Capacity Bounds for Full-Duplex Two-Way Relay Channel with Feedback

Silas L. Fong
Department of Information Engineering
The Chinese University of Hong Kong
Shatin, N.T., Hong Kong
Email: lhfong5@ie.cuhk.edu.hk

Raymond Yeung
Department of Information Engineering
The Chinese University of Hong Kong
Shatin, N.T., Hong Kong
Email: whyeung@ie.cuhk.edu.hk

***Abstract*—We consider a two-way relay channel (TRC) in which two terminals exchange their messages with the help of a relay between them. The two terminals transmit messages to the relay through the Multiple Access Channel (MAC) and the relay transmits messages to the two terminals through the Broadcast Channel (BC). We assume that the MAC and the BC do not interfere with each other, and each terminal receives signals only from the relay but not the other terminal. All the nodes are assumed to be full-duplex, which means that they can transmit and receive information at the same time. We present with detailed proofs an outer bound on the capacity region of each of the discrete memoryless TRC with feedback and the Gaussian TRC with feedback. We show that the outer bound for the discrete memoryless TRC with feedback is loose for some TRC.**

I. Introduction

We consider a two-way relay channel (TRC) [1], in which two terminals exchange their messages with the help of a relay between them. The two terminals transmit messages to the relay through the Multiple Access Channel (MAC) and the relay transmits messages to the two terminals through the Broadcast Channel (BC). We assume that the MAC and the BC do not interfere with each other, and each terminal receives signals only from the relay but not the other terminal. All the nodes are assumed to be full-duplex, which means that they can transmit and receive information at the same time. The terminal nodes can use their previously received messages for encoding their messages, and the TRC described above is said to be *with feedback*.

Although several achievable rate regions for the TRC with feedback have been obtained in [2] and [3], the capacity region of the TRC is unknown. Knopp [4] stated an outer bound on the capacity region of each of the discrete memoryless TRC with feedback and the Gaussian TRC with feedback. However, the proofs in [4] are incomplete, which motivates us to present detailed proofs of the outer bounds stated in [4]. The proofs are nontrivial.

This paper is organized as follows. Section II presents the notation of this paper. Section III presents the proof of the outer bound on the capacity region of the discrete memoryless TRC with feedback. Section IV shows that the outer bound for the discrete memoryless TRC with feedback is loose for some TRC. Section V presents the proof of the outer bound on the capacity region of the Gaussian TRC with feedback. Section VI concludes this paper.

II. Notation

We use $Pr\{E\}$ to represent the probability of an event E. We use a capital letter X to denote a random variable with alphabet $\mathcal{X}$, and use the small letter x to denote the realization of X. We use $E[X]$ to represent the expectation of a random variable X. We use X^n to denote a random column vector $[X_1, X_2, \ldots, X_n]^T$, where the components X_k have the same alphabet. We let $p_X(x)$ and $p_{X^n}(x^n)$ denote the probability mass functions of the discrete random variables X and X^n respectively. We let $p_{Y|X}(y|x)$ denote the conditional probability $Pr\{Y = y|X = x\}$ for any general random variable X and any discrete random variable Y. We let $f_X(x)$ and $f_{X^n}(x^n)$ denote the probability density functions of the random variables X and X^n respectively. We let $f_{Y|X}(y|x)$ denote the probability density function of Y conditioned on the event $\{X = x\}$ for any continuous random variable Y and any general random variable X. For simplicity, we drop the subscript of a notation if there is no ambiguity.

Let $H(X) = E[-\log_2 p(x)]$ and $H(X|Y) = E[H(X|Y = y)]$ be the entropy of X and the entropy of X conditioned on Y respectively for any discrete random variable X and any general random variable Y. Let $h(X)$ and $h(X|Y) = E[h(X|Y = y)]$ be the differential entropy of X and the differential entropy of X conditioned on Y respectively for any continuous random variable X and any general random variable Y. Let $I(X;Y) = E[\log_2(p(x|y)/p(x))]$ be the mutual information of X and Y, where X is a discrete random variable and Y is either a discrete or a continuous random variable.

III. Full-Duplex Discrete Memoryless Two-Way Relay Channel with Feedback

In this section, we study the full-duplex discrete memoryless TRC in which the terminal nodes can use their previously received messages for encoding their messages. Knopp [4] stated an outer bound on the capacity region of the discrete memoryless TRC. However, the proof in [4] is incomplete. Therefore, we present a detailed proof of the outer bound in

[4] in this section. The formulation of the discrete memoryless TRC is as follows.

The TRC consists of two terminal nodes t_1 and t_2 and a relay node r between them. Node t_1 and node t_2 do not communicate directly, but communicate through node r using two different channels. In each time slot, node t_1 and node t_2 transmit a symbol to node r through the Multiple Access Channel (MAC), and node r transmits a symbol to node t_1 and node t_2 through the Broadcast Channel (BC). The MAC and the BC do not interfere with each other. Suppose node t_1 and node t_2 exchange information in n time slots. Then, the variables involved in the communication through the TRC are:

- $W_i \in \{1, 2, \ldots, M_i\}$: messages of node t_i, $i = 1, 2$,
- $X_i^n = [X_{i,1}\ X_{i,2}\ \ldots\ X_{i,n}]^T$: codewords transmitted by node t_i, $i = 1, 2$,
- $Y_r^n = [Y_{r,1}\ Y_{r,2}\ \ldots\ Y_{r,n}]^T$: channel outputs at the relay,
- $X_r^n = [X_{r,1}\ X_{r,2}\ \ldots\ X_{r,n}]^T$: codewords transmitted by the relay,
- $Y_i^n = [Y_{i,1}\ Y_{i,2}\ \ldots\ Y_{i,n}]^T$: channel outputs at node t_i, $i = 1, 2$,
- $\hat{W}_i \in \{1, 2, \ldots, M_i\}$: message estimates of message W_i, $i = 1, 2$.

Node t_1 and node t_2 choose messages W_1 and W_2 independently according to the uniform distribution. In the kth time slot, node t_i constructs and transmits $X_{i,k}$, which is a function of W_i and Y_i^{k-1} for $i = 1, 2$, to node r through the MAC. The received symbol at node r is $Y_{r,k}$. In the same time slot, node r constructs and transmits $X_{r,k}$, which depends on Y_r^{k-1}, to node t_1 and node t_2 through the BC. The received symbol at node t_i is $Y_{i,k}$ for $i = 1, 2$. After n time slots, node t_1 declares $\hat{W}_2$ to be the transmitted W_2 based on Y_1^n and W_1, and node t_2 declares $\hat{W}_1$ to be the transmitted W_1 based on Y_2^n and W_2.

Definition 1: The discrete memoryless TRC consists of six finite sets $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_r$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_r$, one probability mass function $p_1(y_r|x_1, x_2)$ representing the MAC and one probability mass function $p_2(y_1, y_2|x_r)$ representing the BC. For any two inputs X_1 and X_2 to the MAC with a joint distribution $p(x_1, x_2)$ and any input X_r to the BC with a distribution $p(x_r)$, the relationship among X_1, X_2, X_r, the output Y_r of the MAC and the outputs Y_1 and Y_2 of the BC satisfies

$$p(x_1, x_2, x_r, y_1, y_2, y_r) = p(x_1, x_2, y_r)p(x_r, y_1, y_2)$$

for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, $x_r \in \mathcal{X}_r$, $y_r \in \mathcal{Y}_r$, $y_1 \in \mathcal{Y}_1$ and $y_2 \in \mathcal{Y}_2$, where

$$p(x_1, x_2, y_r) = p(x_1, x_2)p_1(y_r|x_1, x_2)$$

and

$$p(x_r, y_1, y_2) = p(x_r)p_2(y_1, y_2|x_r).$$

The discrete memoryless TRC is denoted by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r, p_1(y_r|x_1, x_2), p_2(y_1, y_2|x_r), \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_r)$.

