Limit Theorems for the Sample Entropy of Hidden Markov Chains

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Abstract

Recently, based on the Shannon-McMillan-Breiman theorem, efficient Monte Carlo methods for approximating the entropy rate of a hidden Markov chain were proposed. The convergence behavior of such algorithms is governed by the limiting behavior of the sample entropy sequence, which is typically characterized by classical limit theorems such as the Central Limit Theorem (CLT) and the Law of Iterated Logarithm (LIL). In this paper, for the sample entropy sequence of hidden Markov chains under some positivity assumptions, we establish a CLT with some Berry-Esseen bound (such bound characterizes rate of convergence of CLT), and we use this CLT can to establish a LIL.

1 Introduction and Notations

Consider a bi-infinite stationary stochastic process $Y = (Y_i, i \in \mathbb{Z})$ on a finite alphabet $\mathcal{Y} = \{1, 2, \cdots, B\}$. The entropy rate of $Y$ is defined to be

$$H(Y) = \lim_{n \to \infty} H(Y_{-n}^0)/(n + 1),$$

where

$$H(Y_{-n}) = -\sum_{y_{-n}^0} p(y_{-n}^0) \log p(y_{-n}^0),$$

here $y_{-n}^0 := (y_{-n}, y_{-n+1}, \cdots, y_0)$ denotes an instance of $Y_{-n}^0 := (Y_{-n}, Y_{-n+1}, \cdots, Y_0)$ and $p(y_{-n}^0)$ denotes the probability mass at $z_{-n}^0$. It is well known that $H(Y)$ can also be written as

$$H(Y) = \lim_{n \to \infty} H(Y_0|Y_{-n}^{-1}),$$

where

$$H(Y_0|Y_{-n}^{-1}) = -\sum_{y_{-n}^0} p(y_{-n}^0) \log p(y_0|y_{-n}^{-1}),$$

here $p(y_0|y_{-n}^{-1})$ denotes the conditional probability mass at $y_0$ given $y_{-n}$. 
From now on, assume that $Y$ is a stationary finite-state Markov chain with transition probability matrix $\Delta$ with entries

$$\Delta(i, j) = P(Y_1 = j | Y_0 = i), \quad 1 \leq i, j \leq B.$$ 

A hidden Markov chain $Z$ is a process of the form $Z = \Phi(Y)$, where $\Phi$ is a function defined on $Y$ with values from a finite alphabet $Z = \{1, 2, \cdots, A\}$. Often a hidden Markov chain is alternatively defined as a Markov chain observed when passing through a discrete-time memoryless noisy channel. It is well known that the two definitions are equivalent. For the Markov chain $Y$, $H(Y)$ has a simple analytic form:

$$H(Y) = - \sum_{i,j} P(Y_0 = i) \Delta(i, j) \log \Delta(i, j).$$

For the hidden Markov chain $Z$, Blackwell [9] showed that $H(Z)$ can be written as an integral of an explicit function on a simplex with respect to Blackwell’s Measure $Q$. However, the measure $Q$ seems to be rather complicated for effective computation of $H(Z)$. So far, there is no simple and explicit formula for $H(Z)$.

Recently there has been a rebirth of interest in computing and estimating the entropy rate of a hidden Markov chain, and many approaches have been adopted to tackle this problem: Blackwell’s measure has been used to bound the entropy rate [32], a variation on the classical Birch bounds [8] can be found in [13] and a new numerical approximation of $H(Z)$ has been derived in [17]. In another direction, [1, 41, 21, 22, 23, 24, 1, 37, 20, 21, 22, 23, 24, 1, 37] have studied the variation of the entropy rate as parameters of the underlying Markov chain vary. By the Shannon-McMillian-Breiman theorem, the $n$-th order sample entropy $-\log P(Z_1^n)/n$ converges to $H(Z)$ with probability 1 as $n$ tends to infinity. Based on this fact, efficient Monte Carlo methods for approximating $H(Z)$ were proposed independently by Arnold and Loeliger [2], Pfister et. al. [35], and Sharma and Singh [40]. The convergence behavior of such algorithms is governed by the limiting behavior of the sample entropy sequence, which is typically characterized by classical limit theorems such as the Central Limit Theorem (CLT) and the Law of Iterated Logarithm (LIL). In this direction, a CLT [46] for the sample entropy sequence is derived as a corollary of a rather general CLT for the top Lyapunov exponent of a product of random matrices; a functional CLT is also established in [25]. In essence, both of the two CLTs are proved using effective Martingale approximations of the sample entropy (see [11] for this standard technique).

In this paper, adapting some standard techniques for proving limit theorems for mixing sequences, we further characterize the limiting behavior of the sample entropy of $Z$ under certain positivity assumptions. In Section 3, we establish a CLT with some Berry-Esseen bound (such bound [3, 15] characterizes rate of convergence of CLT) for the sample entropy sequence, and we use this CLT can to establish a LIL in Section 4.

Formally, for $i = 1, 2, \cdots$, define $X_i$ as the “centered” version of $-\log P(Z_i|Z_i^{-1})$, that is,

$$X_i = -\log P(Z_i|Z_i^{-1}) - E[-\log P(Z_i|Z_i^{-1})] = -\log P(Z_i|Z_i^{-1}) - H(Z_i|Z_i^{-1}).$$
And define
\[ S_n = \sum_{i=1}^{n} X_i, \quad \sigma_n^2 = \text{Var}(S_n); \]

obviously \( S_n \) is the “centered” version of \( -\log P(Z_1^n) \), and \( S_n = -\log P(Z_1^n) - H(Z_1^n) \).

Unless specified otherwise, we assume, throughout the paper, that

(I) \( \Delta \) is a strictly positive matrix; and

(II) \( \sigma > 0 \), where \( \sigma^2 = \lim_{n \to \infty} \sigma_n^2/n \) (the existence of the limit under Condition (I) will be established in Lemma 2.6 and Remark 2.7).

Under Conditions (I) and (II), we will prove the following central limit theorem with a Berry-Esseen bound.

**Theorem 1.1.** Under Conditions (I) and (II), for any \( \varepsilon > 0 \), there exists \( C > 0 \) such that for any \( n \)
\[ \sup_x |P(S_n/\sigma_n < x) - \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2)dy| \leq Cn^{-1/2} \]

We will also prove the following law of iterated logarithm theorem.

**Theorem 1.2.** Under Conditions (I) and (II), we have
\[ \limsup_{n \to \infty} \frac{S_n}{(2n\sigma^2 \log \log n\sigma^2)^{1/2}} = 1 \quad \text{a.s.} \]

### 2 Key Lemmas

This section includes several key lemmas, among which Lemmas 2.1, 2.2, 2.4 require Condition (I) only.

With the fact that a \( n \times n \) positive matrix induces a contraction mapping on the interior of the \((n - 1)\)-dimensional real simplex under the Hilbert metric [39], the following well-known lemma can be established (see, e.g., [18] for a rigorous proof).

**Lemma 2.1.** There exist \( C > 0 \) and \( 0 < \rho < 1 \) such that for any two hidden Markov sequences \( z_{-m}^0, \hat{z}_{-m}^0 \) with \( z_{-n}^0 = \hat{z}_{-n}^0 \) (here \( m, \hat{m} \geq n \geq 0 \)), we have
\[ |p(z_0|z_{-m}^0) - p(\hat{z}_0|\hat{z}_{-m}^0)| \leq C\rho^n. \]

Consequently, there exists \( C > 0 \) and \( 0 < \rho < 1 \) such that for any \( n, l \geq 0 \),
\[ |\log p(z_0|z_{-n-l}^0) - \log p(z_0|z_{-n}^0)| \leq C\rho^n, |H(Z_0|Z_{-n-l}^0) - H(Z_0|Z_{-n}^0)| \leq C\rho^n. \]

For a stationary stochastic process \( T = T_{-\infty}^\infty \), let \( \mathcal{B}(T_i^j) \) denote the \( \sigma \)-field generated by \( T_k, k = i, i + 1, \ldots, j \). Define
\[ \psi(n) = \sup_{U \in \mathcal{B}(T_i^n), V \in \mathcal{B}(T_0^n), P(U) > 0, P(V) > 0} |P(V|U) - P(V)|/P(V). \]
$T$ is said to be a $\psi$-mixing sequence if $\psi(n) \to 0$ as $n \to \infty$. It is well known that a finite-state irreducible and aperiodic Markov chain is a $\psi$-mixing sequence, and the corresponding $\psi(n)$ exponentially decays as $n \to \infty$. The following lemma asserts that under Condition (I), $Z$ is a $\psi$-mixing sequence and the corresponding $\psi(n)$ exponentially decays as $n \to \infty$. An excellent survey on various mixing sequences can be found in [10]; for a comprehensive exposition to the vast literature on this subject, we refer to [11].

