Spectrum White Space Trade in Cognitive Radio Networks

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***Abstract*—We study price competition among primaries in a Cognitive Radio Network (CRN) with multiple primaries and secondaries located in a large region. In every slot, each primary has unused bandwidth with some probability, which may be different for different primaries. Also, there may be a random number of secondaries. A primary can lease out its unused bandwidth to a secondary in exchange for a fee. Each primary tries to attract secondaries by setting a lower price for its bandwidth than the other primaries. Radio spectrum has the distinctive feature that transmissions at neighboring locations on the same channel interfere with each other, whereas the same channel can be used at far-off locations without mutual interference. So in the above price competition scenario, each primary must jointly select a set of mutually non-interfering locations within the region (which corresponds to an independent set in the conflict graph representing the region) at which to offer bandwidth and the price at each location. In this paper, we analyze this price competition scenario as a game and seek a Nash Equilibrium (NE). For the game at a single location, we explicitly compute a NE and prove its uniqueness. Also, for the game at multiple locations, we identify a class of conflict graphs, which we refer to as mean valid graphs, such that the conflict graphs of a large number of topologies that commonly arise in practice are mean valid. We explicitly compute a NE in mean valid graphs and show that it is unique in the class of NE with symmetric independent set selection strategies of the primaries.**

I. INTRODUCTION

The emerging cognitive radio technology [1] promises efficient usage of the available radio spectrum. In cognitive radio networks (CRNs), there are two types of spectrum users: (i) *primary* users who lease portions (channels or bands) of the spectrum directly from the regulator, and (ii) *secondary* users who lease channels from primaries and can use a channel when it is not in use by the primary. Time is slotted, and in every slot, each primary has unused bandwidth with some probability, which it would like to sell to secondaries. Now, secondaries buy bandwidth from the primaries that offer it at a low price, which results in *price competition* among the primaries. If a primary quotes a low price, it will attract buyers, but will earn lower profit per sale. This is a common feature of an *oligopoly* [4], in which multiple firms sell a common good to a pool of buyers. Price competition in an oligopoly is naturally modeled using *game theory* [2], and has been extensively studied in economics using, for example, the classic *Bertrand game* [4] and its variants.

However, a CRN has several distinguishing features, which makes the price competition very different from oligopolies encountered in economics. First, in every slot, each primary may or may not have unused bandwidth available. Second, the number of secondaries will be random and not known apriori as each secondary may be a local spectrum provider or even a user shopping for spectrum in a futuristic scenario, *e.g.*, users at airports, hotspots, etc. Thus, each primary who *has* unused bandwidth is uncertain about the number of primaries from whom it will face competition as well as the demand for bandwidth; it may only have access to imperfect information such as statistical distributions about either. A low price will result in unnecessarily low revenues in the event that very few other primaries have unused bandwidth or several secondaries are shopping for bandwidth, because even with a higher price the primary's bandwidth would have been bought, and vice versa. Third, spectrum is a commodity that allows *spatial reuse*: the same band can be simultaneously used at far-off locations without interference; on the other hand, simultaneous transmissions at neighboring locations on the same band interfere with each other. Thus, spatial reuse provides an *opportunity* to primaries to increase their profit by selling the same band to secondaries at different locations, which they can utilize subject to satisfying the interference *constraints*. So when multiple primaries own bandwidth in a large region, each needs to decide on a set of non-interfering locations in the region, which corresponds to an *independent set* in the *conflict graph* representing the region, at which to offer bandwidth. This is another source of strategic interaction among the primaries– each primary would like to select a maximum-sized independent set to offer bandwidth at; but if a lot of primaries offer bandwidth at the same locations, there is intense competition at those locations. So a primary would have benefited by instead offering bandwidth at a smaller independent set and charging high prices at those locations.