Definition 2: An (n, M_1, M_2)-code on the channel $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r, p_1(y_r|x_1, x_2), p_2(y_1, y_2|x_r), \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_r)$ consists of the following:

1) A message set $\mathcal{W}_1 = \{1, 2, \ldots, M_1\}$ at node t_1 and a message set $\mathcal{W}_2 = \{1, 2, \ldots, M_2\}$ at node t_2.
2) A set of n encoding functions $f_{1,k} : \mathcal{W}_1 \times \mathcal{Y}_1^{k-1} \to \mathcal{X}_1$ at node t_1 for $k = 1, 2, \ldots, n$, where $f_{1,k}$ is the encoding function in the kth time slot such that $X_{1,k} = f_{1,k}(W_1, Y_1^{k-1})$.
3) A set of n encoding functions $f_{2,k} : \mathcal{W}_2 \times \mathcal{Y}_2^{k-1} \to \mathcal{X}_2$ at node t_2 for $k = 1, 2, \ldots, n$, where $f_{2,k}$ is the encoding function in the kth time slot such that $X_{2,k} = f_{2,k}(W_2, Y_2^{k-1})$.
4) A set of n encoding functions $f_{r,k} : \mathcal{Y}_r^{k-1} \to \mathcal{X}_r$ at r for $k = 1, 2, \ldots, n$, where $f_{r,k}$ is the encoding function in the kth time slot such that $X_{r,k} = f_{r,k}(X_r^{k-1})$.
5) A decoding function $g_1 : \mathcal{W}_1 \times \mathcal{Y}_1^n \to \mathcal{W}_2$ at node t_1 such that $g_1(W_1, Y_1^n) = \hat{W}_2$.
6) A decoding function $g_2 : \mathcal{W}_2 \times \mathcal{Y}_2^n \to \mathcal{W}_1$ at node t_2 such that $g_2(W_2, Y_2^n) = \hat{W}_1$.

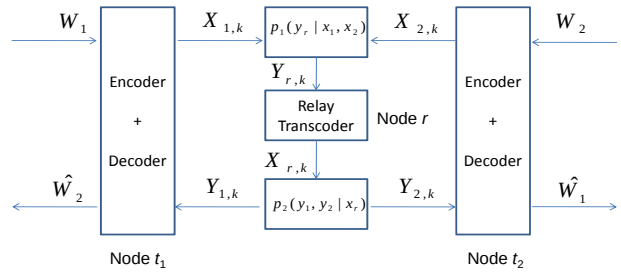

Fig. 1. A discrete memoryless TRC with feedback.

The transmissions of messages in the TRC are illustrated in Figure 1. The following proposition is cited from Proposition 2.5 of [5] to facilitate discussion.

Proposition 1: For the discrete random variables X, Y and Z, $X \to Y \to Z$ forms a Markov Chain if and only if there exists two functions $\chi(x, y)$ and $\varphi(y, z)$ such that $p(x, y, z) = \chi(x, y)\varphi(y, z)$ for all x, y and z where $p(y) > 0$.

Lemma 2: For any (n, M_1, M_2)-code on the discrete memoryless TRC,

$$U \to (X_{1,k}, X_{2,k}) \to Y_{r,k}, \tag{1}$$

$$U \to X_{r,k} \to (Y_{1,k}, Y_{2,k}), \tag{2}$$

$$(U, Y_{1,k}, Y_{2,k}) \to (X_{1,k}, X_{2,k}) \to Y_{r,k} \tag{3}$$

and

$$(U, Y_{r,k}) \to X_{r,k} \to (Y_{1,k}, Y_{2,k}) \tag{4}$$

form four Markov Chains for $k = 1, 2, \ldots, n$, where $U = (W_1, W_2, X_1^{k-1}, X_2^{k-1}, X_r^{k-1}, Y_1^{k-1}, Y_2^{k-1}, Y_r^{k-1})$. In addition, $p(x_{1,k}, x_{2,k}, y_{r,k}) = p(x_{1,k}, x_{2,k})p_1(y_{r,k}|x_{1,k}, x_{2,k})$ and $p(x_{r,k}, y_{1,k}, y_{2,k}) = p(x_{r,k})p_2(y_{1,k}, y_{2,k}|x_{r,k})$ for $k = 1, 2, \ldots, n$.

Proof: Fix an (n, M_1, M_2)-code on the discrete memoryless TRC and consider any $1 \le k \le n$. Figure 2 shows the dependency graph of the variables W_1, W_2, $X_{1,j}$, $X_{2,j}$, $X_{r,j}$,

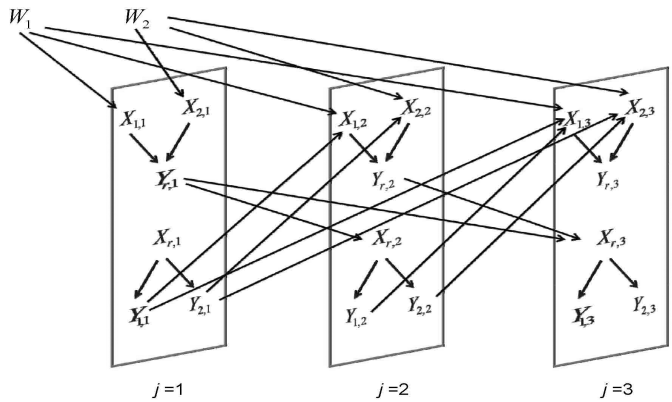

Fig. 2. Dependency graph of the discrete memoryless TRC in three time slots.

$Y_{1,j}$, $Y_{2,j}$ and $Y_{r,j}$ for $j = 1, 2, 3$. We can then extend the dependency graph so that the extended graph includes all the variables by the kth time slot and conclude from the extended graph that for any $w_1 \in \mathcal{W}_1$, $w_2 \in \mathcal{W}_2$, $x_1^k \in \mathcal{X}_1^k$, $x_2^k \in \mathcal{X}_2^k$, $x_r^k \in \mathcal{X}_r^k$, $y_1^k \in \mathcal{Y}_1^k$, $y_2^k \in \mathcal{Y}_2^k$ and $y_r^k \in \mathcal{Y}_r^k$ with $p(u) > 0$ where $u = (w_1, w_2, x_1^{k-1}, x_2^{k-1}, x_r^{k-1}, y_1^{k-1}, y_2^{k-1}, y_r^{k-1})$,

$$\begin{aligned} &p(u, x_{1,k}, x_{2,k}, x_{r,k}, y_{1,k}, y_{2,k}, y_{r,k}) \\ &\quad = p(u)p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k})p(x_{r,k}|u) \\ &\qquad p_2(y_{1,k}, y_{2,k}|x_{r,k}). \end{aligned} \tag{5}$$

If $p(u) > 0$, then

$$\begin{aligned} &p(u, x_{1,k}, x_{2,k}, y_{r,k}) \\ &\quad = \sum_{x_{r,k}, y_{1,k}, y_{2,k}} p(u, x_{1,k}, x_{2,k}, x_{r,k}, y_{1,k}, y_{2,k}, y_{r,k}) \\ &\quad \overset{(a)}{=} \sum_{x_{r,k}, y_{1,k}, y_{2,k}} p(u)p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k}) \\ &\qquad p(x_{r,k}|u)p_2(y_{1,k}, y_{2,k}|x_{r,k}) \\ &\quad = p(u)p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k}) \\ &\qquad \sum_{x_{r,k}, y_{1,k}, y_{2,k}} p(x_{r,k}|u)p_2(y_{1,k}, y_{2,k}|x_{r,k}) \\ &\quad = p(u)p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k}) \\ &\quad = p(u, x_{1,k}, x_{2,k})p_1(y_{r,k}|x_{1,k}, x_{2,k}) \end{aligned} \tag{6}$$

where (a) follows from (5). If $p(u) = 0$, then

$$p(u, x_{1,k}, x_{2,k}, y_{r,k}) = p(u, x_{1,k}, x_{2,k}) = 0. \tag{7}$$

It then follows from (6) and (7) that for all $u \in \mathcal{U}$, $x_{1,k} \in \mathcal{X}_1$, $x_{2,k} \in \mathcal{X}_2$ and $y_{r,k} \in \mathcal{Y}_r$,

$$p(u, x_{1,k}, x_{2,k}, y_{r,k}) = p(u, x_{1,k}, x_{2,k})p_1(y_{r,k}|x_{1,k}, x_{2,k}), \tag{8}$$

which implies from Proposition 1 that (1) is a Markov Chain. Similarly, it can be shown that

$$p(u, x_{r,k}, y_{1,k}, y_{2,k}) = p(u, x_{r,k})p_2(y_{1,k}, y_{2,k}|x_{r,k}), \tag{9}$$

which then implies from Proposition 1 that (2) is a Markov Chain.