Lemma 2.2. $Z$ is a $\psi$-mixing sequence and there exist $C > 0$ and $0 < \lambda < 1$ such that for any positive $n$, $\psi(n) \leq C\lambda^n$.

Proof. For each $z \in Z$, let $\Delta_z$ denote the $B \times B$ matrix such that $\Delta_z(i,j) = \Delta(i,j)$ for $j$ with $\Phi(j) = z$, and $\Delta_z(i,j) = 0$ otherwise. Obviously $\sum_{z \in Z} \Delta_z = \Delta$. One also observes that for any $z_{m1}^m$,

$$p(z_{m2}^m) = \pi \Delta_{z_{m1}^m} 1,$$

where $\pi$ is the stationary vector of $Y$, $1$ denotes the all one column vector and $\Delta_{z_{m1}^m} = \Delta_{z_{m1}^m} \Delta_{z_{m1+1}} \cdots \Delta_{z_{m2}^m}$. It then follows that for any positive $n, m, l$ and any $z_0^n, z_{-n-l}^n$,

$$p(z_0^n | z_{-n-l}^n) = \sum_{z_{n+1}^n} \pi \Delta_{z_{n-l}^n} 1 = \pi \Delta_{z_{n-l}^n} \sum_{z \in Z} \Delta_z^{n-1} \Delta_{z_0^n} 1 = \pi \Delta_{z_{n-l}^n} \Delta_{z_{n-l}^n} 1.$$

Let $\lambda_2$ denote the second largest (in modulus) eigenvalue of $\Delta$. By the Perron-Frobenius theory (see, e.g., [9]), $|\Delta_2| < 1$; furthermore, for any $\lambda$ with $|\lambda_2| < \lambda < 1$, there exists $C_1 > 0$ such that for any probability vector $x$, we have

$$|x \Delta^n - \pi| \leq C_1 \lambda^n.$$

It then follows that

$$p(z_0^n | z_{-n-l}^n) = \pi \Delta_{z_0^n} 1 + O(\lambda^n) \Delta_{z_0^n} 1 = p(z_0^n) + O(\lambda^n) p(z_0^n).$$

Noting that the constant in $O(\lambda^n)$ is independent of $n, m, l$ and $z_0^n, z_{-n-l}^n$, we then conclude that for any $U \in B(Z_{-\infty}), V \in B(Z_0^\infty)$,

$$P(V \mid U) = P(V) + O(\lambda^n) P(V),$$

which immediately implies the lemma.

Remark 2.3. Note that Lemma 2.2 still holds with the same proof if Condition (I) is replaced by “$\Delta$ is an irreducible and aperiodic matrix”. This fact, however, will not be used in this paper.

In the following, we rewrite $- \log p(z_j | z_{i+1}^{j-1}) - H(Z_j | Z_{i+1}^{j-1})$ as $z_j^i$ for notational simplicity.

The following lemma shows that for a fixed $j > 0$, $E[X_i X_{i+j}]$ exponentially converges as $i \to \infty$; and for any $i < j$, $E[X_i X_j]$ exponentially decays in $j - i$.

Lemma 2.4. 1. There exist $C > 0$ and $0 < \rho < 1$ such that for all $i, j \geq 0$,

$$|E[X_{i+1} X_{i+1+j}] - E[X_{i+1} X_{i+1+j}]| \leq C \rho^i.$$
2. There exist $C > 0$ and $0 < \theta < 1$ such that for any positive $i < j$,

$$|E[X_iX_j]| \leq C\theta^{i-j}. $$

**Proof.** 1. Simple computations lead to

$$E[X_{i+1}X_{i+1+j}] - E[X_iX_{i+j}] = \sum_{z_{i+1+j}} p(z_{i+1+j}^i) f(z_{i+1+j}^i) f(z_{i}^i) - \sum_{z_{i+j}} p(z_{i+j}^i) f(z_{i+j}^i) f(z_{i}^i)$$

$$= \sum_{z_{i-j}} p(z_{i-j}^0) f(z_{i-j}^0) f(z_{i-j}^0) - \sum_{z_{i-j+1}} p(z_{i-j+1}^0) f(z_{i-j+1}^0) f(z_{i-j+1}^0)$$

$$= \sum_{z_{i-j}} p(z_{i-j}^0) (f(z_{i-j}^0) f(z_{i-j}^0) - f(z_{i-j+1}^0) f(z_{i-j+1}^0))$$

$$= \sum_{z_{i-j}} p(z_{i-j}^0) (f(z_{i-j}^0) - f(z_{i-j+1}^0))$$

$$+ \sum_{z_{i-j}} p(z_{i-j}^0) (f(z_{i-j}^0) - f(z_{i-j+1}^0)) f(z_{i-j+1}^0). \quad (1)$$

Since $\Delta$ is a strictly positive matrix, $\log p(z_0|z_{i-1}^i)$ and $H(Z_0|Z_0^{0,i})$ are all bounded from above and below uniformly in $i$. It then follows from this fact and Lemma 2.2 that there exist $C > 0$ and $0 < \rho < 1$ such that

$$|E[X_{i+1}X_{i+1+j}] - E[X_iX_{i+j}]| \leq C\rho^i.$$

Part 1 of the lemma then immediately follows.

2. Let $l = [i + j]/2$. By Lemma 2.1 and Lemma 2.2, there exist $0 < \rho, \lambda < 1$ such that

$$E[X_iX_j] = \sum_{z_i^l} p(z_i^l) f(z_i^l)$$

$$= \sum_{z_i^l} p(z_i^l) f(z_i^l) (f(z_i^l) + O(\rho^{-l}))$$

$$= \sum_{z_i^l} p(z_i^l) f(z_i^l) p(z_i^l|z_i^l) f(z_i^l) + O(\rho^{-l})$$

$$= \sum_{z_i^l} p(z_i^l) f(z_i^l) (p(z_i^l) + O(\lambda^{-l})p(z_i^l)) f(z_i^l) + O(\rho^{-l})$$

$$= \sum_{z_i^l} p(z_i^l) f(z_i^l) f(z_i^l) + \sum_{z_i^l} p(z_i^l) f(z_i^l) O(\lambda^{-l})p(z_i^l) f(z_i^l) + O(\rho^{-l})$$

$$= 0 + O(\lambda^{-l}) + O(\rho^{-l}).$$

Notice that the constants in $O(\lambda^{-l}), O(\rho^{-l})$ above does not depend on $z_i^l$. Part 2 then immediately follows. \qed
Remark 2.5. By Part 1 of Lemma 2.4, for any fixed \( j \), the sequence \( E[X_iX_{i+j}], i = 1, 2, \ldots \), is a Cauchy sequence that exponentially converges. For any fixed \( j \), let \( a_j = \lim_{i \to \infty} E[X_iX_{i+j}] \). Then by Part 2, \( |a_j| \) exponentially decays as \( j \to \infty \); consequently, we deduce (for later use) that \( a_0 + 2 \sum_{j=1}^{\infty} a_j \) converges.

