K user fading broadcast channels with CSI at the receivers

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***Abstract*—For an arbitrary number of users, we examine the capacity region of fading broadcast channels with channel state known at the receivers but unknown at the transmitter. Generalizing prior work for two-user broadcast channels, we find the capacity region of the layered erasure broadcast channel. We show that weighted sum rate maximization assigns each signal level to the user that derives maximum weighted expected rate from that level. The outer bound is based on a channel enhancement that creates a sequence of degraded component channels in the order of decreasing user weights. This same approach is then used to find an outer bound to the capacity region of the fading Gaussian broadcast channels.**

I. Introduction

In wireless communications, the Gaussian broadcast channel (AWGN BC) is of particular prominence. The capacity region of the AWGN BC is known, for both the case when the channel state is fixed and time-invariant, and for the case when the channel-state is time-varying. This is holds for single antenna [1]–[5] and for multiple antenna channels [6]. The key assumption behind these results is that the channel state information (CSI) is known at the transmitter as well as the receivers. However, the capacity region is unknown once the assumption of perfect CSI at the transmitter is removed, even when the transmitter and each of the receivers has only a single antenna. Despite its practical importance, there are relatively few results on the capacity of the fading BC with receiver-only CSI (see for example [7], [8]); even in the degraded case, the capacity remains unknown [9].

In prior work [10], we examined the simplest scenario with two receivers and a single antenna at the transmitter and at each of the receivers. We proposed a *layered erasure broadcast channel* to approximate the Gaussian fading channel and determined its capacity region exactly. The erasures in this model are *correlated*, and, like the Gaussian fading BC, the layered erasure BC is neither degraded nor parallel with degraded components. Using insights from the erasure model, we derived a new outer bound to the Gaussian fading BC capacity region and demonstrated a binary expansion superposition (BES) scheme that achieves rates within a constant gap to the outer bound over *all* channel fading distributions.

In this work, we generalize the converse results to $K > 2$ users. For the K-user layered erasure BC, we find the capacity region. For the K-user fading Gaussian BC, we generalize our prior outer bounds. It follows that constant gap results derived in [10] for the two user fading Gaussian BC can be extended to more than two users. While the prior proofs were specific to two users, those readers familiar with channel enhancement methods for the Gaussian BC may not be surprised by this generalization. Nevertheless, while the basic channel enhancement ideas for the two-user BC remain the same, the generalization to K users is not completely straightforward. In fact, the approach here for K users simplifies the proofs, although perhaps at the expense of obscuring the role of the auxiliary variables.

In the following, we introduce our channel models in Section II and provide a summary of results in Section III. Some necessary background on the broadcast channel appears in Section IV. The proof of the layered erasure BC converse is given in Section V. Section VI derives the outer bound is on the fading Gaussian broadcast channel.

In this paper, we adopt several conventions. For a random variable N, the probability mass function (PMF) is $P_N(n) := P\{N = n\}$ and the complementary cumulative distribution function (CCDF) is $\overline{F}_N(n) := P\{N \geq n\}$. Random variable N_1 is *stochastically larger* than N_2, denoted $N_1 \geq_{\text{st}} N_2$, if $\overline{F}_{N_1}(x) \geq \overline{F}_{N_2}(x)$ for all $x \geq 0$. We use $\mathcal{K}$ to denote the set $\{1, \ldots, K\}$. Proofs of various lemmas appear in the appendix.

II. Channel Models

The primary motivation for this work is the complex Gaussian fading broadcast channel

$$Y_k = \sqrt{S_k} e^{j\theta_k} X + Z_k, \qquad k \in \mathcal{K}, \tag{1}$$

with time-varying channel state (S_k, θ_k) known at receiver k but not known at the transmitter. The state S_k is a real-valued non-negative channel gain and $\theta_k \in [0, 2\pi]$ is the channel phase. Random variables S_k, θ_k are assumed to be independent of each other and interpreted to be independent from symbol time to symbol time. The channel input X is a complex baseband transmitted signal with unit power constraint and $Z_k \sim \mathcal{CN}(0, 1)$. We will refer to (1) as the fading Gaussian BC $(S_1, \ldots, S_K)$.

Since the phase θ_k is known at the receiver, user k can post-rotate its received signal by $-\theta_k$. This can be shown to yield identical in-phase and quadrature fading broadcast channels, each with unit-power additive Gaussian noise. Hence, we will

evaluate a communication scheme on the real-valued broadcast channel

$$Y_k = \sqrt{S_k}X + Z_k, \qquad k \in \mathcal{K}, \tag{2}$$

that corresponds to the (post-rotated) in-phase and quadrature channels of the complex fading Gaussian BC (1). The achievable rates and outer bounds of the complex fading Gaussian BC (1) will be precisely double those obtained in the real-valued channel (2).