Pricing related issues have been extensively studied in the context of wired networks and the Internet; see [9] for an overview. Price competition among spectrum providers in wireless networks has been studied in [10], [11], [12], [13], [15], [16], [17]. Specifically, price competition among multiple primaries in CRNs is analyzed in [15], [16], [17]. However, neither uncertain bandwidth availability, nor spatial reuse is modeled in any of the above papers. Also, most of these papers do not explicitly find a Nash Equilibrium (NE) (exceptions are [11], [15], [17]). Our model incorporates both uncertain bandwidth availability and spatial reuse, which makes the problem challenging; despite this, we are able to explicitly compute a NE. In [18], auction mechanisms are proposed for a CRN, using which a primary can choose an allocation of its channel to multiple secondaries based on their bids and taking into account spatial reuse of spectrum. However, the mechanisms in [18] are for

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the case of a single primary, unlike in our model, which applies to multiple primaries. Zhou *et. al.* [19] have designed double auction based spectrum trades in which an auctioneer chooses an allocation taking into account spatial reuse and bids. However, in the price competition model we consider, each primary independently sells bandwidth, and hence a central entity such as an auctioneer is not required. In the economics literature, the Bertrand game [4] and several of its variants [5], [6], [7], [8], [20] have been used to study price competition. Osborne et al [5] consider price competition in a duopoly, when the capacity of each firm is constrained. Chawla *et al.* [20] consider price competition in networks where each seller owns a capacity-constrained link, and decides the price for using it; the consumers choose paths they would use in the networks based on the prices declared and pay the sellers accordingly. The capacities in both cases are deterministic, whereas the availability of bandwidth is random in our model.

The closest to our work are [7], [8], which analyze price competition where each seller may be inactive with some probability, as also our prior work [22], [26], [27] in which we analyzed price competition in a CRN. The above body of work however suffers from the limitation that they either consider (i) only the *symmetric* model where the bandwidth availability probability of each seller is the same [7], [8], [22], [27] or (ii) primaries and secondaries located at a single location [7], [8], [26] (i.e., no spatial reuse) [1]. In addition, the results in [7], [8] are restricted to the case of one buyer, and [26] assumes a fixed, and apriori known number of secondaries, whereas a CRN is likely to have an unknown and random number of secondaries, which we consider in this paper. Characterizing the Nash Equilibrium (NE) in either *asymmetric* games (i.e., when different primaries have different bandwidth availability probabilities in our context) or in games over graphs (i.e., in presence of spatial reuse in our context) is usually quite challenging, and the combination of the above often turns out to be analytically untractable. This is the space where we seek to contribute in this paper.

We consider price competition in a CRN with multiple primaries and multiple secondaries, where each primary has available bandwidth in a slot with a certain probability, which may be different for different primaries. Also, the number of secondaries may be random and unknown to the primaries, with only their distribution being known. First, we analyze the case of primaries and secondaries in a single location (Section III). Since prices can take real values, the *strategy sets of players are continuous*. In addition, the utilities of the primaries are not continuous functions of their actions. Thus, classical results, including those for concave and potential games, do not establish the existence and uniqueness of NE in the resulting game, and there is no standard algorithm for finding a NE. Nevertheless, we are able to explicitly compute a NE and show that it is unique in the class of *all* NE, even allowing for player strategies that are arbitrary mixtures of continuous and discrete probability distributions (Section III).

We subsequently model the scenario where each primary owns bandwidth across multiple locations using a conflict graph in which there is an edge between each pair of mutually interfering locations (Section II-A). Each primary must simultaneously select a set of mutually non-interfering locations (independent set) at which to offer bandwidth and the prices at those locations. We focus on a class of conflict graphs that we refer to as *mean valid graphs*. As we show in Section IV-B, it turns out that the conflict graphs of a large number of topologies that arise in practice are mean valid. We show that a mean valid graph has a *unique* NE in the class of NE with symmetric independent set selection strategies of the primaries (Section IV-C). Also, this NE has a simple form and the NE strategies can be explicitly computed by solving a system of equations that we provide. Finally, we prove that in the limit as the numbers of primaries and secondaries go to infinity, the NE structure exhibits interesting threshold behavior: in particular, the efficiency of this NE, which is the ratio of the aggregate revenue of all the primaries under the NE and the maximum possible aggregate revenue, changes from 1 to 0 as the average bandwidth availability increases relative to the average bandwidth demand at each location.

Due to space constraints, we relegate all proofs to our technical report [23].