Lemma 2.6. For any \( 0 < \alpha < 1 \), there exists \( C > 0 \) such that for any \( m \) and \( n \),
\[
\left| \frac{E[(S_{n+m} - S_m)^2]}{n} - (a_0 + 2 \sum_{j=1}^{\infty} a_j) \right| \leq C n^{-\alpha}
\]
here, recall that, as defined in Remark 2.3, \( a_j = \lim_{i \to \infty} E[X_iX_{i+j}] \).

Proof. Letting \( \beta = n^{-\alpha} \) for a fixed \( 0 < \alpha < 1 \), we then have
\[
\frac{E[(S_{n+m} - S_m)^2]}{n} = \frac{E[(\sum_{i=m+1}^{n+m} X_i)^2]}{n} = \sum_{m+1 \leq i, i+j \leq n+m} (\sum_{j=0}^{2} \sum_{0 < j < \beta n} + 2 \sum_{j > \beta n}) E[X_iX_{i+j}].
\]
By Part 1 of Lemma 2.4 and Remark 2.3, for any \( j > 0 \), \( E[X_iX_{i+j}] - a_j = O(\rho^j) \) for some \( 0 < \rho < 1 \). It then follows that for \( 0 \leq j \leq \beta n \),
\[
\sum_{m+1 \leq i, i+j \leq n+m} E[X_iX_{i+j}] = (n - j) a_j + O(1);
\]
here the constant in \( O(1) \) does not depend on \( j \). Also, by Part 2 of Lemma 2.4 and Remark 2.3, there exists \( 0 < \theta < 1 \) such that for all \( j > \beta n \), \( E[X_iX_{i+j}] = O(\theta^{\beta n}) \), and thus \( a_j = O(\theta^{\beta n}) \). Continuing the computation, we have
\[
\frac{E[(S_{n+m} - S_m)^2]}{n} = \frac{(na_0 + O(1)) + (2(n - 1)a_1 + O(1)) + \cdots + (2(n - \beta n)a_{\beta n} + O(1))}{n} + O(n^2 \theta^{3n})
\]
\[
= a_0 + 2a_1 + \cdots + 2a_{\beta n} - 2a_1 + 2a_2 + \cdots + \beta na_{\beta n} + \beta O(1) + O(n \theta^{\beta n}).
\]
The lemma then immediately follows. \( \square \)

Remark 2.7. Choosing \( m \) in Lemma 2.6 to be 0, we deduce that \( \lim_{n \to \infty} \sigma_n^2/n \) exists and is equal to \( \sigma^2 = a_0 + 2 \sum_{j=1}^{\infty} a_j \).

Lemma 2.8. There exists \( C > 0 \) such that for all \( m \) and \( n \)
\[
E[|S_{n+m} - S_m|^2] \leq C n^{3/2}.
\]

Proof. By Lemmas 2.4, 2.6 and the stationarity of \( Z \), we observe that for any \( m \),
\[
E[|S_{n+m} - S_m|^2] = E[|\sum_{i=m+1}^{n+m} f(Z_i)|^2] = E[|\sum_{i=m+1}^{n+m} (f(Z_{m+i}) + O(\rho^{i-m-1}))|^2]
\]
\[
= E[|S_n|^2] + O(E[|S_n|^2]) + O(E[|S_n|]) + O(1) = E[|S_n|^2] + O(n), \tag{2}
\]

where the constant in $O(n)$ does not depend on $m$. So, to prove the lemma, it suffices to prove that there exists $C > 0$ such that for all $n$,

$$E[|S_n|^3] \leq Cn^{3/2}. \quad (3)$$

First, we observe that

$$E[|S_{2n}|^3] = E[|S_n| + (S_{2n} - S_n)|^3] \leq E[|S_n|^3] + E[|S_{2n} - S_n|^3] + 3E[|S_n|^2|S_{2n} - S_n|] + 3E[|S_n||S_{2n} - S_n|^2].$$

It immediately follows from (E) that

$$E[|S_{2n} - S_n|^3] = E[|S_n|^3] + O(n).$$

And by Lemmas (2.2), (2.1) and the Holder inequality, there exist $0 < \rho < 1$ such that

$$E[|S_n|^2|S_{2n} - S_n|] = E[|S_n|^2] \sum_{i=n+1}^{2n} f(Z_i^n)$$

$$= E[|S_n|^2] \sum_{i=n+1}^{2n} (f(Z_{n+1}) + O(\rho^{i-n-1}))$$

$$= E[|S_n|^2] \sum_{i=n+1}^{2n} f(Z_{n+1}^i) + O(1)E[|S_n|^2]$$

$$= \sum_{z_1^n, z_2^n} (p(z_1^n)p(z_2^{n+1}|z_1^n)|S_n|^2) \sum_{i=n+1}^{2n} f(Z_{n+1}^i) + O(1)E[|S_n|^2]$$

$$= O(1) \sum_{z_1^n, z_2^n} (p(z_1^n)p(z_2^{n+1})|S_n|^2) \sum_{i=n+1}^{2n} f(Z_{n+1}^i) + O(1)E[|S_n|^2]$$

$$= O(1)E[|S_n|^2]E[\sum_{i=n+1}^{2n} f(Z_{n+1}^i)] + O(1)E[|S_n|^2]$$

$$= O(1)E[|S_n|^2]E[|S_{2n} - S_n|] + O(1)E[|S_n|^2]$$

$$= O(1)E[|S_n|^2]E[|S_{2n} - S_n|^2]^{1/2} + O(1)E[|S_n|^2].$$

We then deduce that

$$E[|S_n|^2|S_{2n} - S_n|] = O(n^{3/2}).$$

With a similar argument, we also deduce that

$$E[|S_n||S_{2n} - S_n|^2] = O(n^{3/2}).$$

It then follows that there exists $C_1 > 0$ such that for all $n$

$$E[|S_{2n}|^3] \leq 2E[|S_n|^3] + C_1n^{3/2}. \quad (4)$$

An iterative application of (3) with $n = 2^{r-1}, 2^{r-2}, \ldots, 2^0$ gives us

$$E[|S_{2^r}|^3] \leq 2E[|S_{2^{r-1}}|^3] + C_12^{3(r-1)/2} \leq 2^2E[|S_{2^{r-2}}|^3] + C_1(2 \cdot 2^{3(r-2)/2} + 2^{3(r-1)/2})$$
\[ \leq \cdots \leq 2^r E[|S_1|^3]+C_1 (2^{r-1}+2^{r-2}2^{3/2}+\cdots+2^{3(r-2)/2}+2^{3(r-1)/2}) \leq 2^r E[|S_1|^3]+C_1 \frac{2^{3(r-1)/2}}{1-2^{-3/2}}. \]

It then follows that there exists \( C_2 > 0 \) such that for all \( r \)
\[ E[|S_{2^r}|^3] \leq C_2 (2^r)^{3/2}. \]
and by (2), choosing \( C_2 \) to be larger if necessary, we have for all \( r \) and all \( m \)
\[ E[|S_{m+2^r} - S_m|^3] \leq C_2 (2^r)^{3/2}. \quad (5) \]

Now, consider the general case when \( 2^r \leq n < 2^{r+1} \). Expand
\[ n = v_0 2^r + v_1 2^{r-1} + \cdots + v_r \]
where \( v_0 = 1 \), and \( v_j = 0 \) or 1 for \( j = 1, 2, \cdots, r \). And define
\[ w_i = v_{r-i} 2^i + v_{r-i+1} 2^{i-1} + \cdots + v_r. \]

Applying the Minkowski inequality and (1), we deduce that
\[ E[|S_n|^3] \leq E[\left| \sum_{i=0}^r (S_{w_i} - S_{w_{i-1}}) \right|^3] \leq \left( \sum_{i=0}^r E[|S_{w_i} - S_{w_{i-1}}|^3] \right)^{1/3} \]
\[ \leq (\sum_{i=0}^r (C_2 2^{i/2}))^3 \leq C_2^3 (2^{r/2} \frac{1}{1-1/2})^3 = 8C_2^3 2^{3r/2} \leq 8C_2^3 n^{3/2}. \]

Inequality (2) is then immediately established if we choose \( C = 8C_2^3 \), \( \square \)