Our starting point, however, is the layered erasure broadcast channel. In each symbol period, the transmitted signal is a q-element vector of bits $X^q = \begin{bmatrix} X_1 & X_2 & \cdots & X_q \end{bmatrix}$ from the most significant bit X_1 to less significant bits. The bits can be viewed as layers of the transmitted signal. Fading is modeled as erasures of the less significant bits; receiver i observes

$$Y_k = X^{N_k} = \begin{bmatrix} X_1 & X_2 & \cdots X_{N_k} \end{bmatrix}, \qquad k \in \mathcal{K}. \tag{3}$$

where N_k with distribution $\overline{F}_{N_k}(n) = P\{N_k \geq n\}$ characterizes the instantaneous channel strength to user i. Erasures are correlated because when a bit is erased, all the less significant bits are also erased. When N_k is fixed, the layered erasure model reduces to the binary expansion deterministic channel model introduced by Avestimehr, Diggavi and Tse [11]. While modeling fading as erasures has appeared in the literature (see for example [12]), typically these models regard the *entire* transmitted signal as erased and thus cannot capture the continuous nature of the channel strength in the Gaussian model.

III. Summary of Results

For the broadcast channels (1) and (3), we explore the boundaries of the respective capacity regions $\mathcal{C}$ via the non-negative weight vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K)$ and weighted sum rate maximization problem

$$R^*(\boldsymbol{\omega}) = \max_{(R_1,\ldots,R_K)\in\mathcal{C}} \sum_{k=1}^{K} \omega_k R_k. \tag{4}$$

For the layered erasure BC, we find that the capacity region is specified in terms of a partition of the set of signal levels $\cup_{k=1}^{K} \mathcal{I}_k(\boldsymbol{\omega}) = \{1, \ldots, q\}$ such that

$$\mathcal{I}_k(\boldsymbol{\omega}) = \left\{ n | \omega_k \overline{F}_{N_k}(n) \geq \max_j \omega_j \overline{F}_{N_j}(n) \right\}. \tag{5}$$

Theorem 1: The capacity region $\mathcal{C}$ of the K-user q-bit layered erasure broadcast channel $(N_1, \ldots, N_K)$ is the convex hull of the union over all non-negative weight vectors $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_K)$ of rate vectors $(R_1, \ldots, R_K)$ satisfying

$$R_k \leq \sum_{n\in\mathcal{I}_k(\boldsymbol{\omega})} \overline{F}_{N_k}(n), \qquad k \in \mathcal{K}. \tag{6}$$

We note that the channel for bit X_n to receiver k is simply an erasure channel with erasure probability $\overline{F}_{N_k}(n)$. Thus, $\omega_k \overline{F}_{N_k}(n)$ represents the expected "weighted return" for the assignment of bit level n to receiver k. Thus bit level n is greedily assigned to the receiver with the highest expected weighted return.

It should be apparent that the upper bound rate (6) is achievable by user k with the assignment of bits $n \in \mathcal{I}_k(\boldsymbol{\omega})$ and independent signaling on each bit X_n. We verify this is the capacity region in Section V by using the method of channel enhancement to create a K-user degraded broadcast channel that provides the outer bound.

A similar channel enhancement approach is used in Section VI for an outer bound to the real-valued fading Gaussian BC (2). Employing the connection between MMSE estimation and Shannon capacity for the Gaussian channel [13], an analogous partitioning of channel states in which

$$\mathcal{I}_k(\boldsymbol{\omega}) = \left\{ s \geq 0 | \omega_k \overline{F}_{S_k}(s) \geq \max_j \omega_j \overline{F}_{S_j}(s) \right\} \tag{7}$$

yields the following outer bound to the capacity region.

Theorem 2: The capacity region of the fading Gaussian broadcast channel (1) is contained in the convex hull of the union over all $\boldsymbol{\omega} \geq 0$ of rate vectors $(R_1, \ldots, R_K)$ satisfying

$$R_k \leq R_k^*(\boldsymbol{\omega}) := \log e \int_{\mathcal{I}_k(\omega)} \overline{F}_{S_k}(s) \frac{1}{1+s}\, ds, \quad k \in \mathcal{K}. \tag{8}$$

This outer bound can be interpreted as a partition of capacity by SNR. When $\mathcal{I}_k(\omega) = [0, \infty)$ and all channel states are assigned to user k, the bound reduces to the ergodic capacity of the user k fading channel. In general, this partition does not appear to be physically attainable and the bound is loose. However, we will see in Section VI that the proof has striking parallels to the converse for the layered erasure BC. In addition, some practical signaling schemes can be within a few bits of the outer bound at high SNR [10], [14].