In addition, if $p(u) > 0$, then

$$\begin{aligned} &p(u, x_{1,k}, x_{2,k}, y_{1,k}, y_{2,k}, y_{r,k}) \\ &\quad = \sum_{x_{r,k}} p(u, x_{1,k}, x_{2,k}, x_{r,k}, y_{1,k}, y_{2,k}, y_{r,k}) \\ &\quad \overset{(a)}{=} \sum_{x_{r,k}} p(u)p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k})p(x_{r,k}|u) \\ &\qquad p_2(y_{1,k}, y_{2,k}|x_{r,k}) \\ &\quad \overset{(b)}{=} \sum_{x_{r,k}} p(u, x_{r,k}, y_{1,k}, y_{2,k})p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k}) \\ &\quad = p(u, y_{1,k}, y_{2,k})p(x_{1,k}, x_{2,k}|u)p_1(y_{r,k}|x_{1,k}, x_{2,k}), \end{aligned} \tag{10}$$

where (a) follows from (5) and (b) follows from (9). If $p(u) = 0$,

$$p(u, x_{1,k}, x_{2,k}, y_{1,k}, y_{2,k}, y_{r,k}) = 0. \tag{11}$$

Let χ and φ be two functions such that

$$\begin{aligned} &\chi(u, x_{1,k}, x_{2,k}, y_{1,k}, y_{2,k}) \\ &\quad = \begin{cases} p(u, y_{1,k}, y_{2,k})p(x_{1,k}, x_{2,k}|u) & \text{if } p(u) > 0, \\ 0 & \text{if } p(u) = 0 \end{cases} \end{aligned}$$

and $\varphi(x_{1,k}, x_{2,k}, y_{r,k}) = p_1(y_{r,k}|x_{1,k}, x_{2,k})$. It then follows from (10) and (11) that for all $u \in \mathcal{U}$, $x_{1,k} \in \mathcal{X}_1$, $x_{2,k} \in \mathcal{X}_2$, $y_{1,k} \in \mathcal{Y}_1$, $y_{2,k} \in \mathcal{Y}_2$ and $y_{r,k} \in \mathcal{Y}_r$,

$$\begin{aligned} &p(u, x_{1,k}, x_{2,k}, y_{1,k}, y_{2,k}, y_{r,k}) \\ &\quad = \chi(u, x_{1,k}, x_{2,k}, y_{1,k}, y_{2,k})\varphi(x_{1,k}, x_{2,k}, y_{r,k}), \end{aligned}$$

which then implies from Proposition 1 that (3) is a Markov Chain. Similarly, it can be shown from (5) and (8) that (4) is a Markov Chain.

The second statement of the lemma follows from summing u on both sides of (8) and (9). ■

Definition 3: For an (n, M_1, M_2)-code on the discrete memoryless TRC, the average probabilities of decoding error of W_1 and W_2 are defined as $P_{e,1}^n = Pr\{g_2(W_2, Y_2^n) \neq W_1\}$ and $P_{e,2}^n = Pr\{g_1(W_1, Y_1^n) \neq W_2\}$ respectively.

Definition 4: A rate pair (R_1, R_2) is achievable if there exists a sequence of (n, M_1, M_2)-codes with $\lim_{n\to\infty} \frac{\log_2 M_1}{n} \geq R_1$ and $\lim_{n\to\infty} \frac{\log_2 M_2}{n} \geq R_2$ such that $\lim_{n\to\infty} P_{e,1}^n = 0$ and $\lim_{n\to\infty} P_{e,2}^n = 0$.

Definition 5: The capacity region $\mathcal{R}'$ of the discrete memoryless TRC with feedback is the closure of the set of all achievable rate pairs.

Let $\mathcal{R}^*$ denote

$$\left\{ \begin{array}{c|c} (R_1, R_2) \in \mathbb{R}^2 & \begin{array}{c} R_1 \geq 0, R_2 \geq 0, \\ R_1 \leq \min\{I(X_1; Y_r|X_2), I(X_r; Y_2)\}, \\ R_2 \leq \min\{I(X_2; Y_r|X_1), I(X_r; Y_1)\} \\ \text{where } p(x_1, x_2, x_r, y_1, y_2, y_r) = \\ p(x_1, x_2)p(x_r)p_1(y_r|x_1, x_2)p_2(y_1, y_2|x_r) \\ \text{for some input distribution } p(x_1, x_2) \\ \text{for the MAC } p_1(y_r|x_1, x_2) \text{ and some} \\ \text{input distribution } p(x_r) \text{ for the BC} \\ p_2(y_1, y_2|x_r). \end{array} \end{array} \right\}, \tag{12}$$

which is stated in [4] as an outer bound on $\mathcal{R}'$. We prove this outer bound in detail in the following theorem.

Theorem 1: $\mathcal{R}' \subset \mathcal{R}^*$.

Proof: Suppose (R_1, R_2) is in $\mathcal{R}'$. By Definition 4 and Definition 5, there exists a sequence of (n, M_1, M_2)-codes with

$$\lim_{n\to\infty} \frac{\log_2 M_1}{n} \geq R_1 \tag{13}$$

and

$$\lim_{n\to\infty} \frac{\log_2 M_2}{n} \geq R_2 \tag{14}$$

such that

$$\lim_{n\to\infty} P_{e,1}^n = 0 \tag{15}$$

and

$$\lim_{n\to\infty} P_{e,2}^n = 0. \tag{16}$$

Fix n and the corresponding (n, M_1, M_2)-code. It then follows from Lemma 2 that

$$U \to (X_{1,k}, X_{2,k}) \to Y_{r,k}, \tag{17}$$

$$U \to X_{r,k} \to Y_{2,k} \tag{18}$$

and

$$(U, Y_{r,k}) \to X_{r,k} \to Y_{2,k} \tag{19}$$

form three Markov Chains, where $U = (W_1, W_2, X_1^{k-1}, X_2^{k-1}, X_r^{k-1}, Y_1^{k-1}, Y_2^{k-1}, Y_r^{k-1})$. Since W_1 and W_2 are independent, we have

$$\begin{aligned}\log_2 M_1 &= H(W_1|W_2)\\ &= I(W_1; Y_2^n, Y_r^n|W_2) + H(W_1|Y_2^n, Y_r^n, W_2)\\ &\leq I(W_1; Y_2^n, Y_r^n|W_2) + H(W_1|Y_2^n, W_2)\\ &\leq I(W_1; Y_2^n, Y_r^n|W_2) + 1 + P_{e,1}^n \log_2 M_1,\end{aligned} \tag{20}$$

where the last inequality follows from the Fano's inequality.

Consider the following chain of inequalities:

$$\begin{aligned}&I(W_1; Y_2^n, Y_r^n|W_2)\\ &= \sum_{k=1}^n I(W_1; Y_{2,k}, Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1})\\ &= \sum_{k=1}^n I(W_1; Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1}) + I(W_1; Y_{2,k}|W_2, Y_2^{k-1}, Y_r^k)\\ &= \sum_{k=1}^n I(W_1; Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1}) + H(Y_{2,k}|W_2, Y_2^{k-1}, Y_r^k)\\ &\quad - H(Y_{2,k}|W_1, W_2, Y_2^{k-1}, Y_r^k)\\ &\overset{(a)}{=} \sum_{k=1}^n I(W_1; Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1}) + H(Y_{2,k}|W_2, X_{r,k}, Y_2^{k-1}, Y_r^k) - H(Y_{2,k}|W_1, W_2, Y_2^{k-1}, Y_r^k)\\ &\leq \sum_{k=1}^n I(W_1; Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1}) + H(Y_{2,k}|X_{r,k}) - H(Y_{2,k}|W_1, W_2, X_1^{k-1}, X_2^{k-1}, X_r^k, Y_1^{k-1}, Y_2^{k-1}, Y_r^k)\\ &\overset{(b)}{=} \sum_{k=1}^n I(W_1; Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1}) + H(Y_{2,k}|X_{r,k}) - H(Y_{2,k}|X_{r,k})\\ &= \sum_{k=1}^n I(W_1; Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1})\\ &= \sum_{k=1}^n H(Y_{r,k}|W_2, Y_2^{k-1}, Y_r^{k-1}) - H(Y_{r,k}|W_1, W_2, Y_2^{k-1}, Y_r^{k-1})\\ &\overset{(c)}{=} \sum_{k=1}^n H(Y_{r,k}|W_2, X_{2,k}, Y_2^{k-1}, Y_r^{k-1}) - H(Y_{r,k}|W_1, W_2, Y_2^{k-1}, Y_r^{k-1})\\ &\leq \sum_{k=1}^n H(Y_{r,k}|X_{2,k}) - H(Y_{r,k}|W_1, W_2, X_1^k, X_2^k, X_r^{k-1}, Y_1^{k-1}, Y_2^{k-1}, Y_r^{k-1})\\ &\overset{(d)}{=} \sum_{k=1}^n H(Y_{r,k}|X_{2,k}) - H(Y_{r,k}|X_{1,k}, X_{2,k})\\ &= \sum_{k=1}^n I(X_{1,k}; Y_{r,k}|X_{2,k}),\end{aligned} \tag{21}$$

