of a truncated convolutional code encoded by $\overleftarrow{\boldsymbol{G}}_t^\perp$.

Since truncated convolutional codes and their dual codes are block codes, it is evident that they satisfy MacWilliams identity [5]:

Theorem 1: Let $\mathcal{C}_t$ be a binary convolutional code of rate $R=b/c$ truncated after t c-tuples with dual code $\mathcal{C}_t^\perp$. Then

$$\sum_{k=0}^{ct} A_k^\perp x^{ct-k}y^k = \frac{1}{2^{bt}}\sum_{i=0}^{ct} A_i(x+y)^{ct-i}(x-y)^i$$

with

$$\boldsymbol{z}A(W)^t\mathbf{1}^T = \sum_{i=0}^{ct} A_i W^i$$

and

$$\mathbf{1}A^{\perp}(W)^t\boldsymbol{z}^T = \sum_{i=0}^{ct} A_i^{\perp} W^i$$

where $\boldsymbol{z} = (1,0,\ldots,0)$ and $\mathbf{1} = (1,1,,\ldots,1)$; $A(W)$ and $A^{\perp}(W)$ are weight adjacency matrices obtained from the state-transition diagram for the minimal encoders of $\mathcal{C}_t$ and $\mathcal{C}_t^{\perp}$, respectively.

Shearer and McEliece [4] considered the generator matrix

$$G(D) = \begin{pmatrix} 1 & D & 1+D \end{pmatrix}$$

and its convolutional dual [12]

$$G_{\perp}(D) = \begin{pmatrix} 1 & 1 & 1 \\ D & 1 & 0 \end{pmatrix}$$

where

$$G(D)\Big(G_{\perp}(D)\Big)^T = \mathbf{0}$$

and showed that MacWilliams identity does not hold for the free distance spectra of their minimal encoders.

We consider the same generator matrix $G(D)$ and the generator matrix of its dual, that is,

$$G^{\perp}(D) = \begin{pmatrix} D & D & D \\ 1 & D & 0 \end{pmatrix}$$

where

$$G(D)\left(G^{\perp}(D)\right)^T \neq \mathbf{0}.$$

But for the corresponding codewords we have

$$\boldsymbol{v}\left(\boldsymbol{v}^{\perp}\right)^T = 0.$$

The generator matrix $G(D)$ has the semi-infinite binary generator matrix

$$\boldsymbol{G} = \begin{pmatrix} 101 & 011 & & \\ & 101 & 011 & \\ & & \ddots & \ddots \end{pmatrix}.$$

As a simple example, we consider the truncated convolutional code with truncation length $t = 2$. The generator matrix for the truncated convolutional code $\mathcal{C}_{t=2}$ is

$$G_{t=2} = \begin{pmatrix} 101 & 011 \\ 000 & 101 \end{pmatrix}.$$

Next we convert $G^{\perp}(D)$ to its minimal-basic form [12] and obtain

$$\begin{aligned} G^{\perp}_{\rm mb}(D) &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1+D & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} D. \end{aligned}$$

The corresponding semi-infinite generator matrix is

$$\boldsymbol{G}^{\perp}_{\rm mb} = \begin{pmatrix} 111 & 000 & & \\ 011 & 010 & & \\ & 111 & 000 & \\ & 011 & 010 & \\ & & \ddots & \ddots \end{pmatrix}$$

which yields the following generator matrix for the reversal of the dual code:

$$\begin{aligned} \overleftarrow{G}^{\perp}(D) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} D \\ &= \begin{pmatrix} D & D & D \\ 0 & 1+D & D \end{pmatrix} \end{aligned}$$

or in minimal-basic form:

$$\overleftarrow{G}^{\perp}_{\rm mb}(D) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1+D & D \end{pmatrix}.$$

Hence, we have

$$\overleftarrow{G}^{\perp}_{t=2} = \begin{pmatrix} 111 & 000 \\ 010 & 011 \\ 000 & 111 \\ 000 & 010 \end{pmatrix}$$

or, equivalently, the generator matrix for the dual of the truncated convolutional code $\mathcal{C}_{t=2}$ is

$$G^{\perp}_{t=2} = \begin{pmatrix} 000 & 111 \\ 011 & 010 \\ 111 & 000 \\ 010 & 000 \end{pmatrix}.$$

It is easy to verify that

$$G_{t=2}\left(G^{\perp}_{t=2}\right)^T = \begin{pmatrix} 101 & 011 \\ 000 & 101 \end{pmatrix} \begin{pmatrix} 00 & 10 \\ 01 & 11 \\ 01 & 10 \\ 10 & 00 \\ 11 & 00 \\ 10 & 00 \end{pmatrix} = \mathbf{0}.$$

The weight adjacency matrix for $G(D)$ realized in controller canonical form is

$$A(W) = \begin{pmatrix} 1 & W^2 \\ W^2 & W^2 \end{pmatrix}$$

and we obtain the spectrum for $\mathcal{C}_{t=2}$ as

$$\boldsymbol{z}A(W)^2\mathbf{1}^T = 1 + W^2 + 2W^4$$

where $\boldsymbol{z} = (1,0,0,\ldots,0)$ and $\mathbf{1} = (1,1,\ldots,1)$.

For $\overleftarrow{G}^{\perp}_{\rm mb}(D)$ we have

$$A^{\perp}(W) = \begin{pmatrix} 1+W^2 & W+W^2 \\ W+W^2 & W+W^2 \end{pmatrix}$$

and the spectrum for the dual $\mathcal{C}^{\perp}_{t=2}$ follows as

$$\begin{aligned} &\mathbf{1}A^{\perp}(W)^2\boldsymbol{z}^T = \\ &1 + W + 3W^2 + 6W^3 + 3W^4 + W^5 + W^6. \end{aligned}$$

It is easily verified that MacWilliams identity holds.

V. Infinite sequences of spectra

Let the sequence of the (block code) spectra for the truncated convolutional codes $\mathcal{C}_t$ be given by

$$B_{\mathcal{C}_t}(W) = A_0 + A_1 W + \cdots + A_{ct} W^{ct}, \quad t = 1, 2, \ldots .$$

In [13] we prove that $B_{\mathcal{C}_t}(W)$ can be obtained from the recursion

$$B_{\mathcal{C}_t}(W) = \sum_{i=1}^{2^\nu} a_i(W) B_{\mathcal{C}_{t-i}}(W)$$

where ν is the overall constraint length and $a_i(W)$, $i = 1, 2, \ldots, 2^\nu$, are the coefficients of the characteristic equation for $A(W)$.

Moreover, it is also proven, that the spectral components A_k for the truncated convolutional code $\mathcal{C}_t$ can be obtained from those for the dual code $\mathcal{C}_t^\perp$ as

$$A_k = \frac{1}{2^{ct}} \sum_{i=0}^{ct} A_i^\perp P_k(i)$$

where $P_k(i)$ is a Krawtchouk polynomial.

VI. Conclusion

We defined those convolutional codes whose semi-infinite generator matrices differ only by a column permutation to be equivalent. Generator matrices that encode equivalent convolutional codes were defined to be weakly equivalent (WE). Tailbiting terminated codes of equivalent convolutional codes have the same spectra. The spectra for truncated convolutional codes and their dual codes were shown to be related via MacWilliams identity. Finally, a recursion for the spectra of truncated convolutional codes was presented.

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