II. Model and Objective

A. Model

Suppose there are $n \geq 2$ primaries, each of whom owns a channel throughout a large region which is a geographically well-separated or separately administered area, such as a state or a country [2]. The channels owned by the primaries are all orthogonal to each other. Time is divided into slots of equal duration. In every slot, each primary independently either uses its channel throughout the region to satisfy its own subscriber demand, or does not use it anywhere in the region. A typical scenario where this happens is when primaries broadcast the same signal over the entire region, *e.g.*, if they are television broadcasters. For $i \in \{1, \ldots, n\}$, let $q_i \in (0, 1)$ be the probability that primary i does *not* use its channel in a slot (to satisfy its subscriber demand). Without loss of generality, we assume that:

$$q_1 \geq q_2 \geq \ldots \geq q_n. \tag{1}$$

Now, the region contains smaller parts, which we refer to as *locations*. For example, the large region may be a state, and the locations may be towns within it.

Each secondary may be a local spectrum provider or even a user seeking to lease spectrum bands to transmit data on an on-demand basis at a location. In practice, the number of secondaries seeking to buy bandwidth may be random and unequal at different locations and also apriori unknown to the primaries, due to user mobility, varying bandwidth requirements of the secondaries, etc. Thus, the number of secondaries seeking to buy bandwidth (henceforth referred to

[1] In [22], the asymmetric case is considered only for a toy model with two primaries and one secondary; [26] largely focuses on a single location game, except for a limited analysis of spatial reuse in the setting of a linear conflict graph.

[2] We assume that all the primaries own bandwidth in the same region.

as the number of secondaries for simplicity) at a location v is K_v, where K_v is a random variable with probability mass function (p.m.f.) $Pr(K_v = k) = \gamma_k$. Also, the random variables K_v at different nodes v may be correlated. The primaries apriori know only the γ_ks, but not the values of K_v for any given location v. We will make some technical assumptions on the p.m.f. $\{\gamma_k\}$: (i) $\sum_{k=0}^{n-1} \gamma_k > 0$ (i.e., the total number of primaries exceeds the number of secondaries with positive probability, but not necessarily probability 1) (ii) if $\gamma_0 > 0$, then $\gamma_1 > 0$ (if the event that no secondary requires bandwidth has positive probability, then the event that only 1 secondary requires bandwidth also has positive probability). A large class of probability mass functions, including those generated from the most common scenario, where each local provider or user from a given pool requires bandwidth with a positive probability independent of others, satisfy both the above assumptions.

A primary who has unused bandwidth in a slot can lease it out to secondaries at a subset of the locations, provided this subset satisfies the *spatial reuse constraints*, which we describe next. The overall region can be represented by an undirected graph [3] $G = (V, E)$, where V is the set of nodes and E is the set of edges, called the *conflict graph*, in which each node represents a location, and there is an edge between two nodes iff transmissions at the corresponding locations interfere with each other. Note that graphs have been widely used to model ad hoc networks, wherein wireless devices are modeled as nodes in an undirected graph, with mutually interfering nodes being connected by an edge (*e.g.*, see [24]). However, the concept of spatial reuse in our paper is more closely related to the corresponding notion in cellular networks, where cells are represented by nodes in an undirected graph, with interfering cells corresponding to neighbors in the graph [25]. Recall that an *independent set* [3] (I.S.) in a graph is a set of nodes such that there is no edge between any pair of nodes in the set. Now, a primary who is not using its channel must offer it at a set of mutually non-interfering locations, or equivalently, at an I.S. of nodes; otherwise secondaries[3] will not be able to successfully transmit simultaneously using the bandwidth they purchase, owing to mutual interference.

A primary i who offers bandwidth at an I.S. I, must also determine for each node $v \in I$, the access fee, $p_{i,v}$, to be charged to a secondary if the latter leases the bandwidth at node v. A primary incurs a cost of $c \geq 0$ per slot per node for leasing out bandwidth. This cost may arise, for example, if the secondary uses its infrastructure to access the Internet. We assume that $p_{i,v} \leq \nu$ for each primary i and each node v, for some constant $\nu > c$. This upper bound ν may arise as follows. (1) The spectrum regulator may impose this upper bound to ensure that primaries do not excessively overprice bandwidth even when competition is limited owing to bandwidth scarcity or high demands from secondaries, or when the primaries collude. (2) Alternatively, the *valuation* of each secondary for 1 unit of bandwidth may be ν, and no secondary will buy bandwidth at a price that exceeds its valuation. We assume that the primaries know this upper limit ν.