where

- (a) follows from the fact that $X_{r,k}$ is a function of Y_r^{k-1},
- (b) follows from the Markov Chain in (19),
- (c) follows from the fact that $X_{2,k}$ is a function of W_2 and Y_2^{k-1},
- (d) follows from the Markov Chain in (17).

Let Q be a new timesharing random variable distributed uniformly on $\{1, 2, \ldots, n\}$ such that Q is independent of the collection of random variables $\{X_{1,k}, X_{2,k}, Y_{r,k}, X_{r,k}, Y_{1,k}, Y_{2,k} \mid k = 1, 2, \ldots, n\}$. Then,

$$\begin{aligned}&p_{X_{1,Q}, X_{2,Q}, Y_{r,Q}|Q}(x_1, x_2, y_r|k)\\ &\quad = p_{X_{1,k}, X_{2,k}, Y_{r,k}|Q}(x_1, x_2, y_r|k)\\ &\quad \overset{(a)}{=} p_{X_{1,k}, X_{2,k}, Y_{r,k}}(x_1, x_2, y_r)\end{aligned} \tag{22}$$

and

$$\begin{aligned}&p_{Q, X_{1,Q}, X_{2,Q}, Y_{r,Q}}(k, x_1, x_2, y_r)\\ &\quad = p_Q(k) p_{X_{1,Q}, X_{2,Q}, Y_{r,Q}|Q}(x_1, x_2, y_r|k)\\ &\quad \overset{(b)}{=} p_Q(k) p_{X_{1,k}, X_{2,k}, Y_{r,k}}(x_1, x_2, y_r)\\ &\quad \overset{(c)}{=} p_Q(k) p_{X_{1,k}, X_{2,k}}(x_1, x_2) p_1(y_r|x_1, x_2) \qquad (23)\\ &\quad = \chi(k, x_1, x_2)\varphi(x_1, x_2, y_r) \qquad (24)\end{aligned}$$

for some functions χ and φ, where

- (a) follows from the construction of Q that Q is independent of $X_{1,k}$, $X_{2,k}$ and $Y_{r,k}$,
- (b) follows from (22),
- (c) follows from the second statement of Lemma 2.

Following from the above,

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} I(X_{1,k};Y_{r,k}|X_{2,k}) \\
&\quad\overset{(a)}{=} \sum_{k=1}^{n}\frac{1}{n} I(X_{1,Q};Y_{r,Q}|X_{2,Q},Q=k) \\
&\quad= I(X_{1,Q};Y_{r,Q}|X_{2,Q},Q) \\
&\quad= H(Y_{r,Q}|X_{2,Q},Q) - H(Y_{r,Q}|X_{1,Q},X_{2,Q},Q) \\
&\quad\le H(Y_{r,Q}|X_{2,Q}) - H(Y_{r,Q}|X_{1,Q},X_{2,Q},Q) \\
&\quad\overset{(b)}{=} H(Y_{r,Q}|X_{2,Q}) - H(Y_{r,Q}|X_{1,Q},X_{2,Q}) \\
&\quad= I(X_{1,Q};Y_{r,Q}|X_{2,Q}) \qquad (25)
\end{aligned}$$

where

- (a) follows from (22),
- (b) follows from the fact that $Q \to (X_{1,Q},X_{2,Q}) \to Y_{r,Q}$ forms a Markov Chain (cf. (24) and Proposition 1).

Using (20), (21) and (25), we obtain

$$\log_2 M_1 \le 1 + P_{e,1}^n \log_2 M_1 + nI(X_{1,Q};Y_{r,Q}|X_{2,Q}). \quad (26)$$

By symmetry, we obtain

$$\log_2 M_2 \le 1 + P_{e,2}^n \log_2 M_2 + nI(X_{2,Q};Y_{r,Q}|X_{1,Q}). \quad (27)$$

Since

$$\begin{aligned}
&p_{X_{1,Q},X_{2,Q},Y_{r,Q}}(x_1,x_2,y_r) \\
&\quad= \sum_{k=1}^{n} p_{Q,X_{1,Q},X_{2,Q},Y_{r,Q}}(k,x_1,x_2,y_r) \\
&\quad\overset{(a)}{=} \sum_{k=1}^{n} p_Q(k) p_{X_{1,k},X_{2,k}}(x_1,x_2) p_1(y_r|x_1,x_2) \\
&\quad= \sum_{k=1}^{n}\frac{1}{n} p_{X_{1,k},X_{2,k}}(x_1,x_2) p_1(y_r|x_1,x_2)
\end{aligned}$$

where (a) follows from (23), it follows that $I(X_{1,Q};Y_{r,Q}|X_{2,Q})$ and $I(X_{2,Q};Y_{r,Q}|X_{1,Q})$ are achieved by the distribution $p_{X_{1,Q},X_{2,Q},Y_{r,Q}}(x_1,x_2,y_r) = q_n(x_1,x_2)$ $p_1(y_r|x_1,x_2)$ where $q_n(x_1,x_2) = \sum_{k=1}^{n}\frac{1}{n} p_{X_{1,k},X_{2,k}}(x_1,x_2)$ is the input distribution for the MAC. Using (26), (27), (15), (16), (13) and (14), we obtain

$$R_1 \le \liminf_{n\to\infty} I(X_{1,Q};Y_{r,Q}|X_{2,Q}) \quad (28)$$

and

$$R_2 \le \liminf_{n\to\infty} I(X_{2,Q};Y_{r,Q}|X_{1,Q}). \quad (29)$$

Consider each distribution of (X_1,X_2) as a point in an $|\mathcal{X}_1||\mathcal{X}_2|$-dimensional Euclidean space. Let $\{q_{n_k}(x_1,x_2)\}_{k=1,2,\ldots}$ be a convergent subsequence of $\{q_n(x_1,x_2)\}_{n=1,2,\ldots}$ with respect to the $\mathcal{L}_1$-distance, where the $\mathcal{L}_1$-distance between two distributions $u(x)$ and $v(x)$ on the same discrete alphabet $\mathcal{X}$ is defined as $\sum_{x\in\mathcal{X}}|u(x)-v(x)|$. Since the set of all joint distributions $\{p_{X_1,X_2}(x_1,x_2)\}$ is closed with respect to the $\mathcal{L}_1$-distance, there exists a joint distribution $\bar{q}(x_1,x_2)$ such that $\lim_{k\to\infty} q_{n_k}(x_1,x_2) = \bar{q}(x_1,x_2)$. Since $I(X_1;Y_r|X_2)$ is a continuous functional of $p_{X_1,X_2}(x_1,x_2)$, it then follows from (28) and (29) that

$$R_1 \le \liminf_{n\to\infty} I(X_{1,Q};Y_{r,Q}|X_{2,Q}) \le I(X_1;Y_r|X_2) \quad (30)$$

and

$$R_2 \le \liminf_{n\to\infty} I(X_{2,Q};Y_{r,Q}|X_{1,Q}) \le I(X_2;Y_r|X_1) \quad (31)$$

where $p_{X_1,X_2,Y_r}(x_1,x_2,y_r) = \bar{q}(x_1,x_2)p_1(y_r|x_1,x_2)$.