IV. Background and Definitions

As introduced by Cover [1] for two users, the K-user memoryless broadcast channel (BC) consists of a transmitter with input X and receiver observations $Y_1, \ldots, Y_K$ described by a channel transition probability $P_{Y_1,\ldots,Y_K|X}(y_1, \ldots, y_k|x)$. Through this multiuser channel, the sender wishes to communicate private messages at rate R_i to receiver i as well as a common message at rate R_0 to all receivers.

In this work, we focus on the $R_0 = 0$ case where there is no common message. Even in this case, the general BC capacity region is unknown. However, the capacity region of the important special case of the degraded channel is known. A K-user broadcast channel is *degraded* if there exists a Markov chain X—Y_1—Y_2—$\cdots$—Y_K that yields the marginal conditional distributions $P_{Y_k|X}$, $k = 1, \ldots, K$, that are consistent with $P_{Y_1,\ldots,Y_K|X}$ [15]. As a generalization of the two-user result given in [2], [16], the capacity region of the K-user degraded memoryless BC was derived in [17] and is repeated here.

Theorem 3: The capacity region $\mathcal{R}$ of the degraded memoryless BC $P_{Y_1,\ldots,Y_K|X}$ is the union, over all $V_1, \ldots, V_K, X$ satisfying the Markov chain V_K—V_{K-1}—$\cdots$—V_1—X, of rate vectors $(R_1, \ldots, R_K)$ satisfying

$$R_K \leq I(V_K; Y_K), \tag{9a}$$

$$R_k \leq I(V_k; Y_k|V_{k+1}), \quad k = 1, \ldots, K-1. \tag{9b}$$

In the following, we will exploit the structure of the degraded K-user broadcast channel to prove the converse to Theorem 1 as well as the outer bound of Theorem 2. In the context of layered erasure channels, the following definition and lemma show that degradedness is associated with stochastic dominance of the q-bit fading distributions.

Lemma 1: The q-bit K-user layered erasure broadcast channel $(N_1, \ldots, N_K)$ satisfying $N_k \geq_{\text{st}} N_{k+1}$ is a degraded broadcast channel.

The proof appears in the appendix. The degraded fading Gaussian BC has an equally simple characterization.

Lemma 2: The fading Gaussian BC (S_1, S_2) satisfying $S_1 \geq_{\text{st}} S_2$ is a degraded broadcast channel.

The proof is analogous to the layered erasure channel.

V. Layered Erasure Broadcast Channel

Each component channel of the layered erasure BC (3) input $X = X^q$ is a layered erasure channel of the form $Y = X^N$. A key property of this channel is given in the next lemma.

Lemma 3: For a q-bit layered erasure channel with output X^N and Markov chain U—V—X^N,

$$I(V; X^N|U) = \sum_{n=1}^{q} \overline{F}_N(n) I(V; X_n|X^{n-1}, U).$$

The proof appears in the Appendix.

For arbitrary $N_1, \ldots, N_K$, the K-user q-bit layered erasure broadcast channel is not degraded. Our approach is to successively enhance the channels to successively less-favored receivers by replacing the fading distribution N_k by $\tilde{N}_k$ such that $\tilde{N}_k \geq_{\text{st}} N_k$ and $\tilde{N}_k \geq_{\text{st}} \tilde{N}_{k+1}$. It follows that any reliable broadcast communication strategy that communicates at rates $(R_1, \ldots, R_K)$ through the q-bit layered erasure BC $(N_1, \ldots, N_K)$ is also a reliable strategy through the q-bit fading BC $(\tilde{N}_1, \ldots, \tilde{N}_K)$, implying $\mathcal{R}(N_1, \ldots, N_K) \subseteq \mathcal{R}(\tilde{N}_1, \ldots, \tilde{N}_K)$. We will show for each $\boldsymbol{\omega}$ that the enhancement of the channels will not result in the transmission of additional information through those enhanced channels. Consequently, the maximum weighted sum rate $\sum_{k=1}^{K} \omega_k R_k$ will be the same whether the maximization is over $\mathcal{R}(N_1, \ldots, N_K)$ or $\mathcal{R}(\tilde{N}_1, \ldots, \tilde{N}_K)$.