3 Central Limit Theorem

Recall that
\[ X_i = -\log P(Z_i|Z_i^{-1}) - H(Z_i|Z_i^{-1}), \]
and
\[ S_n = \sum_{i=1}^n X_i, \quad \sigma_n^2 = Var(S_n). \]

Fix \( \varepsilon_0 > 0 \) (we will choose \( \varepsilon_0 \) to be small later), and let \( p = p(n) = \lfloor r^{3/(11+\varepsilon_0)} \rfloor, q = q(n) = \lfloor n^{\varepsilon_0} \rfloor \). Choose \( k = k(n) \) such that \( kp + (k-1)q \leq n < (k+1)p + kq \); one easily checks that \( k = O(n^{8/(11-\varepsilon_0)}) \). Then, for \( 1 \leq i \leq k \), define
\[ \zeta_i = X_{(i-1)(p+q)+1} + \cdots + X_{ip+(i-1)q}. \]
For \( 1 \leq i \leq k-1 \), define
\[ \eta_i = X_{ip+(i-1)q+1} + \cdots + X_{ip+q}, \]
and define
\[ \eta_k = \begin{cases} X_{kp+(k-1)q+1} + \cdots + X_n & \text{if } kp + (k-1)q + 1 \leq n \\ 0 & \text{otherwise}. \end{cases} \]
Now $S_n$ can be rewritten as a sum of $\zeta$-“blocks” and $\eta$-“blocks”:

$$S_n = S'_n + S''_n := \sum_{i=1}^{k} \zeta_i + \sum_{i=1}^{k} \eta_i.$$ 

The above so called “Bernstein blocking method” [3] is a standard technique to the proof of limit theorems for a variety of mixing sequences. Roughly speaking, the partial sum $S_n$ is partitioned into “long blocks” $\zeta_1, \zeta_2, \cdots, \zeta_k$ and “short blocks” $\eta_1, \eta_2, \cdots, \eta_k$. Under certain mixing conditions, all long blocks are “weakly dependent” on each other, while all short blocks are “negligible” in some sense.

With lemmas in Section 2 established, the remainder of the proof of Theorem 1.1 becomes more or less standard, which can be roughly outlined as follows:

1. We first show $E[\exp(itS'_n/\sigma_n)]$ and $\prod_{j=1}^{k} E[\exp(it\zeta_j/\sigma_n)]$ are “close” (see Lemma 3.1).
2. Standard analysis shows that $\prod_{j=1}^{k} E[\exp(it\zeta_j/\sigma_n)]$ and $\exp(-t^2/2)$ are “close” (see Lemma 3.2).
3. Then by the standard Esseen’s Lemma, $P(S'_n/\sigma_n < x)$ and $\int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2) dy$ are “close” (see Lemma 3.3).
4. Finally, since $S''_n$ are “negligible”, we conclude, in the proof of Theorem 1.1, that $P(S_n/\sigma_n < x)$ and $P(S''_n/\sigma_n < x)$ are “close”, and thus $P(S_n/\sigma_n < x)$ and $\int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2) dy$ are “close”.

**Lemma 3.1.** There exists $C > 0$ and $0 < \rho_1 < 1$ such that for all $n$ and $|t| \leq n^{1/11}$,

$$|E[\exp(itS'_n/\sigma_n)] - \prod_{j=1}^{k} E[\exp(it\zeta_j/\sigma_n)]| \leq C\rho_1^{n}.$$ 

**Proof.** Let $l = (k-1)p + (k-2)q + q/2$. By Lemma 2.1 and Lemma 2.2, there exist $0 < \lambda, \rho < 1$ such that

$$E[\exp(it \sum_{j=1}^{k} \zeta_j/\sigma_n)] = E[\exp(it \sum_{j=1}^{k-1} \zeta_j/\sigma_n) \exp(it\zeta_k/\sigma_n)]$$

$$= E[\exp(it \sum_{j=1}^{k-1} \zeta_j/\sigma_n) \exp(it \sum_{i=(k-1)(p+q)+1}^{kp+(k-1)q} f(z_i')/\sigma_n)]$$

$$= E[\exp(it \sum_{j=1}^{k-1} \zeta_j/\sigma_n) \exp(it \sum_{i=(k-1)(p+q)+1}^{kp+(k-1)q} f(z_i')/\sigma_n)] + O(n^{1/11}\rho^{q(n)/2}/\sigma_n)$$

$$= E[\exp(it \sum_{j=1}^{k-1} \zeta_j/\sigma_n)]E[\exp(it \sum_{i=(k-1)(p+q)+1}^{kp+(k-1)q} f(z_i')/\sigma_n)] + O(\lambda^{q(n)/2}) + O(n^{1/11}\rho^{q(n)/2}/\sigma_n)$$

$$= E[\exp(it \sum_{j=1}^{k-1} \zeta_j/\sigma_n)]E[\exp(it\zeta_k/\sigma_n)] + O(\lambda^{q(n)/2}) + O(n^{1/11}\rho^{q(n)/2}/\sigma_n),$$
here, again, \(-\log p(z_j|z_i^{j-1}) - H(Z_j|Z_i^{j-1})\) is rewritten as \(f(z_j^1)\). Noticing that \(|E[\exp(it\zeta_j/\sigma_n)]| \leq 1\) and applying an inductive argument, we conclude that

\[
E[\exp(itS_n'/\sigma_n)] = E[\exp(it \sum_{j=1}^k \zeta_j/\sigma_n)] = \prod_{j=1}^k E[\exp(it\zeta_j/\sigma_n)] + O(\lambda^{q(n)/2}) + O(n^{1/11} \rho^{q(n)/2}/\sigma_n),
\]

which immediately implies the lemma.

\[\square\]

**Lemma 3.2.** There exists \(C > 0\) such that for all \(n\) and \(|t| \leq n^{1/11}\),

\[
|\prod_{j=1}^k E[\exp(it\zeta_j/\sigma_n)] - \exp(-t^2/2)| \leq C n^{-1/11+\varepsilon_0/2}.
\]

**Proof.** It is well known (see, e.g., page 343 of [1]) that for any random variable \(X\) and any \(t \in \mathbb{R}\), we have

\[
|E[\exp(itX)] - \sum_{k=0}^n (it)^k/k! E[X^k]| \leq E[|tX|^{n+1}/(n+1)!].
\]

Replacing \(X\) by \(\zeta_j/\sigma_n\), we deduce that

\[
E[\exp(it\zeta_j/\sigma_n)] = 1 - E[\zeta_j^2]/(2\sigma_n^2) + O(E[|\zeta_j|^3]t^3/(6\sigma_n^3)).
\]

With Lemmas 2.7 and 2.8, one checks that for any \(|t| \leq n^{1/11}\),

\[
E[|\zeta_j|^2]/(2\sigma_n^2), E[|\zeta_j|^3]t^3/(6\sigma_n^3) \to 0 \text{ as } n \to \infty.
\]

Using the fact that \(\log(1-x) = -x + O(x^2)\) for \(|x| < 1\), we deduce that

\[
\log E[\exp(it\zeta_j/\sigma_n)] = -E[\zeta_j^2]/(2\sigma_n^2) + O(E[|\zeta_j|^3]t^3/\sigma_n^3) + O(E^2[\zeta_j^2]t^4/\sigma_n^4) + O(E^2[\zeta_j^3]t^6/\sigma_n^6) + O(E[\zeta_j^2]E[|\zeta_j|^3]t^5/\sigma_n^5).
\]

Now,

\[
\log \prod_{j=1}^k E[\exp(it\zeta_j/\sigma_n)] = \sum_{j=1}^k \log E[\exp(it\zeta_j/\sigma_n)]
\]

\[= -t^2/2 - t^2(\sum_{j=1}^k E[\zeta_j^2]/\sigma_n^2 - 1)/2 + O(\sum_{j=1}^k E[|\zeta_j|^3]t^3/\sigma_n^3) + O(\sum_{j=1}^k E^2[\zeta_j^2]t^4/\sigma_n^4)
\]

\[+ O(\sum_{j=1}^k E^2[|\zeta_j|^3]t^6/\sigma_n^6) + O(\sum_{j=1}^k E[\zeta_j^2]E[|\zeta_j|^3]t^5/\sigma_n^5) \tag{7}
\]

It follows from Lemma 2.6 that for any \(\alpha > 0\)

\[
E[\zeta_j^2] = p(n)(\sigma^2 + O(p(n)^{-\alpha})), \quad i = 1, 2, \ldots, k,
\]
Lemma 3.4. for every $T > C$, there exists $C > 0$ such that for all $n$

$$\sup_x \left| P(S_n'/\sigma_n < x) - \int_{-\infty}^x (2\pi)^{-1/2} \exp(-y^2/2) dy \right| \leq Cn^{-1/11+\varepsilon}.$$ 

The following lemma is a version of Esseen’s lemma, which gives upper bounds on the difference between two distribution functions using the difference between the two corresponding characteristic functions. We refer to page 314 of [11] for a standard proof.