On the other hand,

$$\begin{aligned}
\log_2 M_1 &= H(W_1|W_2) \\
&= I(W_1;Y_2^n|W_2) + H(W_1|Y_2^n,W_2) \\
&\le I(W_1;Y_2^n|W_2) + 1 + P_{e,1}^n \log_2 M_1 \qquad (32)
\end{aligned}$$

where the last inequality follows from the Fano's inequality.

Consider the following chain of inequalities:

$$\begin{aligned}
&I(W_1;Y_2^n|W_2) \\
&= \sum_{k=1}^{n} I(W_1;Y_{2,k}|W_2,Y_2^{k-1}) \\
&= \sum_{k=1}^{n} H(Y_{2,k}|W_2,Y_2^{k-1}) - H(Y_{2,k}|W_1,W_2,Y_2^{k-1}) \\
&\le \sum_{k=1}^{n} H(Y_{2,k}) - H(Y_{2,k}|W_1,W_2,X_1^{k-1},X_2^{k-1},X_r^k,Y_1^{k-1}, \\
&\qquad Y_2^{k-1},Y_r^{k-1}) \\
&\overset{(a)}{=} \sum_{k=1}^{n} H(Y_{2,k}) - H(Y_{2,k}|X_{r,k}) \\
&= \sum_{k=1}^{n} I(X_{r,k};Y_{2,k}) \qquad (33)
\end{aligned}$$

where (a) follows from the Markov Chain in (18). In addition,

$$\begin{aligned}
&p_{X_{r,Q},Y_{1,Q},Y_{2,Q}|Q}(x_r,y_1,y_2|k) \\
&\quad= p_{X_{r,k},Y_{1,k},Y_{2,k}|Q}(x_r,y_1,y_2|k) \\
&\quad\overset{(a)}{=} p_{X_{r,k},Y_{1,k},Y_{2,k}}(x_r,y_1,y_2) \qquad (34)
\end{aligned}$$

and

$$\begin{aligned}
&p_{Q,X_{r,Q},Y_{1,Q},Y_{2,Q}}(k,x_r,y_1,y_2) \\
&\quad= p_Q(k) p_{X_{r,Q},Y_{1,Q},Y_{2,Q}|Q}(x_r,y_1,y_2|k) \\
&\quad\overset{(b)}{=} p_Q(k) p_{X_{r,k},Y_{1,k},Y_{2,k}}(x_r,y_1,y_2) \\
&\quad\overset{(c)}{=} p_Q(k) p_{X_{r,k}}(x_r) p_2(y_1,y_2|x_r) \qquad (35) \\
&\quad= \chi(k,x_r)\varphi(x_r,y_1,y_2) \qquad (36)
\end{aligned}$$

for some functions χ and φ, where

- (a) follows from the construction of Q that Q is independent of $X_{r,k}$, $Y_{1,k}$ and $Y_{2,k}$,
- (b) follows from (34),
- (c) follows from the second statement of Lemma 2.

Following from the above,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} I(X_{r,k};Y_{2,k}) &\overset{(a)}{=} \sum_{k=1}^{n}\frac{1}{n} I(X_{r,Q};Y_{2,Q}|Q=k)\\ &= I(X_{r,Q};Y_{2,Q}|Q)\\ &= H(Y_{2,Q}|Q) - H(Y_{2,Q}|X_{r,Q},Q)\\ &\le H(Y_{2,Q}) - H(Y_{2,Q}|X_{r,Q},Q)\\ &\overset{(b)}{=} H(Y_{2,Q}) - H(Y_{2,Q}|X_{r,Q})\\ &= I(X_{r,Q};Y_{2,Q}) \end{aligned} \tag{37}$$

where

- (a) follows from (34),
- (b) follows from the fact that $Q \to X_{r,Q} \to Y_{2,Q}$ forms a Markov Chain (cf. (36) and Proposition 1).

Using (32), (33) and (37), we obtain

$$\log_2 M_1 \le 1 + P_{e,1}^n \log_2 M_1 + nI(X_{r,Q};Y_{2,Q}). \tag{38}$$

By symmetry, we obtain

$$\log_2 M_2 \le 1 + P_{e,2}^n \log_2 M_2 + nI(X_{r,Q};Y_{1,Q}). \tag{39}$$

Since

$$\begin{aligned} &p_{X_{r,Q},Y_{1,Q},Y_{2,Q}}(x_r,y_1,y_2)\\ &\quad= \sum_{k=1}^{n} p_{Q,X_{r,Q},Y_{1,Q},Y_{2,Q}}(k,x_r,y_1,y_2)\\ &\quad\overset{(a)}{=} \sum_{k=1}^{n} p_Q(k)p_{X_{r,k}}(x_r)p_2(y_1,y_2|x_r)\\ &\quad= \sum_{k=1}^{n}\frac{1}{n}p_{X_{r,k}}(x_r)p_2(y_1,y_2|x_r)\end{aligned}$$

where (a) follows from (35), it follows that $I(X_{r,Q};Y_{1,Q})$ and $I(X_{r,Q};Y_{2,Q})$ are achieved by the distribution $p_{X_{r,Q},Y_{1,Q},Y_{2,Q}}(x_r,y_1,y_2) = s_n(x_r)p_2(y_1,y_2|x_r)$ where $s_n(x_r) = \sum_{k=1}^{n}\frac{1}{n}p_{X_{r,k}}(x_r)$ is the input distribution for the BC. Using (38), (39), (15), (16), (13) and (14), we obtain

$$R_1 \le \liminf_{n\to\infty} I(X_{r,Q};Y_{2,Q}) \tag{40}$$

and

$$R_2 \le \liminf_{n\to\infty} I(X_{r,Q};Y_{1,Q}). \tag{41}$$

Consider each distribution of X_r as a point in an $|X_r|$-dimensional Euclidean space. Let $\{s_{n_k}(x_r)\}_{k=1,2,\ldots}$ be a convergent subsequence of $\{s_n(x_r)\}_{n=1,2,\ldots}$ with respect to the $\mathcal{L}_1$-distance and let $\bar{s}(x_r)$ be the distribution such that $\lim_{k\to\infty} s_{n_k}(x_r) = \bar{s}(x_r)$. Since $I(X_r;Y_2)$ is a continuous functional of $p_{X_r}(x_r)$, it then follows from (40) and (41) that

$$R_1 \le \liminf_{n\to\infty} I(X_{r,Q};Y_{2,Q}) \le I(X_r;Y_2) \tag{42}$$

and

$$R_2 \le \liminf_{n\to\infty} I(X_{r,Q};Y_{1,Q}) \le I(X_r;Y_1) \tag{43}$$

where $p_{X_r,Y_1,Y_2}(x_r,y_1,y_2) = \bar{s}(x_r)p_2(y_1,y_2|x_r)$.

Let $p(x_1,x_2,x_r,y_1,y_2,y_r) = \bar{q}(x_1,x_2)\bar{s}(x_r)p_1(y_r|x_1,x_2)$ $p_2(y_1,y_2|x_r)$, i.e. (X_1,X_2,Y_r) and (X_r,Y_1,Y_2) are independent. Then we see that $p(x_1,x_r,y_r) = \bar{q}(x_1,x_2)p_1(y_r|x_1,x_2)$ and $p(x_r,y_1,y_2) = \bar{s}(x_r)p_2(y_1,y_2|x_r)$. It then follows from (30), (31), (42) and (43) that for this choice of $p(x_1,x_2,x_r,y_1,y_2,y_r)$, $R_1 \le \min\{I(X_1;Y_r|X_2), I(X_r;Y_2)\}$ and $R_2 \le \min\{I(X_2;Y_r|X_1), I(X_r;Y_1)\}$. ■

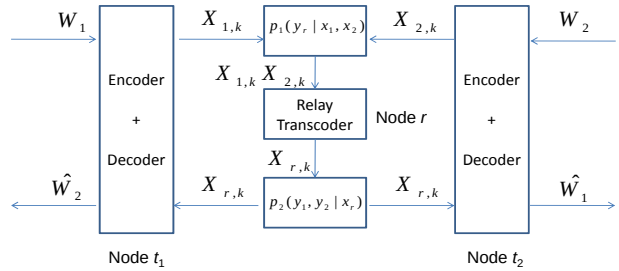

Fig. 3. Binary multiplying relay channel (BMRC).