To define the enhancement, we assume wlog that

$$\omega_1 \leq \omega_2 \leq \cdots \leq \omega_K. \tag{10}$$

For convenience, we define $\omega_{K+1} := 0$ and the trivial $N_{K+1} = 0$. Given that each receiver $j > k$ has an enhanced channel given by $\tilde{N}_j$, the enhanced channel for receiver k is given by

$$\overline{F}_{\tilde{N}_k}(n) = \min\left[1, \max\left(\overline{F}_{N_k}(n), \frac{\omega_{k+1}}{\omega_k}\overline{F}_{\tilde{N}_{k+1}}(n)\right)\right]. \tag{11}$$

As $\omega_{K+1} = 0$, the channel for receiver K is unchanged: $\overline{F}_{\tilde{N}_K}(n) = \overline{F}_{N_K}(n)$. We summarize key properties of these successive enhancements in the following lemma.

Lemma 4: The layered erasure BC $(\tilde{N}_1, \ldots, \tilde{N}_K)$ has the following properties:

(a) If $\omega_k \overline{F}_{\tilde{N}_k}(n) > \omega_{k+1}\overline{F}_{\tilde{N}_{k+1}}(n)$, then

$$\omega_k \overline{F}_{N_k}(n) = \omega_k \overline{F}_{\tilde{N}_k}(n) > \max_{j>k} \omega_j \overline{F}_{N_j}(n). \tag{12}$$

(b) If $\omega_k \overline{F}_{\tilde{N}_k}(n) < \omega_{k+1}\overline{F}_{\tilde{N}_{k+1}}(n)$, then

$$\overline{F}_{\tilde{N}_j}(n) = 1, \qquad j \leq k. \tag{13}$$

(c) Receiver $k^*(n)$ satisfying

$$k^*(n) = \min\left\{j | \omega_j \overline{F}_{\tilde{N}_j}(n) > \omega_{j+1}\overline{F}_{\tilde{N}_{j+1}}(n)\right\} \tag{14}$$

has the property that

$$\omega_{k^*(n)}\overline{F}_{N_{k^*(n)}}(n) = \max_j \omega_j \overline{F}_{N_j}(n). \tag{15}$$

In Lemma 4, properties (a) and (b) follow directly from the construction of the enhanced channels. Further, $k^*(n)$ is the minimum over the set of indices k for which channel k was not enhanced at level n and thus its weighted return $\omega_k \overline{F}_{\tilde{N}_k}(n)$ is the maximum over all channels. From this argument, property (c) also follows.

Before proceeding, we note that it is easy to verify that $\overline{F}_{\tilde{N}_k}(n)$ is a well-defined complementary CDF for a q-bit fading random variable such that $\tilde{N}_k \geq_{\text{st}} N_k$. Further, we observe that (10) and (11) imply $\tilde{N}_k \geq_{\text{st}} \tilde{N}_{k+1}$. It then follows from Lemma 1 that $(\tilde{N}_1, \ldots, \tilde{N}_K)$ is a degraded q-bit broadcast channel. We now use Lemma 3 to apply Theorem 3 to the q-bit layered erasure broadcast channel $(\tilde{N}_1, \ldots, \tilde{N}_K)$. For convenience we define V_{K+1} to be a trivial random variable so that the capacity region $\tilde{\mathcal{R}} = \mathcal{R}(\tilde{N}_1, \ldots, \tilde{N}_K)$ is the set of all rate vectors $(\tilde{R}_1, \ldots, \tilde{R}_K)$ satisfying

$$\tilde{R}_k \leq I(V_k; X^{\tilde{N}_k}|V_{k+1}), \quad k \in \mathcal{K}. \tag{16}$$

We see that the weighted sum rate achieved by any feasible rate vector $(R_1, \ldots, R_K) \in \tilde{\mathcal{R}}$ is upper bounded by the weighted sum rate

$$\tilde{R}^*(\boldsymbol{\omega}, \mathbf{V}) = \sum_{k=1}^{K} \omega_k \tilde{R}_k^*(\mathbf{V}) \tag{17}$$

associated with the extreme point rate vector $(\tilde{R}_1^*, \ldots \tilde{R}_K^*)$ defined by

$$\tilde{R}_k^*(\mathbf{V}) = I(V_k; X^{\tilde{N}_k}|V_{k+1}), \quad k \in \mathcal{K}, \tag{18}$$

for some auxiliary $\mathbf{V} = (V_1, \ldots, V_K)$.