**Lemma 3.3.** Let $F$ and $G$ be distribution functions with characteristic functions $\phi_F$ and $\phi_G$, respectively. Suppose that $F$ and $G$ each has mean 0, and $G$ has a derivative $g$ such that $|g| \leq M$. Then

$$\sup_x |F(x) - G(x)| \leq \frac{1}{\pi} \int_{-T}^T \left| \frac{\phi_F(t) - \phi_G(t)}{t} \right| dt + \frac{24M}{\pi T}$$

for every $T > 0$.

**Lemma 3.4.** For any $\varepsilon > 0$, there exists $C > 0$ such that for all $n$

$$\sup_x \left| P(S_n'/\sigma_n < x) - \int_{-\infty}^x (2\pi)^{-1/2} \exp(-y^2/2) dy \right| \leq Cn^{-1/11+\varepsilon}.$$
Proof. Let \( \phi_0(t) \) denote the characteristic function of a standard normal random variable, that is, \( \phi_0(t) = \exp(-t^2/2) \). Let \( \phi_1(t) \) denote the characteristic function of \( S'_n/\sigma_n \), that is, \( \phi_1(t) = E[\exp(itS'_n/\sigma_n)] \). Let \( \phi_2(t) \) denote the function \( \prod_{j=1}^k E[\exp(it\zeta_j/\sigma_n)] \). Applying Lemma \( \ref{lem:characteristic-function} \) with \( T = n^{1/11} \), we have

\[
\sup_{x} \left| P(S'_n/\sigma_n < x) - \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-y^2/2) dy \right| = O \left( \int_{-n^{1/11}}^{n^{1/11}} \left| \frac{\phi_1(t) - \phi_0(t)}{t} \right| dt + n^{-1/11} \right)
\]
\[
= O \left( \int_{-n^{1/11}}^{n^{1/11}} \left| \frac{\phi_1(t) - \phi_2(t)}{t} \right| dt + \int_{-n^{1/11}}^{n^{1/11}} \left| \frac{\phi_2(t) - \phi_0(t)}{t} \right| dt + n^{-1/11} \right)
\]
\[
= O \left( \int_{|t|<n^{-1/2}} \left| \frac{\phi_1(t) - \phi_2(t)}{t} \right| dt + \int_{n^{-1/2}<|t|<n^{1/11}} \left| \frac{\phi_1(t) - \phi_2(t)}{t} \right| dt + n^{-1/11} \right)
\]
\[
+ \int_{|t|<n^{-1/2}} \left| \frac{\phi_2(t) - \phi_0(t)}{t} \right| dt + \int_{n^{-1/2}<|t|<n^{1/11}} \left| \frac{\phi_2(t) - \phi_0(t)}{t} \right| dt + n^{-1/11} \right).
\]

Note that by Lemmas \( \ref{lem:uniform-bound} \), \( \ref{lem:characteristic-function} \), we deduce that for any \( \alpha > 0 \) and some \( 0 < \theta < 1 \),

\[
\frac{E[S'^2_n]}{\sigma_n^2} = \sum_{i=1}^k E[\zeta_i^2] + 2 \sum_{i<j} E[\zeta_i \zeta_j] = \frac{k(n)p(n)(\sigma^2 + O(p(n)^{-\alpha})) + O(n^2\theta^q(n))}{n(\sigma^2 + O(n^{-\alpha}))},
\]

which implies that

\[
\frac{E[S'^2_n]}{\sigma_n^2} = O(1);
\]
uniformly over all \( n \). It then follows that for all \( |t| \leq 1 \),

\[
\phi_1(t) = E[\exp(itS'_n/\sigma_n)] = 1 + t^2 O(E[S'^2_n]/\sigma_n^2) = 1 + O(t^2),
\]

where we applied \( \ref{lem:weak-convergence} \). Also, it follows from \( \ref{lem:convergence_1}, \ref{lem:convergence_2}, \ref{lem:convergence_3} \) and \( \ref{lem:convergence_4} \) that for all \( |t| \leq 1 \),

\[
\log \phi_2(t) = O(t^2),
\]
uniformly over all \( n \), and thus for all \( |t| \leq 1 \),

\[
\phi_2(t) = 1 + O(t^2),
\]
uniformly over all \( n \). Obviously, we also have for all \( |t| \leq 1 \),

\[
\phi_0(t) = 1 + O(t^2).
\]

It then follows from \( \ref{lem:convergence_5}, \ref{lem:convergence_6} \) and \( \ref{lem:convergence_7} \) that there exists \( C_1 > 0 \) such that for all \( |t| \leq 1 \) and all \( n \),

\[
|\phi_1(t) - \phi_2(t)| \leq C_1 t^2, \quad |\phi_2(t) - \phi_0(t)| \leq C_1 t^2,
\]
which implies that

\[
\int_{|t|<n^{-1/2}} \left| \frac{\phi_1(t) - \phi_2(t)}{t} \right| dt = O(n^{-1}), \quad \int_{|t|<n^{-1/2}} \left| \frac{\phi_2(t) - \phi_0(t)}{t} \right| dt = O(n^{-1}).
\]
It follows from Lemma 3.2 that
\[
\int_{n^{-1/2} < |t| < n^{1/11}} \left| \frac{\phi_2(t) - \phi_0(t)}{t} \right| \, dt = O(n^{-1/11 + \varepsilon_0/2} \log n),
\] (15)
and from Lemma 3.1 that
\[
\int_{n^{-1/2} < |t| < n^{1/11}} \left| \frac{\phi_1(t) - \phi_2(t)}{t} \right| \, dt = O(\rho_1^{\varepsilon(n)} \log n)
\] (16)
for some 0 < \rho_1 < 1. The lemma then follows from (14), (15), (16) and the fact that \varepsilon_0 can be chosen to be arbitrarily small.

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( F \) denote the event \(|S''_n|/\sigma_n \leq n^{-1/11}\). Then
\[
|P(S''_n/\sigma_n \leq x) - P(S_n/\sigma_n \leq x)|
\leq |P(S_n/\sigma_n \leq x, F) - P(S_n/\sigma_n \leq x, F^c)| + |P(S''_n/\sigma_n \leq x, F^c) - P(S_n/\sigma_n \leq x, F^c)|
\leq |P(S_n/\sigma_n \leq x, F) - P(S_n/\sigma_n \leq x, F)| + P(|S''_n|/\sigma_n > n^{-1/11}).
\]
Applying Lemma 3.1, we have, for any \( \varepsilon > 0 \), there exists \( C_1 > 0 \) such that for any \( n \)
\[
|P(S''_n/\sigma_n \leq x, F) - P(S_n/\sigma_n \leq x, F)|
\leq \max\{P(S_n/\sigma_n \leq x + n^{-1/11}, F) - P(S'_n/\sigma_n \leq x, F), P(S'_n/\sigma_n \leq x, F) - P(S''_n/\sigma_n \leq x - n^{-1/11}, F)\}
\leq C_1 n^{-1/11 + \varepsilon} + \int_{-n^{-1/11}}^{n^{-1/11}} (2\pi)^{-1/2} \exp(-y^2/2) \, dy = O(n^{-1/11 + \varepsilon}) + O(n^{-1/11}) = O(n^{-1/11 + \varepsilon}).
\]
Applying Lemma 3.3 and Lemma 3.1, we deduce that for some 0 < \( \theta < 1 \),
\[
E[(S''_n)^2]/\sigma_n^2 = \sum_{i=1}^k E[\eta^2_i] + \sum_{i<j} E[\eta_i \eta_j]/\sigma_n^2 = k(n)q(n)\sigma^2(1 + o(1)) + O(n^2\theta^{q(n)}) = O(n^{-3/11}).
\]
Also, by the Markov inequality, we have
\[
P(|S''_n|/\sigma_n > n^{-1/11}) \leq E[(S''_n)^2]/\sigma_n^2n^{-2/11} = O(n^{-1/11}).
\]
The theorem then immediately follows.