IV. Looseness of the Outer Bound

The outer bound $\mathcal{R}^*$ is not tight in general. Consider the TRC with deterministic $p_1(y_r|x_1,x_2)$ and $p_2(y_1,y_2|x_r)$ such that $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_r = \mathcal{X}_r = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0,1\}$, $Pr\{Y_r = X_1X_2\} = 1$ and $Pr\{Y_1 = Y_2 = X_r\} = 1$. We call the TRC described above the *binary multiplying relay channel* (BMRC), which is illustrated in Figure 3. Then,

$$I(X_r;Y_1) = I(X_r;Y_2) = H(X_r). \tag{44}$$

In addition,

$$I(X_1;Y_r|X_2) \le H(Y_r) \le 1. \tag{45}$$

Similarly,

$$I(X_2;Y_r|X_1) \le 1. \tag{46}$$

Then,

$$\mathcal{R}^* \overset{(a)}{=} \left\{ (R_1,R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} R_1 \ge 0,\ R_2 \ge 0,\\ R_1 \le \min\{I(X_1;Y_r|X_2), H(X_r)\},\\ R_2 \le \min\{I(X_2;Y_r|X_1), H(X_r)\}\\ \text{where } p(x_1,x_2,x_r,y_1,y_2,y_r) = \\ p(x_1,x_2)p(x_r)p_1(y_r|x_1,x_2)p_2(y_1,y_2|x_r)\\ \text{for some input distribution } p(x_1,x_2)\\ \text{for the MAC } p_1(y_r|x_1,x_2) \text{ and some}\\ \text{input distribution } p(x_r) \text{ for the BC}\\ p_2(y_1,y_2|x_r). \end{array} \right\} \tag{47}$$

$$\overset{(b)}{\supset} \left\{ (R_1,R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} R_1 \ge 0,\ R_2 \ge 0,\\ R_1 \le \min\{I(X_1;Y_r|X_2), H(X_r)\},\\ R_2 \le \min\{I(X_2;Y_r|X_1), H(X_r)\}\\ \text{where } p(x_1,x_2,x_r,y_1,y_2,y_r) = p(x_1,x_2)\\ p(x_r)p_1(y_r|x_1,x_2)p_2(y_1,y_2|x_r) \text{ for some}\\ \text{input distribution } p(x_1,x_2) \text{ for the MAC}\\ p_1(y_r|x_1,x_2) \text{ and the input distribu-}\\ \text{tion } p(x_r) \text{ is Bernoulli(1/2) for the BC}\\ p_2(y_1,y_2|x_r). \end{array} \right\}$$

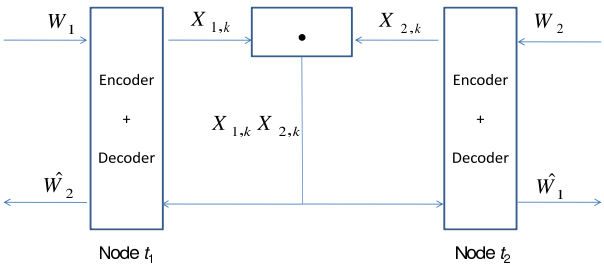

Fig. 4. Binary multiplying channel (BMC).

$$\overset{(c)}{=} \left\{ (R_1, R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} R_1 \geq 0,\ R_2 \geq 0, \\ R_1 \leq I(X_1; Y_r|X_2), \\ R_2 \leq I(X_2; Y_r|X_1) \\ \text{where } p(x_1, x_2, y_r) = p(x_1, x_2) \\ p_1(y_r|x_1, x_2) \text{ for some input} \\ \text{distribution } p(x_1, x_2) \text{ for the} \\ \text{MAC } p_1(y_r|x_1, x_2). \end{array} \right\}, \quad (48)$$

where

(a) follows from (12) and (44),

(b) follows from restricting the input distribution for the BC in (47) to be Bernoulli(1/2),

(c) follows from (45) and (46).

Let G denote the set in (48), which is equivalent to the Shannon's outer bound on the capacity region of the binary multiplying channel (BMC) in [6]. The transmissions of messages in the BMC are illustrated in Figure 4.

We prove the looseness of $\mathcal{R}^*$ by presenting another outer bound for the BMRC that does not contain $\mathcal{R}^*$. In the BMRC, $Y_{1,k} = Y_{2,k} = X_{r,k}$ with probability 1 and $X_{r,k}$ is a function of $Y_r^{k-1} = [X_{1,1}X_{2,1}, X_{1,2}X_{2,2}, \ldots, X_{1,k-1}X_{2,k-1}]$ for any positive integer k. In the BMC, node t_1 and node t_2 receive $X_{1,k}X_{2,k}$ in the kth time slot with probability 1, which implies that node t_1 and node t_2 in the BMC have received $[X_{1,1}X_{2,1}, X_{1,2}X_{2,2}, \ldots, X_{1,k}X_{2,k}]$ by the end of the kth time slot. Therefore, the operations performed by the transcoder in any transmission scheme in the BMRC can be imitated identically at the terminal nodes in the BMC, which implies that the terminal nodes in the BMC can construct $X_{r,k}$ in the kth time slot. It then follows that any transmission scheme in the BMRC can be regarded as a transmission scheme in the BMC. Consequently, the capacity region of the BMRC is a subset of the capacity region of the BMC, which implies that any outer bound for the BMC is an outer bound for the BMRC. Hekstra and Willems [7] obtained an outer bound for the BMC and the maximum equal-rate pair in their outer bound, which is 0.64628, is strictly less than the maximum equal-rate pair in G, which is 0.69424 [6]. Consequently, G is not inside the Hekstra's outer bound for the BMC as well as the BMRC, which implies from (48) that $\mathcal{R}^*$ is not inside the Hekstra's outer bound for the BMRC.

V. Full-Duplex Gaussian TRC with feedback

In this section, we study the full-duplex Gaussian TRC in which the terminal nodes can use their previously received messages for encoding their messages. Knopp [4] stated an outer bound on the capacity region of the Gaussian TRC. However, the proof in [4] is incomplete. Therefore, we present a detailed proof of the outer bound in [4] in this section. The formulation of the Gaussian TRC is as follows.

If $X_{1,k}$ is transmitted at node t_1 and $X_{2,k}$ is transmitted at node t_2 in the kth time slot, the received signal at the relay in the time slot is assumed to be $Y_{r,k} = X_{1,k} + X_{2,k} + Z_{r,k}$ where $Z_{r,k}$ is a Gaussian random variable independent of $(X_{1,k}, X_{2,k})$. If $X_{r,k}$ is transmitted at r in the same time slot, the received signal at node t_1 in the time slot is assumed to be $Y_{1,k} = X_{r,k} + Z_{1,k}$, where $Z_{1,k}$ is a Gaussian random variable independent of $X_{r,k}$, and the received signal at node t_2 in the time slot is assumed to be $Y_{2,k} = X_{r,k} + Z_{2,k}$, where $Z_{2,k}$ is a Gaussian random variable independent of $X_{r,k}$. For any three codewords $(x_{1,1}, x_{1,2}, \ldots, x_{1,n})$, $(x_{2,1}, x_{2,2}, \ldots, x_{2,n})$ and $(x_{r,1}, x_{r,2}, \ldots, x_{r,n})$ that are transmitted over the channel, we require that $\frac{1}{n}\sum_{k=1}^{n} x_{1,k}^2 \leq P_1$, $\frac{1}{n}\sum_{k=1}^{n} x_{2,k}^2 \leq P_2$ and $\frac{1}{n}\sum_{k=1}^{n} x_{r,k}^2 \leq P_r$ where P_1, P_2 and P_r represent the maximum power constraints for node t_1, node t_2 and node r respectively.