We now maximize $\tilde{R}^*(\boldsymbol{\omega}, \mathbf{V})$ over the auxiliary $\mathbf{V}$. Applying Lemma 3 with $V = V_k$ and $U = V_{k+1}$, we obtain the

weighted rate

$$\begin{aligned}\tilde{R}^*(\boldsymbol{\omega},\mathbf{V}) &= \sum_{k=1}^{K}\omega_k I(V_k;X^{\tilde{N}_k}|V_{k+1}) &(19)\\ &= \sum_{k=1}^{K}\omega_k\sum_{n=1}^{q}\overline{F}_{\tilde{N}_k}(n)I(V_k;X_n|X^{n-1},V_{k+1}) &(20)\\ &= \sum_{n=1}^{q}\sum_{k=1}^{K}\omega_k\overline{F}_{\tilde{N}_k}(n)\left[H(X_n|X^{n-1},V_{k+1})\right.\\ &\qquad\left.-H(X_n|X^{n-1},V_k,V_{k+1})\right]. &(21)\end{aligned}$$

The Markov chain V_{k+1}—V_k—X^n implies

$$H(X_n|X^{n-1},V_k,V_{k+1}) = H(X_n|X^{n-1},V_k). \quad (22)$$

Thus,

$$\tilde{R}^*(\boldsymbol{\omega},\mathbf{V}) = \sum_{n=1}^{q}\sum_{k=1}^{K}\omega_k\overline{F}_{\tilde{N}_k}(n)g_{n,k}, \quad (23)$$

where

$$g_{n,k} := H(X_n|X^{n-1},V_{k+1}) - H(X_n|X^{n-1},V_k). \quad (24)$$

The Markov chain V_{k+1}—V_k—X^n implies

$$g_{n,k} \geq 0. \quad (25)$$

In addition $\sum_{k=1}^{K} g_{n,k}$ is a telescoping sum such that

$$\sum_{k=1}^{K} g_{n,k} = H(X_n|X^{n-1}) - H(X_n|X^{n-1},V_1) \leq 1. \quad (26)$$

Define

$$G = \left\{(g_1,\ldots,g_K) \geq 0 | \sum_{k=1}^{K} g_k \leq 1\right\} \quad (27)$$

Since $(g_{n,1},\ldots,g_{n,K}) \in G$, it follows from (23) that

$$\begin{aligned}\tilde{R}^*(\omega,\mathbf{V}) &\leq \sum_{n=1}^{q}\max_{(g_1,\ldots,g_K)\in G}\sum_{k=1}^{K}\omega_k\overline{F}_{\tilde{N}_k}(n)g_k &(28)\\ &= \sum_{n=1}^{q}\max_k \omega_k\overline{F}_{\tilde{N}_k}(n). &(29)\end{aligned}$$

By construction of the enhanced channels $\tilde{N}$, for each n the maximum $\omega_k\overline{F}_{\tilde{N}_k}(n)$ is not unique. In particular, given n, we choose index $k = k^*(n)$. By Lemma 4(c), we can define $\mathcal{I}_k(\omega) = \{n|k^*(n) = k\}$. By Lemma 4(a), $\overline{F}_{\tilde{N}_k}(n) = \overline{F}_{N_k}(n)$ for $n \in \mathcal{I}_k(\omega)$ enabling us to write

$$\tilde{R}^*(\omega,\mathbf{V}) = \sum_{k=1}^{K}\omega_k\sum_{n\in\mathcal{I}_k(\omega)}\overline{F}_{\tilde{N}_k}(n). \quad (30)$$

Hence, we have the desired converse for Theorem 1.

VI. Fading Gaussian BC

From the weighted sum rate maximization (4), we examine $\mathcal{C}$, the capacity region of the real-valued fading broadcast channel (2). As in the layered erasure BC, this channel is not degraded. However, we can successively enhance the channels of less favored users to make a K-user degraded BC. Following enhancement, the fading state of user k has complementary CDF

$$\overline{F}_{\tilde{S}_k}(s) := \min\left[1, \max\left(\overline{F}_{S_k}(s), \frac{\omega_{k+1}}{\omega_k}\overline{F}_{\tilde{S}_{k+1}}(s)\right)\right]. \quad (31)$$

As $\omega_{K+1} = 0$, the channel for receiver K is unchanged: $\overline{F}_{\tilde{S}_K}(n) = \overline{F}_{S_K}(n)$. We summarize key properties of these successive enhancements in the following lemma.

Lemma 5: The enhanced channels $\tilde{S}_1,\ldots,\tilde{S}_K$ have the following properties:

(a) If $\omega_k\overline{F}_{\tilde{S}_k}(s) > \omega_{k+1}\overline{F}_{\tilde{N}_{s+1}}(s)$, then

$$\omega_k\overline{F}_{S_k}(s) = \omega_k\overline{F}_{\tilde{S}_k}(s) > \max_{j>k}\omega_j\overline{F}_{S_j}(s). \quad (32)$$