**Remark 3.5.** If Condition (II) fails, i.e., \( \lim_{n \to \infty} \sigma_n^2/n = 0 \), then a CLT of degenerated form holds for \((X_i, i \in \mathbb{N})\); more precisely, the distribution of \((X_1 + X_2 + \cdots + X_n)/\sqrt{n}\) converges to that of a centered normal distribution with variance 0, i.e., a point mass at 0, as \( n \to \infty \). This is can be readily checked since for any \( \varepsilon > 0 \), by the Markov inequality, we have
\[
P(|(X_1 + X_2 + \cdots + X_n)/\sqrt{n} \geq \varepsilon|) \leq \sigma^2_n/(n\varepsilon^2) \to 0 \text{ as } n \to \infty.
\]
Law of Iterated Logarithm

From the central limit theorem with a Berry-Esseen bound (Theorem 1.1), we only need to follow a standard “track” to establish the law of iterated logarithm. In particular, we closely follow the proof of Reznik’s law of the iterated logarithm (for a stationary $\phi$-mixing sequence) (see page 307 of [31]):

1. As an immediately corollary of Theorem 1.1, the following Lemma 4.1 gives bounds on the tail probability of $S_n$.

2. We then slightly modified Reznik’s maximal inequality to to obtain our maximal inequality in Lemma 4.2.

3. Finally, we are ready for the proof of Theorem 1.2, where some necessary modifications are incorporated into the original Reznik’s proof to deal with the complications resulted from the fact that $X$ is not stationary.

Lemma 4.1. For any $|\delta| < 1$ and $\alpha > 0$, we have

$$(\log \sigma_n^2)^{-(1+\delta)^2(1+\alpha)} < P(S_n > (1 + \delta)(2\sigma_n^2 \log \log \sigma_n^2)^{1/2}) < (\log \sigma_n^2)^{-(1+\delta)^2(1-\alpha)}$$

for $n$ sufficiently large.

Proof. By Theorem 1.1, we have for any $\varepsilon > 0$

$$P(S_n/\sigma_n > (1 + \delta)(2 \log \log \sigma_n^2)^{1/2}) - \int_{(1+\delta)(2 \log \log \sigma_n^2)^{1/2}}^{\infty} (2\pi)^{-1/2} \exp(-y^2/2)dy = O(n^{-1/11+\varepsilon}).$$

It can be verified that for any $x > 0$

$$\int_x^{\infty} (2\pi)^{-1/2} \exp(-y^2/2)dy \leq \exp(-x^2/2),$$

which implies that

$$\int_{(1+\delta)(2 \log \log \sigma_n^2)^{1/2}}^{\infty} (2\pi)^{-1/2} \exp(-y^2/2)dy \leq (\log \sigma_n^2)^{-(1+\delta)^2}.$$ 

It can also be verified that for any $\alpha > 0$,

$$\exp(-(1+\alpha)x^2/2) \leq \int_x^{\infty} (2\pi)^{-1/2} \exp(-y^2/2)dy$$

for $x$ large enough, which implies that for any $\alpha > 0$,

$$(\log \sigma_n^2)^{-(1+\delta)^2(1+\alpha)} \leq \int_{(1+\delta)(2 \log \log \sigma_n^2)^{1/2}}^{\infty} (2\pi)^{-1/2} \exp(-y^2/2)dy,$$

for $n$ large enough. The lemma then follows from $\sigma_n^2 = n\sigma^2(1 + o(1))$, by Remark 2.7. □
Lemma 4.2. For any \( x > 0, 0 < \alpha < 1/2 \) and \( C > 0 \), we have

\[
P(\max_{j \leq n} S_j > x) \leq 2P(S_n > x - 2\sigma_n) + Cn^{-\alpha},
\]

for sufficiently large \( n \).

Proof. For \( j = 1, 2, \cdots, n \), let \( F_j \) be the event “\( S_1, S_2, \cdots, S_{j-1} \leq x, S_j > x \)”, that is, \( F_j \) is the event “\( j \) is the smallest index such that \( S_j > x \)”. Then, for any \( x > 0, 0 < \alpha < 1/2 \) and \( C > 0 \), we have

\[
P(\max_{j \leq n} S_j > x) = \sum_{j=1}^{n} P(F_j) = (\sum_{P(F_j) > Cn^{-(1+\alpha)}} + \sum_{P(F_j) \leq Cn^{-(1+\alpha)}})P(F_j) \leq \sum_{P(F_j) > Cn^{-(1+\alpha)}} P(F_j) + Cn^{-\alpha}.
\]

Since

\[
P(S_n > x - 2\sigma_n) \geq \sum_{j=1}^{n} P(|S_n - S_j| \leq 2\sigma_n, F_j) = \sum_{j=1}^{n} P(|S_n - S_j| \leq 2\sigma_n|F_j)P(F_j),
\]

we only need to prove that for \( n \) sufficiently large, \( P(|S_n - S_j| > 2\sigma_n|F_j) \leq 1/2 \) for any \( F_j \) with \( P(F_j) > Cn^{-(1+\alpha)} \).

Now fix a positive integer \( n_0 \) (we will choose \( n_0 \) large enough later). For the case when \( n - j \leq n_0 \), applying the Markov inequality, we have

\[
P(|S_n - S_j| > 2\sigma_n|F_j) \leq P(|S_n - S_j| > 2\sigma_n)/p(F_j) \leq E[\sum_{i=j+1}^{n} X_i^2](8P(F_j)\sigma_n^2)^{-1} = O(n^{\alpha-1/2}),
\]

where we used \( P(F_j) > Cn^{-(1+\alpha)} \). For the case when \( n - j > n_0 \), we have

\[
P(|S_n - S_j| > 2\sigma_n|F_j) \leq P(|\sum_{i=1}^{j+n_0} X_i| > \sigma_n/2|F_j) + P(|\sum_{i=j+1+n_0}^{n} X_i| > 3\sigma_n/2|F_j).
\]

Again, using \( P(F_j) > Cn^{-(1+\alpha)} \) and the Markov inequality, we have

\[
P(|\sum_{i=1}^{j+n_0} X_i| > \sigma_n/2|F_j) \leq P(|\sum_{i=1}^{j+n_0} X_i| > \sigma_n/2)/P(F_j) \leq 8E[\sum_{i=1}^{j+n_0} X_i^2](P(F_j)\sigma_n^2)^{-1} = O(n^{\alpha-1/2}).
\]

Also, it follows from Lemma 2.2 and Lemma 2.6 that

\[
P(|\sum_{i=j+1+n_0}^{n} X_i| > 3\sigma_n/2|F_j) \leq \psi(n_0) + P(|\sum_{i=j+1+n_0}^{n} X_i| > 3\sigma_n/2)
\]

\[
\leq \psi(n_0) + 4E[\sum_{i=j+1+n_0}^{n} X_i^2]/9\sigma_n^2 = \psi(n_0) + 4/9 + o(1).
\]

Apparently, for both cases, choosing \( n_0 \) sufficiently large so that \( \psi(n_0) \) is small enough, and then choosing \( n \) sufficiently large, we deduce that for any \( F_j \) with \( P(F_j) > Cn^{-(1+\alpha)} \),

\[
P(|S_n - S_j| > 2\sigma_n|F_j) \leq 1/2.
\]

The lemma then immediately follows. \( \square \)
We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We first show that

\[
\limsup_{n \to \infty} \frac{S_n}{(2n\sigma^2 \log \log n\sigma^2)^{1/2}} \leq 1 \quad \text{a.s.;} \quad (17)
\]
equivalently, we show that for any \( \delta > 0 \),

\[
P\left( \frac{S_n}{(2\sigma_n^2 \log \log \sigma_n^2)^{1/2}} > 1 + \delta \ i.o. \right) = 0,
\]

here we remind the reader that by Remark 2.7, \( \sigma_n^2 = n(\sigma^2 + o(1)) \) and “i.o.” means “infinitely often”.