Secondaries buy bandwidth from the primaries that offer the lowest price. More precisely, in a given slot, let Z_v be the number of primaries who offer unused bandwidth at node v. Then, since there are K_v secondaries at the node, the bandwidth of the $\min(Z_v, K_v)$ primaries that offer the lowest prices is bought (ties are resolved at random) at the node.

If primary i has unused bandwidth, then the *utility* or *payoff* of primary i is defined to be its net revenue [4]. Also, we consider an additive utility function, which is natural in the context of monetary profits. So the utility of a primary i who offers bandwidth at an I.S. I and sets a price of $p_{i,v}$ at node $v \in I$ is given by $\sum(p_{i,v} - c)$, where the summation is over the nodes $v \in I$ at which primary i's bandwidth is bought. (The utility is 0 if bandwidth is not bought at any node).

Thus, each primary must jointly select an I.S. at which to offer bandwidth, and the prices to set at the nodes in it. Both the I.S. and price selection may be random. *Thus, a strategy, say ψ_i, of a primary i provides a probability mass function (p.m.f.) for selection among the I.S., and the price distribution it uses at each node (both selections contingent on having unused bandwidth).* Note that we allow a primary to use different (and arbitrary) price distributions for different nodes (and therefore allow, but do not require, the selection of different prices at different nodes), and arbitrary p.m.f. (i.e., discrete distributions) for selection among the different I.S. The vector $(\psi_1, \ldots, \psi_n)$ of strategies of the primaries is called a *strategy profile* [4]. Let $\psi_{-i} = (\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots, \psi_n)$ denote the vector of strategies of primaries other than i. Let $E\{u_i(\psi_i, \psi_{-i})\}$ denote the expected utility of primary i when it adopts strategy ψ_i and the other primaries adopt ψ_{-i}.

B. Nash Equilibrium

We use the Nash Equilibrium solution concept, which has been extensively used in game theory in general and wireless network applications in particular to predict the outcome of a game.

Definition 1 (Nash Equilibrium (NE)): A *Nash equilibrium* (NE) is a strategy profile such that no player can improve its expected utility by unilaterally deviating from its strategy [4]. Thus, $(\psi_1^*, \ldots, \psi_n^*)$ is a NE if for each primary i:

$$E\{u_i(\psi_i^*, \psi_{-i}^*)\} \geq E\{u_i(\widetilde{\psi}_i, \psi_{-i}^*)\}, \ \forall \ \widetilde{\psi}_i \qquad (2)$$

Equation (2) says that when players other than i play ψ_{-i}^*, ψ_i^* maximizes i's expected utility; ψ_i^* is said to be its *best response* [4] to ψ_{-i}^*.

Note that the existence of a NE is not apriori clear even in the simplest possible setting of a single location, far less the uniqueness and characterization of NE strategy profiles. This is because the prices can take real values and hence the strategy sets of players are not finite. In addition, the utilities of the primaries are not continuous functions of their actions. For example, consider the game in which there is a single location

[3]Note that secondaries usually purchase bandwidth for communication (and not television broadcasts). Thus, two secondaries can not use the same band simultaneously at interfering locations.

[4]If instead, the utility were defined to be primary i's net revenue, unconditional on whether it has unused bandwidth or not, then the expected utilities of primary i in the game analysis would all be scaled by q_i.

v, $n = 2$ primaries and $K_v = 1$ secondary with probability 1. If primary 1 has unused bandwidth, its expected utility is

$$\begin{cases} p_{1,v} - c & \text{if } p_{1,v} < p_{2,v} \\ (p_{1,v} - c)/2 & \text{if } p_{1,v} = p_{2,v} \\ (1 - q_2)(p_{1,v} - c) & \text{if } p_{1,v} > p_{2,v}. \end{cases}$$

which is a discontinuous function of the prices. Thus, classical results, including those for concave and potential games, do not establish the existence of NE in the resulting game, and there is no standard algorithm for finding a NE.