(b) If $\omega_k\overline{F}_{\tilde{S}_k}(s) < \omega_{k+1}\overline{F}_{\tilde{S}_{k+1}}(s)$, then

$$\overline{F}_{\tilde{S}_j}(s) = 1, \qquad j \leq k. \quad (33)$$

(c) Receiver $k^*(s)$ satisfying

$$k^*(s) = \min\left\{j|\omega_j\overline{F}_{\tilde{S}_j}(s) > \omega_{j+1}\overline{F}_{\tilde{S}_{j+1}}(s)\right\} \quad (34)$$

has the property that

$$\omega_{k^*(s)}\overline{F}_{S_{k^*(s)}}(s) = \max_j \omega_j\overline{F}_{S_j}(s). \quad (35)$$

In Lemma 5, properties (a) and (b) follow directly from the construction of the enhanced channels. Further, $k^*(s)$ is the minimum over the set of indices k for which channel k was not enhanced at state s and thus its weighted return $\omega_k\overline{F}_{\tilde{S}_k}(s)$ is the maximum over all channels. From this argument, property (c) also follows.

Before proceeding, we observe that (10) and (31) imply $\tilde{S}_k \geq_{\text{st}} \tilde{S}_{k+1}$. It then follows from Lemma 1 that $(\tilde{S}_1,\ldots,\tilde{S}_K)$ is a degraded fading Gaussian broadcast channel.

As the channel state is known at each receiver, we view $(Y_k,\tilde{S}_k)$ as the observed outputs at receiver k respectively, and apply Theorem 3 to the q-bit layered erasure broadcast channel $(\tilde{N}_1,\ldots,\tilde{N}_K)$. For convenience we define V_{K+1} to be a trivial random variable so that the capacity region $\tilde{\mathcal{R}} = \mathcal{R}(\tilde{N}_1,\ldots,\tilde{N}_K)$ is the set of all rate vectors $(\tilde{R}_1,\ldots,\tilde{R}_K)$ satisfying

$$\tilde{R}_k \leq I(V_k;Y_k,\tilde{S}_k|V_{k+1}) = I(V_k;Y_k|V_{k+1},\tilde{S}_k). \quad (36)$$

We see that the weighted sum rate achieved by any feasible rate vector $(R_1,\ldots,R_K) \in \tilde{\mathcal{R}}$ is upper bounded by the weighted sum rate

$$\tilde{R}^*(\boldsymbol{\omega},\mathbf{V}) = \sum_{k=1}^{K}\omega_k\tilde{R}_k^*(\mathbf{V}) \quad (37)$$

associated with the extreme point rate vector $(\tilde{R}_1^*, \ldots \tilde{R}_K^*)$ defined by

$$\tilde{R}_k^*(\mathbf{V}) = I(V_k; Y_k | V_{k+1} \tilde{S}_k) \tag{38}$$

$$= \int_0^\infty f_{\tilde{S}_k}(s)\, I(V_k; Y_k | V_{k+1} \tilde{S}_k = s)\, ds \tag{39}$$

for some auxiliary $\mathbf{V} = (V_1, \ldots, V_K)$.

We now optimize $\tilde{R}^*(\boldsymbol{\omega}, \mathbf{V})$ over the auxiliary $\mathbf{V}$. When channel k is in state s, receiver k observes an output identically distributed as

$$Y^{(s)} := \sqrt{s} X + Z, \tag{40}$$

where Z denotes a $\mathcal{N}(0,1)$ random variable identical to each Z_k. Thus (39) implies

$$\tilde{R}_k^*(\mathbf{V}) = \int_0^\infty f_{\tilde{S}_k}(s)\, I(V_k; Y^{(s)} | V_{k+1})\, ds. \tag{41}$$

We note $Y^{(s)}\big|_{s=0} = Z$, implying $I(V_k; Y^{(s)}|V_{k+1})\big|_{s=0} = 0$. Applying integration by parts to (41), it follows that

$$\tilde{R}_k^*(\mathbf{V}) = \int_0^\infty \overline{F}_{\tilde{S}_k}(s) I'(V_k; Y^{(s)} | V_{k+1})\, ds, \tag{42}$$

where $I'(V_k; Y^{(s)}|V_{k+1})$ denotes the derivative with respect to channel state s.