Fox fixed \( M > 1 \), define \( n_j = M^j, j = 1, 2, \ldots \). One then checks that

\[
\frac{S_n}{(2\sigma_n^2 \log \log \sigma_n^2)^{1/2}} > 1 + \delta \ i.o. \quad \text{⊂} \quad \max_{n \leq n_{j+1}} \frac{S_n}{(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2}} > 1 + \delta \ i.o. \quad .
\]

So, to prove (18), it suffices (by the Borel-Cantelli Lemma) to show that

\[
\sum_{j=1}^{\infty} P\left( \max_{n \leq n_{j+1}} \frac{S_n}{(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2}} > 1 + \delta \right) < \infty. \quad (19)
\]

Now, by Lemma 4.2,

\[
\sum_{j=1}^{\infty} P\left( \max_{n \leq n_{j+1}} \frac{S_n}{(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2}} > 1 + \delta \right) \leq \sum_{j=1}^{\infty} P(S_{n_{j+1}} > (1 + \delta)(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2} - 2\sigma_{n_{j+1}}). \quad (20)
\]

Note that there exists \( 0 < \delta_1 < \delta \) such that for \( j \) sufficiently large,

\[
(1 + \delta)(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2} - 2\sigma_{n_{j+1}} > (1 + \delta_1)(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2} . \quad (21)
\]

Applying Lemma 4.1 with \( \alpha \) chosen such that \( (1 + \delta_1)^2(1 - \alpha) > 1 \), we deduce that

\[
\sum_{j=1}^{\infty} P(S_{n_{j+1}} > (1 + \delta_1)(2\sigma_{n_j}^2 \log \log \sigma_{n_j}^2)^{1/2}) \leq \sum_{j=1}^{\infty} (\log \sigma_{n_j}^2)^{-(1+\delta_1)^2(1-\alpha)} = \sum_{j=1}^{\infty} O(j^{-(1+\delta_1)^2(1-\alpha)}) < \infty. \quad (22)
\]

Immediately, (18) follows from (20), (21) and (22). Here, we remark that the same argument as above with \( X_i \) replaced by \( -X_i \) leads to

\[
\liminf_{n \to \infty} \frac{S_n}{(2n\sigma^2 \log \log n\sigma^2)^{1/2}} \geq -1 \quad \text{a.s.} \quad (23)
\]

For the other direction, we next show that

\[
\limsup_{n \to \infty} \frac{S_n}{(2n\sigma^2 \log \log n\sigma^2)^{1/2}} \geq 1 \quad \text{a.s.;}
\]
equivalently, we show that for any $\delta > 0$,
\[
P\left( \frac{S_n}{(2\sigma_n^2 \log \log \sigma_n^2)^{1/2}} > 1 - \delta \ i.o. \right) = 1. \tag{24}
\]
For fixed $N > 1$ and $\delta > 0$, let $C_n(\delta)$ be the event
\[
"S_{N^n} - S_{N^n-1+N^{n/2}} > (1 - \delta)g(N^n - N^{n-1} - N^{n/2})",
\]
where $g(n) = (2n\sigma^2 \log n\sigma^2)^{1/2}$. With Lemmas 2.1 and 4.1, one checks that there exists $0 < \delta_2 < \delta$ such that for a given $\alpha > 0$
\[
P(C_n(\delta)) \geq P(S_{N^n - N^{n-1} - N^{n/2}} > (1 - \delta_2)g(N^n - N^{n-1} - N^{n/2})) \geq \log(N^n - N^{n-1} - N^{n/2})^{-\frac{1}{2}(1+\alpha)/2}
\tag{25}
\]
for sufficiently large $n$. From now on, we choose $\alpha > 0$ such that $(1 - \delta_2)^2(1 + \alpha) < 1$. If $n$ and $N$ are large enough, we have
\[
N^n - N^{n-1} - N^{n/2} \geq N^n/2,
\]
which, together with (25), implies that for any $\delta > 0$
\[
\sum_{n=1}^{\infty} P(C_n(\delta)) = \infty. \tag{26}
\]
Similarly, let $\hat{C}_n(\delta)$ be the event
\[
"\sum_{i=N^{n-1}+N^{n/2}+1}^{N^n} - \log p(Z_i|Z_{N^{n-1}+N^{n/4}}^{i-1}) - H(Z_i|Z_{N^{n-1}+N^{n/4}}^{i-1}) > (1 - \delta)g(N^n - N^{n-1} - N^{n/2})",
\]
Applying Lemma 2.1, we deduce that that for any $\delta' > 0$, there exists $0 < \delta < \delta'$ such that for sufficiently large $n$,
\[
\hat{C}_n(\delta') \supset C_n(\delta),
\]
which, together with (26), implies that for any $\delta' > 0$
\[
\sum_{n=1}^{\infty} P(\hat{C}_n(\delta')) = \infty.
\]
Again, by Lemma 2.1, for any $\delta > 0$, there exists $0 < \delta'' < \delta$ such that for sufficiently large $n$,
\[
\hat{C}_n(\delta'') \subset C_n(\delta).
\]
It then follows from an iterative application of Lemma 2.2 that there exists $0 < \theta < 1$ such that for any $n, l$,
\[
P(\cap_{m=n}^{n+l} C_m^c(\delta)) \leq P(\cap_{m=n}^{n+l} \hat{C}_m(\delta'')) = \prod_{m=n}^{n+l} P(\hat{C}_m(\delta'')) + \sum_{m=n}^{n+l} O(\theta^{N^n/4}),
\]
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\[
\prod_{m=n}^{n+l} (1 - P(\hat{C}_m(\delta''))) + \sum_{m=n}^{n+l} O(\theta^{N^{m/4}}) \leq \exp\left(-\sum_{m=n}^{n+l} P(\hat{C}_m(\delta'')) \right) + \sum_{m=n}^{l} O(\theta^{N^{m/4}}).
\]

So, as \( l,n \to \infty \), \( P(\cap_{m=n}^{n+l} C_m^c(\delta)) \to 0 \), or equivalently, for any \( \delta > 0 \),

\[
P(C_n(\delta) \ i.o. = 1. \tag{27}
\]

Let \( B_n \) be the event \( "S_{N_{n-1}+N_{n-2}+2} > -2g(N_{n-1} + N_{n-2} + N/2)" \). It then follows from (28) that

\[
P(B_n \ i.o. = 1,
\]

which, together with (22), implies that for any \( \hat{\delta} > 0 \)

\[
P(B_n \cap C_n(\hat{\delta}) \ i.o. = 1. \tag{28}
\]

One then checks that for \( \delta > 0 \), there exists \( 0 < \hat{\delta} < \delta \) such that for sufficiently large \( n \),

\[
"S_{N_n} > (1-\delta)g(N^n) \ i.o." \supset \("S_{N_n} > (1-\hat{\delta})g(N^n - N_{n-1} - N_{n-2}) - 2g(N_{n-1} + N_{n-2}) \ i.o." \supset "B_n \cap C_n(\hat{\delta}) \ i.o.".
\]

It then follows from (28) that

\[
P(S_{N_n} > (1-\delta)g(N^n) \ i.o. = 1,
\]

which immediately implies (23).