III. Single Location

In this section, we analyze price competition when all the primaries and secondaries are present in a single location. Let the (random) number of secondaries at this location be denoted as K. Since there is only one location, there are no spatial reuse constraints, and the strategy of a primary i is a distribution function (d.f.) [5] $\psi_i(.)$, which it uses to select the price p_i. For convenience, we define the pseudo-price of primary $i \in \{1, \ldots, n\}$, p'_i, as the price it selects if it has unused bandwidth and $p'_i = \nu + 1$ otherwise [6]. Also, let $\phi_i(.)$ be the d.f. of p'_i. For $c \leq x \leq \nu$, $p'_i \leq x$ for a primary i iff it has unused bandwidth and sets a price $p_i \leq x$. So $\phi_i(x) = q_i P(p_i \leq x) = q_i \psi_i(x)$. Thus, $\psi_i(.)$ and $\phi_i(.)$ differ only by a constant factor on $[c, \nu]$ and we use them interchangeably wherever applicable.

A. Necessary Conditions for a NE

Consider a NE under which the d.f. of the price (respectively, pseudo-price) of primary i is $\psi_i(.)$ (respectively, $\phi_i(.)$). In Theorem 1 below, we show that the NE strategies must have a particular structure. Before stating Theorem 1, we describe some basic properties of the NE strategies.

Property 1: $\phi_2(.), \ldots, \phi_n(.)$ are continuous on $[c, \nu]$. $\phi_1(.)$ is continuous at every $x \in [c, \nu)$, has a jump [7] of size $q_1 - q_2$ at ν if $q_1 > q_2$ and is continuous at ν if $q_1 = q_2$.

Thus, there does not exist a pure strategy NE (one in which every primary selects a single price with probability (w.p.) 1).

Now, let $u_{i,max}$ be the expected payoff that primary i gets in the NE and L_i be the lower endpoint of the support set [8] of $\psi_i(.)$, *i.e.*:

$$L_i = \inf\{x : \psi_i(x) > 0\}. \tag{3}$$

Also, let w_i be the probability of the event that at least K primaries among $\{1, \ldots, n\} \backslash i$ have unused bandwidth. Let r be the probability that $K \geq 1$. Note that $r = 1 - \gamma_0$, and w_i can be easily computed using the p.m.f $\{\gamma_k\}$ and the fact that each primary j independently has unused bandwidth w.p. q_j.

Property 2: $L_1 = \ldots L_n = \tilde{p}$, where $\tilde{p} = c + \frac{(\nu - c)(1 - w_1)}{r}$. Also, $u_{i,max} = (\tilde{p} - c)r$, $i = 1, \ldots, n$.

Thus, the lower endpoints of the support sets of the d.f.s $\psi_1(.), \ldots, \psi_n(.)$ of all the primaries are the same.

[5] Recall that the d.f. of a random variable X is the function $f(x) = P(X \leq x), x \in R$, where R denotes the set of real numbers.

[6] The choice $\nu + 1$ is arbitrary. Any other choice greater than ν also works.

[7] A d.f. $f(x)$ is said to have a *jump* (discontinuity) of size $b > 0$ at $x = a$ if $f(a) - f(a-) = b$, where $f(a-) = \lim_{x \uparrow a} f(x)$.

[8] The support set of a d.f. is the smallest closed set such that its complement has probability zero under the d.f.

Theorem 1: The following are necessary conditions for strategies $\phi_1(.), \ldots, \phi_n(.)$ to constitute a NE:
1) $\phi_1(.), \ldots, \phi_n(.)$ satisfy Property 1 and Property 2.
2) There exist numbers $R_j, j = 1, \ldots, n+1$, and a function $\{\phi(x) : x \in [\tilde{p}, \nu)\}$ such that

$$\tilde{p} = R_{n+1} < R_n \leq R_{n-1} \leq \ldots \leq R_1 \leq \nu, \tag{4}$$

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x),\ \tilde{p} \leq x < R_j,\ j \in \{1, \ldots, n\}, \tag{5}$$

$$\text{and } \ \phi_j(R_j) = q_j,\ j = 1, \ldots, n. \tag{6}$$

Also, every point in $[\tilde{p}, R_j)$ is a best response for primary j and it plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability. Finally, $R_1 = R_2 = \nu$.