Definition 6: The Gaussian TRC consists of the following:

1) Three Gaussian random variables $Z_1 = \mathcal{N}(0, N_1)$, $Z_2 = \mathcal{N}(0, N_2)$ and $Z_r = \mathcal{N}(0, N_r)$.
2) A probability density function $f_1(y_r|x_1, x_2)$ defined for all real numbers x_1, x_2 and y_r representing the MAC and a probability density function $f_2(y_1, y_2|x_r)$ defined for all real numbers x_r, y_1 and y_2 representing the BC such that

$$\begin{aligned} f_1(y_r|x_1, x_2) &= f_{Z_r}(y_r - x_1 - x_2|x_1, x_2) \\ &= f_{Z_r}(y_r - x_1 - x_2) \end{aligned} \quad (49)$$

and

$$\begin{aligned} f_2(y_1, y_2|x_r) &= f_{Z_1,Z_2}(y_1 - x_r, y_2 - x_r|x_r) \\ &= f_{Z_1,Z_2}(y_1 - x_r, y_2 - x_r). \end{aligned} \quad (50)$$

In addition, we require that for any two finite discrete random variables X_1 and X_2 with an input distribution $p(x_1, x_2)$ for the MAC and any finite discrete random variable X_r with an input distribution $p(x_r)$ for the BC, the relationship among X_1, X_2, X_r, the output Y_r of the MAC and the outputs Y_1 and Y_2 of the BC satisfies

$$\begin{aligned} &Pr\{X_1 = x_1, X_2 = x_2, X_r = x_r, Y_1 \leq y_1, Y_2 \leq y_2, Y_r \leq y_r\} \\ &\quad = Pr\{X_1 = x_1, X_2 = x_2, Y_r \leq y_r\} Pr\{X_r = x_r, Y_1 \leq y_1, \\ &\qquad Y_2 \leq y_2\} \end{aligned}$$

for all $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, $x_r \in \mathcal{X}_r$, $y_1 \in \mathbb{R}$, $y_2 \in \mathbb{R}$ and $y_r \in \mathbb{R}$, where

$$\begin{aligned} &Pr\{X_1 = x_1, X_2 = x_2, Y_r \leq y_r\} \\ &\quad = \int_{-\infty}^{y_r} f_1(v_r|x_1, x_2) p(x_1, x_2) dv_r \end{aligned}$$

and

$$Pr\{X_r = x_r, Y_1 \le y_1, Y_2 \le y_2\} = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f_2(v_1, v_2|x_r)p(x_r)dv_1dv_2.$$

The Gaussian TRC is denoted by (N_1, N_2, N_r).

Definition 7: An $(n, M_1, M_2, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r)$-code on the channel (N_1, N_2, N_r) with maximum power constraints P_1, P_2 and P_r consists of the following:

1) A message set $\mathcal{W}_1 = \{1, 2, \ldots, M_1\}$ at node t_1 and a message set $\mathcal{W}_2 = \{1, 2, \ldots, M_2\}$ at node t_2.
2) A set of n encoding functions $\alpha_{1,k} : \mathcal{W}_1 \times \mathcal{Y}_1^{k-1} \to \mathcal{X}_1$ at node t_1 for $k = 1, 2, \ldots, n$, where $\alpha_{1,k}$ is the encoding function in the kth time slot such that $X_{1,k} = \alpha_{1,k}(W_1, Y_1^{k-1})$. Without loss of generality, we assume that $\mathcal{X}_1$ is finite. In addition, every codeword x_1^n must satisfy the power constraint
$$\sum_{k=1}^{n} x_{1,k}^2 \le nP_1. \tag{51}$$
3) A set of n encoding functions $\alpha_{2,k} : \mathcal{W}_2 \times \mathcal{Y}_2^{k-1} \to \mathcal{X}_2$ at node t_2 for $k = 1, 2, \ldots, n$, where $\alpha_{2,k}$ is the encoding function in the kth time slot such that $X_{2,k} = \alpha_{2,k}(W_2, Y_2^{k-1})$. Without loss of generality, we assume that $\mathcal{X}_2$ is finite. In addition, every codeword x_2^n must satisfy the power constraint
$$\sum_{k=1}^{n} x_{2,k}^2 \le nP_2. \tag{52}$$
4) A set of n encoding functions $\alpha_{r,k} : \mathcal{Y}_r^{k-1} \to \mathcal{X}_r$ at r for $k = 1, 2, \ldots, n$, where $\alpha_{r,k}$ is the encoding function in the kth time slot such that $X_{r,k} = \alpha_{r,k}(Y_r^{k-1})$. Without loss of generality, we assume that $\mathcal{X}_r$ is finite. In addition, every codeword x_r^n must satisfy the power constraint
$$\sum_{k=1}^{n} x_{r,k}^2 \le nP_r. \tag{53}$$
5) A decoding function $\beta_1 : \mathcal{W}_1 \times \mathcal{Y}_1^n \to \mathcal{W}_2$ at node t_1 such that $\beta_1(W_1, Y_1^n) = \hat{W}_2$.
6) A decoding function $\beta_2 : \mathcal{W}_2 \times \mathcal{Y}_2^n \to \mathcal{W}_1$ at node t_2 such that $\beta_2(W_2, Y_2^n) = \hat{W}_1$.

The transmissions of messages in the Gaussian TRC are illustrated in Figure 5.

Definition 8: For an $(n, M_1, M_2, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r)$-code on the Gaussian TRC (N_1, N_2, N_r) with maximum power constraints P_1, P_2 and P_r, the average probabilities of decoding error of W_1 and W_2 in (N_1, N_2, N_r) are defined as $P_{e,1}^n = Pr\{\beta_2(W_2, Y_2^n) \ne W_1\}$ and $P_{e,2}^n = Pr\{\beta_1(W_1, Y_1^n) \ne W_2\}$ respectively.

Definition 9: A rate pair (R_1, R_2) is Gaussian-achievable if there exists a sequence of $(n, M_1, M_2, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r)$-codes for the Gaussian TRC (N_1, N_2, N_r) with maximum power constraints P_1, P_2 and P_r such $\lim_{n\to\infty} \frac{\log_2 M_1}{n} \ge R_1$, $\lim_{n\to\infty} \frac{\log_2 M_2}{n} \ge R_2$, $\lim_{n\to\infty} P_{e,1}^n = 0$ and $\lim_{n\to\infty} P_{e,2}^n = 0$.

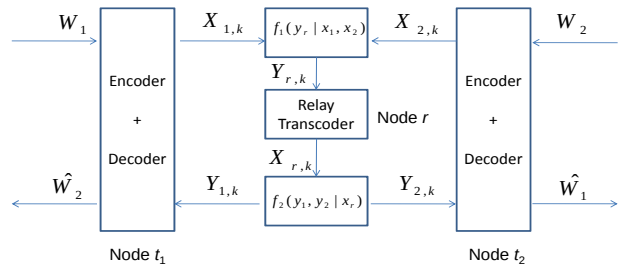

Fig. 5. Gaussian TRC.

Definition 10: The capacity region $\mathcal{R}_G$ of the Gaussian TRC (N_1, N_2, N_r) with maximum power constraints P_1, P_2 and P_r is the closure of the set of all Gaussian-achievable rate pairs.

In the following theorem, we prove in detail the outer bound for $\mathcal{R}_G$ stated in [4].