\section{Alternatives for Condition (II)}

This section only assume Condition (I) and gives alternatives for Condition (II) provided Condition (I) is satisfied.

Let \( (\Omega, \mathcal{F}, P) \) be the probability space which \( Z \) is defined on, and let \( H_0 = H(Z_{k}, k \in \mathbb{Z}) \) be the subspace of \( L^2(\mathcal{F}) \) spanned by the equivalence classes of the random variables \( Z_k, k \in \mathbb{Z} \), with inner product defined as

\[
<V, W> = E[VW],
\]

for any \( V, W \in H_0 \).

\textbf{Theorem 5.1.} If \( \liminf_{n \to \infty} E[S_n^2] < \infty \), then there exist a sequence of random variables \( (V_i, i \in \mathbb{N}) \) as \( n \to \infty \) such that \( X_i = V_i - V_{i+1} \) with \( E[V_i^2] = O(1) \) uniformly for all \( i \), and thus \( \sup_n E[S_n^2] < \infty \).

\textbf{Proof.} Let \( Q \) be an infinite subset of \( \mathbb{N} \) such that \( \sup_{n \in Q} E[S_n^2] < \infty \). Applying Lemma 23, we deduce that there exists \( C > 0 \) such that for all \( i \in \mathbb{N} \),

\[
\sup_{n \in Q} E[(S_{n+i-1} - S_i)^2] \leq C,
\]

where \( S_0 \) is interpreted as 0. It follows from Banach-Alaoglu theorem (which implies that every bounded and closed set in a Hilbert space is weakly compact; see Section 3.15 of [13])

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that for any \( i \in \mathbb{N} \), there exists \( V_i \in H_0 \) with \( E[V_i^2] \leq C \), and \( Q_i \), an infinite subset of \( Q \) such that for all \( W \in H_0 \),
\[
\lim_{n \to \infty, n \in Q_i} <W, S_{n+i-1} - S_{i-1}>=<W, V_i>;
\]
here, without loss of generality, we can assume that \( Q_{i+1} \subseteq Q_i \) for all \( i \). Then one verifies that for any \( W \in H_0 \), we have that for any \( i \),
\[
<W, X_i - V_i + V_{i+1}> = \lim_{n \to \infty, n \in Q_i+1} <W, X_i-(S_{n+i-1} - S_{i-1})+(S_{n+i} - S_i)> = \lim_{n \to \infty, n \in Q} <W, X_{n+i}> = 0,
\]
where we have applied Lemma 2.3 for the last equality. Choosing \( W = X_i - V_i + V_{i+1} \), we then obtain that
\[
\|X_i - V_i + V_{i+1}\|_2 = 0,
\]
which implies that \( X_i = V_i - V_{i+1}, \ a.s. \). It then follows that
\[
E[S_n^2] = E[(V_1 - V_{n+1})^2] = E[V_1^2] + E[V_{n+1}^2] - 2E[V_1V_{n+1}],
\]
which, together with \( E[V_i^2] \leq C \), implies the theorem.

A sequence of positive numbers, \((h(i), i \in \mathbb{N})\), is said to be “slowly varying” if for every positive integer \( m \),
\[
\lim_{n \to \infty} h(mn)/h(n) = 1,
\]
and it is said to be “slowly varying in the strong sense” if
\[
\lim_{m \to \infty} \frac{\min_{m \leq n \leq 2m} h(n)}{\max_{m \leq n \leq 2m} h(n)} = 1.
\]

**Lemma 5.2.** If \( \lim_{n \to \infty} E[S_n^2] = \infty \), then \( E[S_n^2] = nh(n) \), where \((h(i), i \in \mathbb{N})\) is a sequence of slowly varying positive numbers.

**Proof.** We only need to show that for every positive integer \( l \),
\[
\lim_{n \to \infty} \frac{\sigma_n^2}{\sigma_i^2} = l.
\]
Following the proof of Theorem 2.1.2 in [30], let
\[
\zeta_j = \sum_{s=1}^{n} X_{(j-1)n+(j-1)r+s}, \quad j = 1, 2, \cdots, l,
\]
\[
\eta_j = \sum_{s=1}^{r} X_{jn+(j-1)r+s}, \quad j = 1, 2, \cdots, l-1,
\]

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\[ \eta_l = - \sum_{s=1}^{(l-1)r} X_{nl+s}, \]

where \( r = \lfloor \log \sigma_n^2 \rfloor \).

Now

\[ \sigma_{ln}^2 = E[S_{ln}^2] = \sum_{j=1}^l E[\zeta_j^2] + 2 \sum_{i \neq j} E[\zeta_i \zeta_j] + \sum_{i,j} E[\zeta_i \eta_j] + \sum_{i,j} E[\eta_i \eta_j]. \]

It follows from Lemma 2.1 that for any \( j \),

\[ E[\zeta_j^2] = E[\zeta_1^2] + O(1)E[\zeta_1^2]^{1/2} = \sigma_n^2 + O(\sigma_n) \tag{29} \]

uniformly in \( j \). Using an argument similar to the proof for Part 2 of Lemma 2.4, one has that there exists \( 0 < \theta < 1 \) such that for \( i \neq j \),

\[ |E[\zeta_i \zeta_j]| = O(\theta^{\lfloor \log \sigma_n^2 \rfloor}) E[\zeta_1^2], \]

where we also used (29). Using Schwartz inequality and (29), we also have

\[ |E[\zeta_i \eta_j]| \leq E[\zeta_i^2]^{1/2} E[\eta_j^2]^{1/2} = O(\sigma_n \sigma_r) = O(\sigma_n \log \sigma_n), \]

and

\[ |E[\eta_i \eta_j]| \leq O(\sigma_r^2) = O((\log \sigma_n)^2). \]

It then follows that for any positive integer \( l \),

\[ \sigma_{ln}^2 = l \sigma_n^2 + o(\sigma_n^2), \]

which immediately implies the lemma. \( \square \)

**Lemma 5.3.** If \( \lim_{n \to \infty} E[S_n^2] = \infty \), then \( E[S_n^2] = nh(n) \), where \((h(i), i \in \mathbb{N})\) is a sequence of slowly varying positive numbers in the strong sense.

**Proof.** Note that by Lemma 2.1, we have that for any \( j \),

\[ \lim_{n \to \infty} E[(S_{nj} - S_j)^2] = 1 \]

uniformly in \( j \). The lemma then follows from (29), Lemma 5.2 and an almost the same proof for Theorem 8.13 of [11]. \( \square \)

The following lemma is well-known; see, e.g., Proposition 0.16 in [11].

**Lemma 5.4.** Suppose \((h(n), n \in \mathbb{N})\) is a sequence of positive numbers which is slowly varying in the strong sense. Then for every \( \varepsilon > 0 \), one has that \( n^\varepsilon h(n) \to \infty \) as \( n \to \infty \).

**Lemma 5.5.** If \( \lim_{n \to \infty} E[S_n^2] = \infty \), then \( \sigma > 0 \).

**Proof.** Assume, for contraditions, that \( \sigma = 0 \). Since \( \lim_{n \to \infty} E[S_n^2] = \infty \), we deduce, by Lemma 5.4, that \( E[S_n^2]/n \) is slowly varying in the strong sense. Then, by Lemma 5.4, for any \( \alpha > 0 \), \( n^\alpha E[S_n^2]/n \to \infty \) as \( n \to \infty \). However, by Lemma 2.4, when \( \sigma = 0 \), \( n^\alpha E[S_n^2]/n \to 0 \) as \( n \to \infty \) for any \( 0 < \alpha < 1 \), which is a contradiction. \( \square \)
The following theorem immediately follows from Theorem 5.1 and Lemma 5.5, which gives alternatives for Condition (II) given Condition (I) is satisfied.

**Theorem 5.6.** Under Condition (I), the following statements are equivalent

1. $\sigma > 0$.
2. $\lim_{n \to \infty} E[S_n^2] = \infty$.
3. $\limsup_{n \to \infty} E[S_n^2] = \infty$. 
References