The Markov chain V_{k+1}—V_k—X—$Y^{(s)}$ implies

$$I(V_k; Y^{(s)} | V_{k+1}) = h(Y^{(s)}|V_{k+1}) - h(Y^{(s)}|V_k V_{k+1}) \tag{43}$$

$$= h(Y^{(s)}|V_{k+1}) - h(Y^{(s)}|V_k) \tag{44}$$

and

$$I(X; Y^{(s)} | V_k) = h(Y^{(s)}|V_k) - h(Y^{(s)}|V_k, X) \tag{45}$$

$$= h(Y^{(s)}|V_k) - h(Y^{(s)}|X) \tag{46}$$

$$= h(Y^{(s)}|V_k) - h(Z). \tag{47}$$

Combining (44) and (47), we obtain

$$I(V_k; Y^{(s)} | V_{k+1}) = I(X; Y^{(s)} | V_{k+1}) - I(X; Y^{(s)} | V_k). \tag{48}$$

It was shown in [13] that

$$I'(X; Y^{(s)} | V) = \frac{\log e}{2} \mathrm{mmse}\,(s|V)\,, \tag{49}$$

where, given a conditioning variable V,

$$\mathrm{mmse}\,(s|V) := E\left[\left(X - E\left[X|Y^{(s)}, V\right]\right)^2\right] \tag{50}$$

denotes the mean square error of the conditional mean estimator $E\left[X|Y^{(s)}, V\right]$.

Taking a derivative with respect to the channel SNR s in (48) yields

$$I'(V_k; Y^{(s)} | V_{k+1}) = I'(X; Y^{(s)} | V_{k+1}) - I'(X; Y^{(s)} | V_k) \tag{51}$$

$$= \frac{\log e}{2} \left(\mathrm{mmse}\,(s|V_{k+1}) - \mathrm{mmse}\,(s|V_k)\right). \tag{52}$$

With the definition

$$g_k(s) := \mathrm{mmse}\,(s|V_{k+1}) - \mathrm{mmse}\,(s|V_k)\,, \quad k \in \mathcal{K}, \tag{53}$$

we can combine (42) and (52) to write

$$\tilde{R}_k^*(\mathbf{V}) = \frac{\log e}{2} \int_0^\infty \overline{F}_{\tilde{S}_k}(s) g_k(s)\, ds. \tag{54}$$

Combining (37) and (54), we observe that our objective is to choose $\mathbf{V}$ to maximize

$$\tilde{R}^*(\boldsymbol{\omega}, \mathbf{V}) = \frac{\log e}{2} \sum_{k=1}^K \omega_k \int_0^\infty \overline{F}_{\tilde{S}_k}(s) g_k(s)\, ds \tag{55}$$

$$= \frac{\log e}{2} \int_0^\infty \sum_{k=1}^K \omega_k \overline{F}_{\tilde{S}_k}(s) g_k(s)\, ds. \tag{56}$$

The Markov chain V_K—V_{K-1}—$\cdots$—V_1—X, along with the definition of the trivial V_{K+1}, implies

$$\frac{1}{1+s} = \mathrm{mmse}\,(s|V_{K+1}) \geq \cdots \geq \mathrm{mmse}\,(s|V_1) \geq 0. \tag{57}$$

Thus the $g_k(s)$ are non-negative and form a telescoping sum satisfying

$$\sum_{k=1}^K g_k(s) \leq \frac{1}{1+s}. \tag{58}$$

It follows for any choice of auxiliaries $\mathbf{V}$ and channel state s that $(g_1(s), \ldots, g_K(s)) \in G^{(s)}$ defined by

$$G^{(s)} := \left\{ \mathbf{g} = (g_1, \ldots, g_K) \geq 0 | \sum_{k=1}^K g_k \leq \frac{1}{1+s} \right\}. \tag{59}$$

Thus, the choice of $\mathbf{V}$ specifies the $g_k(s)$ and it follows from (56) that

$$\tilde{R}^*(\boldsymbol{\omega}, \mathbf{V}) \leq \frac{\log e}{2} \int_0^\infty \left(\max_{\mathbf{g} \in G^{(s)}} \sum_{k=1}^K \omega_k \overline{F}_{\tilde{S}_k}(s) g_k \right) ds \tag{60}$$

$$= \frac{\log e}{2} \int_0^\infty \left(\max_k \omega_k \overline{F}_{\tilde{S}_k}(s) \right) \frac{1}{1+s}\, ds. \tag{61}$$

In particular, we see that (61) can be interpreted an assignment of channel state s to the the user k that maximizes the incremental return $\omega_k \overline{F}_{\tilde{S}_k}(s)$. By construction of the enhanced channels $\tilde{S}_k$, the maximum $\omega_k \overline{F}_{\tilde{S}_k}(s)$ is not unique. In particular, given s, we choose index $k = k^*(s)$ in (34). By Lemma 5(c), we can define $\mathcal{I}_k(\omega) = \{s | k^*(n) = k\}$. By Lemma 5(a), $= \overline{F}_{\tilde{S}_k}(s) = \overline{F}_{S_k}(s)$ for $s \in \mathcal{I}_k(\omega)$, enabling us to write

$$\tilde{R}^*(\omega, \mathbf{V}) \leq \frac{\log e}{2} \sum_{k=1}^K \omega_k \int_{\mathcal{I}_k(\omega)} \overline{F}_{S_k}(s) \frac{1}{1+s}\, ds. \tag{62}$$

Returning to the complex fading Gaussian BC (1) in which rates are twice those of the in-phase channel, we observe that the components of the upper bound (62) correspond to the extreme points of the outer bound of Theorem 2, thus completing the proof of this claim.

