On Optimal Anticodes over Permutations with the Infinity Norm

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Abstract—Motivated by the set-antiset method for codes over permutations under the infinity norm, we study anticodes under this metric. For half of the parameter range we classify all the optimal anticodes, which is equivalent to finding the maximum permanent of certain $(0,1)$-matrices. For the rest of the cases we show constraints on the structure of optimal anticodes.

I. INTRODUCTION

Codes over permutations have attracted recent interest due storage schemes for flash memories [1], [4], [6], [8]. In particular, correcting limited-magnitude in the rank modulation scheme was discussed in [8], where the metric of choice is the $\ell_\infty$-metric induced by the distance measure $d(f,g) = \max_{1 \leq i \leq n} |f(i) - g(i)|$ defined between any two permutations $f, g \in S_n$.

In analogy to the definition of a code, a subset $A \subseteq S_n$ is an anticode with maximal distance $d$, if any two of its members are at distance at most $d$ apart. The following theorem, which is sometimes referred to as the set-antiset theorem, motivates us to explore anticodes of maximum size. This theorem was also used before over different spaces and distance measures (see [2], [3], [8]).

Theorem 1. [8, Theorem 13] Let $C, A \subseteq S_n$ be a code and an anticode under the $\ell_\infty$-metric, with minimal distance $d$ and maximal distance $d - 1$, respectively. Then $|C| \cdot |A| \leq |S_n|$.

It should be noted that balls are just a special case of anticodes, since a ball of radius $r$ is an anticode with maximal distance $2r$. The size of balls in $S_n$ under the $\ell_\infty$-metric has been studied in [5], [7].

Let $\Gamma_n^d$ denote the set of $(0,1)$-matrices of order $n$ with exactly $d$ non-zero entries in each row which form a contiguous block. Since for any optimal anticode $A \subseteq S_n$ we can assume the identity permutation is in $A$ (since the $\ell_\infty$-metric is right-invariant), and since for any $n \times n$ matrix $B = (b_{ij})$, $\per(B) = \sum_{f \in S_n} \prod_{i=1}^n b_{i,f(i)}$, it is easily seen that the size of an optimal anticode of maximal distance $d - 1$ in $S_n$ is given by the maximum permanent of a matrix in $\Gamma_n^d$. We therefore define, $M_n^d = \{A \in \Gamma_n^d : \per(A) \geq \per(B) \text{ for all } B \in \Gamma_n^d\}$.

II. RESULTS

Our results fall into two cases. First we consider the case of $2d \geq n$, i.e., $n = d + r$, where $0 \leq r \leq d$. Let $A \in \Gamma_{d+r}^d$, $A = (a_{ij})$, and for any row index $i$ we define $x_i = \min \{j : a_{ij} = 1\}$. It can be seen that $A$ is defined uniquely by the vector $(x_1, x_2, \ldots, x_{d+r})$, so by abuse of notation we will write $A = (x_1, x_2, \ldots, x_{d+r})$.

Theorem 2. Let $A \in M_{d+r}^d$, $0 \leq r \leq d$, then the only possible configurations of $A = (x_1, x_2, \ldots, x_{d+r})$, up to a permutation of the rows and columns, are

$$x_i = \begin{cases} 1 & 1 \leq i \leq \left[\frac{d+r}{2}\right] \\ r+1 & \left[\frac{d+r}{2}\right] < i \leq d+r. \end{cases}$$

Note that for $n = d + r$ odd, the value of $x_\left[\frac{d+r}{2}\right]$ is unconstrained. In addition,

$$\per(A) = \left(2d - n\right) \left(\left[\frac{n}{2}\right]\right)! \left(\left[\frac{n}{2}\right]\right)!.$$ 

For the case of $2d < n$ we show periodicity results.

Theorem 3. Every $A \in M_n^d$ is of the form $A \cong a_1 x_{d+d} \oplus C_1 \oplus C_2 \cdots \oplus C_h$ where $a \geq 0$ and $0 \leq h \leq d - 1$. Moreover $C_i \in M_{\ord(C_i)}^d$ and if in addition $C_i$ does not contain $1_{x_{d+d}}$ as a sub-matrix, then $\ord(C_i) \leq b_i^d$, where $b_i^d$ is a constant depending on $d$ only.

Theorem 4. For each positive integer $d$ there exists $\mu_d$ such that $M_n^d$ is periodic for $n \geq \mu_d$ in the sense that $A \in M_n^d$ if and only if $A \oplus 1_{x_{d+d}} \in M_{n+d}^d$.

REFERENCES