Theorem 2: For the Gaussian TRC (N_1, N_2, N_r) with maximum power constraints P_1, P_2 and P_r, $\mathcal{R}_G$ is a subset of

$$\left\{ \begin{array}{l|c} (R_1, R_2) & R_1 \ge 0, R_2 \ge 0, \\ \in \mathbb{R}^2 & R_1 \le \min\left\{\frac{1}{2}\log_2\left(1 + \frac{P_1}{N_r}\right), \frac{1}{2}\log_2\left(1 + \frac{P_r}{N_2}\right)\right\}, \\ & R_2 \le \min\left\{\frac{1}{2}\log_2\left(1 + \frac{P_2}{N_r}\right), \frac{1}{2}\log_2\left(1 + \frac{P_r}{N_1}\right)\right\}. \end{array} \right\}.$$

Proof: Suppose (R_1, R_2) is in $\mathcal{R}_G$. By Definitions 9 and 10, there exists a sequence of $(n, M_1, M_2, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r)$-codes with

$$\lim_{n\to\infty} \frac{\log_2 M_1}{n} \ge R_1 \tag{54}$$

and

$$\lim_{n\to\infty} \frac{\log_2 M_2}{n} \ge R_2 \tag{55}$$

such that

$$\lim_{n\to\infty} P_{e,1}^n = 0 \tag{56}$$

and

$$\lim_{n\to\infty} P_{e,2}^n = 0. \tag{57}$$

Fix n and the corresponding $(n, M_1, M_2, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r)$-code. Following similar procedures for proving (20), (21), (32) and (33), we can obtain

$$\log_2 M_1 \le 1 + P_{e,1}^n \log_2 M_1 + \sum_{k=1}^{n} I(X_{1,k}; Y_{r,k}|X_{2,k}),$$

$$\log_2 M_1 \le 1 + P_{e,1}^n \log_2 M_1 + \sum_{k=1}^{n} I(X_{r,k}; Y_{2,k}),$$

$$\log_2 M_2 \le 1 + P_{e,2}^n \log_2 M_2 + \sum_{k=1}^{n} I(X_{2,k}; Y_{r,k}|X_{1,k})$$

and

$$\log_2 M_2 \leq 1 + P_{e,2}^n \log_2 M_2 + \sum_{k=1}^n I(X_{r,k}; Y_{1,k}).$$

It then follows from (56), (57), (54) and (55) that

$$R_1 \leq \liminf_{n\to\infty} \sum_{k=1}^n \frac{1}{n} I(X_{1,k}; Y_{r,k}|X_{2,k}), \tag{58}$$

$$R_1 \leq \liminf_{n\to\infty} \sum_{k=1}^n \frac{1}{n} I(X_{r,k}; Y_{2,k}), \tag{59}$$

$$R_2 \leq \liminf_{n\to\infty} \sum_{k=1}^n \frac{1}{n} I(X_{2,k}; Y_{r,k}|X_{1,k}) \tag{60}$$

and

$$R_2 \leq \liminf_{n\to\infty} \sum_{k=1}^n \frac{1}{n} I(X_{r,k}; Y_{1,k}). \tag{61}$$

Consider the following chain of inequalities:

$$\begin{aligned}
&I(X_{1,k}; Y_{r,k}|X_{2,k}) \\
&\quad = h(Y_{r,k}|X_{2,k}) - h(Y_{r,k}|X_{1,k}, X_{2,k}) \\
&\quad \overset{(a)}{=} h(X_{1,k} + X_{2,k} + Z_{r,k}|X_{2,k}) - h(Z_{r,k}) \\
&\quad = h(X_{1,k} + Z_{r,k}|X_{2,k}) - h(Z_{r,k}) \\
&\quad \leq h(X_{1,k} + Z_{r,k}) - h(Z_{r,k}) \\
&\quad = h(X_{1,k} + Z_{r,k}) - \frac{1}{2}\log_2 2\pi e N_r \\
&\quad \overset{(b)}{\leq} \frac{1}{2}\log_2(2\pi e(E[X_{1,k}^2] + N_r)) - \frac{1}{2}\log_2(2\pi e N_r) \\
&\quad = \frac{1}{2}\log_2(1 + E[X_{1,k}^2]/N_r),
\end{aligned}$$

where

- (a) follows from $Y_{r,k} = X_{1,k} + X_{2,k} + Z_{r,k}$ and (49),
- (b) follows from the fact that $X_{1,k}$ and $Z_{r,k}$ are independent, and the differential entropy of a random variable is upper bounded by the differential entropy of a Gaussian random variable with the same second moment.

Therefore,

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{n} I(X_{1,k}; Y_{r,k}|X_{2,k}) &\leq \frac{1}{2}\sum_{k=1}^n \frac{1}{n}\log_2(1 + E[X_{1,k}^2]/N_r) \\
&\overset{(a)}{\leq} \frac{1}{2}\log_2\left(1 + \sum_{k=1}^n \frac{E[X_{1,k}^2]}{nN_r}\right) \\
&= \frac{1}{2}\log_2\left(1 + \frac{E[\sum_{k=1}^n X_{1,k}^2]}{nN_r}\right) \\
&\overset{(b)}{\leq} \frac{1}{2}\log_2\left(1 + \frac{P_1}{N_r}\right),
\end{aligned}$$

where

- (a) follows from applying Jenson's inequality to the concave function $\log_2(1+x)$,
- (b) follows from (51) and the fact that $\log_2(1+x)$ is increasing in x.

Consequently,

$$\liminf_{n\to\infty} \sum_{k=1}^n \frac{1}{n} I(X_{1,k}; Y_{r,k}|X_{2,k}) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N_r}\right),$$

which then implies from (58) that

$$R_1 \leq \frac{1}{2}\log_2\left(1 + \frac{P_1}{N_r}\right). \tag{62}$$

By symmetry, we obtain from (60) that

$$R_2 \leq \frac{1}{2}\log_2\left(1 + \frac{P_2}{N_r}\right). \tag{63}$$

Consider the following chain of inequalities:

$$\begin{aligned}
&I(X_{r,k}; Y_{2,k}) \\
&\quad = h(Y_{2,k}) - h(Y_{2,k}|X_{r,k}) \\
&\quad \overset{(a)}{=} h(X_{r,k} + Z_{2,k}) - h(Z_{2,k}) \\
&\quad = h(X_{r,k} + Z_{2,k}) - \frac{1}{2}\log_2 2\pi e N_2 \\
&\quad \overset{(b)}{\leq} \frac{1}{2}\log_2(2\pi e(E[X_{r,k}^2] + N_2)) - \frac{1}{2}\log_2(2\pi e N_2) \\
&\quad = \frac{1}{2}\log_2(1 + E[X_{r,k}^2]/N_2),
\end{aligned}$$

where

- (a) follows from $Y_{2,k} = X_{r,k} + Z_{2,k}$ and (50),
- (b) follows from the fact that $X_{r,k}$ and $Z_{2,k}$ are independent, and the differential entropy of a random variable is upper bounded by the differential entropy of a Gaussian random variable with the same second moment.

Therefore,

$$\begin{aligned}
\sum_{k=1}^n \frac{1}{n} I(X_{r,k}; Y_{2,k}) &\leq \frac{1}{2}\sum_{k=1}^n \frac{1}{n}\log_2(1 + E[X_{r,k}^2]/N_2) \\
&\overset{(a)}{\leq} \frac{1}{2}\log_2\left(1 + \sum_{k=1}^n \frac{E[X_{r,k}^2]}{nN_2}\right) \\
&= \frac{1}{2}\log_2\left(1 + \frac{E[\sum_{k=1}^n X_{r,k}^2]}{nN_2}\right) \\
&\overset{(b)}{\leq} \frac{1}{2}\log_2\left(1 + \frac{P_r}{N_2}\right),
\end{aligned}$$

where

- (a) follows from applying Jenson's inequality to the concave function $\log_2(1+x)$,
- (b) follows from (53) and the fact that $\log_2(1+x)$ is increasing in x.

Consequently,

$$\liminf_{n\to\infty} \sum_{k=1}^n \frac{1}{n} I(X_{r,k}; Y_{2,k}) \leq \frac{1}{2}\log_2\left(1 + \frac{P_r}{N_2}\right),$$

which then implies from (59) that

$$R_1 \leq \frac{1}{2}\log_2\left(1 + \frac{P_r}{N_2}\right). \tag{64}$$

By symmetry, we obtain from (61) that

$$R_2 \leq \frac{1}{2}\log_2\left(1+\frac{P_r}{N_1}\right). \tag{65}$$

The theorem then follows from (62), (63), (64) and (65). ■

VI. Conclusion

We investigate the discrete memoryless TRC with feedback and the Gaussian TRC with feedback. Knopp [4] stated an outer bound on the capacity region of each of the discrete memoryless TRC with feedback and the Gaussian TRC with feedback. However, the proofs in [4] are incomplete. We therefore present detailed proofs of the outer bounds stated in [4]. We also show that the outer bound on the capacity region of the discrete memoryless TRC with feedback is loose for some TRC.

Further research includes obtaining achievable rate regions and capacity bounds for a general TRC, since the capacity region of a general TRC is still an open problem.

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