VII. Conclusion

The parallel proofs in Sections V and VI highlight the connections between the layered erasure channel and the fading AWGN channel. A curious feature of these proofs is that we do not need to deduce the auxiliaries. In the case of the layered erasure channel, it can be resolved from (24) that if bit n is assigned to user k then V_i specifies X_n for users $i \leq k$ while V_j is independent of X_n for users $j > k$. In the fading Gaussian channel, the role of the auxiliary is less clear. The Theorem 2 outer bound uses the auxiliaries to associate SNR levels s with particular users. Unfortunately, this does not appear to be physically realizable.

Appendix

Proof: Lemma 3 As N is a deterministic function of X^N,

$$I(V;X^N|U) = I(V;X^N,N|U) \tag{63}$$
$$= I(V;N|U) + I(V;X^N|U,N). \tag{64}$$

Since U and V are independent of the channel state N, $I(V;N|U) = 0$ and

$$I(V;X^N|U) = I(V;X^N|U,N) \tag{65}$$
$$= \sum_{n=1}^{q} P_N(n)\, I(V;X^n|U,N=n) \tag{66}$$

Since U, V, and X^n are independent of N,

$$I(V;X^N|U) = \sum_{n=1}^{q} P_N(n)\, I(V;X^n|U) \tag{67}$$
$$= \sum_{n=1}^{q}\sum_{j=1}^{n} P_N(n)\, I(V;X_j|X^{j-1},U). \tag{68}$$

Reversing the order of summation then yields

$$I(V;X^N|U) = \sum_{j=1}^{q}\sum_{n=j}^{q} P_N(n)\, I(V;X_n|X^{n-1},U) \tag{69}$$
$$= \sum_{j=1}^{q} \overline{F}_N(j) I(V;X_n|X^{n-1},U). \tag{70}$$

□

Proof: Lemma 1 It is sufficient to prove $N_k \geq_{\text{st}} N_{k+1}$ implies the BC with input X^q and outputs X^{N_k} and $X^{N_{k+1}}$ is degraded. To do so, we define N'_k, N'_{k+1} to have the joint CCDF

$$\overline{F}_{N'_k,N'_{k+1}}(n_k,n_{k+1})$$
$$= P\{N'_k \geq n_k, N'_{k+1} \geq n_{k+1}\} \tag{71}$$
$$= \min\left[\overline{F}_{N_k}(n_k), \overline{F}_{N_{k+1}}(n_{k+1})\right]. \tag{72}$$

We observe that N'_k has marginal CCDF $\overline{F}_{N'_k}(n_k) = \overline{F}_{N'_k,N'_{k+1}}(n_k,0) = \overline{F}_{N_k}(n_k)$. Similarly, $\overline{F}_{N'_{k+1}}(n_{k+1}) = \overline{F}_{N_{k+1}}(n_{k+1})$. Since $\overline{F}_{N_k}(n) \geq \overline{F}_{N_k}(n+1) \geq \overline{F}_{N_{k+1}}(n+1)$, we observe that

$$P\{N'_k = n, N'_{k+1} > n\}$$
$$= \overline{F}_{N'_k,N'_{k+1}}(n,n+1) - \overline{F}_{N'_k,N'_{k+1}}(n+1,n+1) \tag{73}$$
$$= \min\left[\overline{F}_{N_k}(n), \overline{F}_{N_{k+1}}(n+1)\right]$$
$$\quad - \min\left[\overline{F}_{N_k}(n+1), \overline{F}_{N_{k+1}}(n+1)\right] \tag{74}$$
$$= \overline{F}_{N_{k+1}}(n+1) - \overline{F}_{N_{k+1}}(n+1) = 0. \tag{75}$$

Thus, $N'_k \geq N'_{k+1}$ for all random samples. Hence, the layered erasure BC (N'_k, N'_{k+1}) obeys the Markov chain X^q—$X^{N'_k}$—$X^{N'_{k+1}}$ in that $X^{N'_k}$ is passed to receiver 1, which given N'_k draws a sample N'_{k+1} using $\overline{F}_{N'_k,N'_{k+1}}(n_k,n_{k+1})$ and then sends $X^{N'_{k+1}}$ to receiver 2. As N_k and N_{k+1} have the same marginals as N'_k and N'_{k+1}, we conclude that the layered erasure BC (N_k, N_{k+1}) is degraded. □

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