Theorem 1 says that all n primaries play prices in the range $[\tilde{p}, R_n)$, the d.f. $\phi_n(.)$ of primary n stops increasing at R_n, the remaining primaries $1, \ldots, n-1$ also play prices in the range $[R_n, R_{n-1})$, the d.f. $\phi_{n-1}(.)$ of primary $n-1$ stops increasing at R_{n-1}, and so on. Also, primary 1's d.f. $\phi_1(.)$ has a jump of height $q_1 - q_2$ at ν if $q_1 > q_2$. Fig. 1 illustrates the structure.

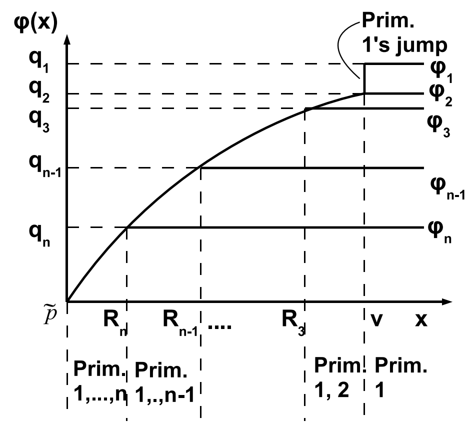

Fig. 1. The figure shows the structure of a NE described in Theorem 1. The horizontal axis shows prices in the range $x \in [\tilde{p}, \nu]$ and the vertical axis shows the functions $\phi(.)$ and $\phi_1(.), \ldots, \phi_n(.)$.

B. Explicit Computation, Uniqueness and Sufficiency

By Theorem 1, for each $i \in \{1, \ldots, n\}$:

$$\phi_i(x) = \begin{cases} \phi(x), & \tilde{p} \leq x < R_i \\ q_i, & x \geq R_i \end{cases} \tag{7}$$

So the candidate NE strategies $\phi_1(.), \ldots, \phi_n(.)$ are completely determined once $\tilde{p}, R_1, \ldots, R_n$ and the function $\phi(.)$ are specified. Also, Property 2 provides the value of $\tilde{p}$, and $R_1 = R_2 = \nu$ by Theorem 1. First, we will show that there also exist unique $R_3, \ldots, R_n$ and $\phi(.)$ satisfying (4), (5), and (6) and will compute them. Then, we will show that the resulting strategies given by (7) indeed constitute a NE (sufficiency).

Let p'_{-i} be the K'th smallest pseudo-price out of the pseudo-prices, $\{p'_l : l \in \{1, \ldots, n\}, l \neq i\}$, of the primaries other than i (with $p'_{-i} = 0$ if $K = 0$ and $p'_{-i} = \nu + 2$ if $K > n - 1$). Also, let $F_{-i}(x)$ denote the d.f. of p'_{-i}. Since there are K secondaries, if primary 1 has unused bandwidth and sets $p_1 = x \in [\tilde{p}, \nu)$, its bandwidth is bought iff [9] $p'_{-1} > x$, which happens w.p. $1 - F_{-1}(x)$. Note that primary 1's payoff is $(x - c)$ if its bandwidth is bought and 0 otherwise. So, letting $E\{u_i(x, \psi_{-i})\}$ denote the expected payoff of primary i if it sets a price x and the other primaries use the strategy profile ψ_{-i}, we have:

$$E\{u_1(x, \psi_{-1})\} = (x - c)(1 - F_{-1}(x)) = (\tilde{p} - c)r, \; x \in [\tilde{p}, \nu) \tag{8}$$

where the second equality follows from the facts that each $x \in [\tilde{p}, \nu)$ is a best response for primary 1 by Theorem 1, and $u_{1,max} = (\tilde{p} - c)r$ by Property 2. By (8), we get:

$$F_{-1}(x) = \frac{x - c - (\tilde{p} - c)r}{x - c}, \; x \in [\tilde{p}, \nu). \tag{9}$$

Next, we calculate $R_i, i = 3, \ldots, n$ and $\phi(.)$ using (9).

1) Computation of $R_i, i = 3, \ldots, n$: For $0 \le y \le 1$, let $f_i(y)$ be the probability of K or more successes out of $n - 1$ independent Bernoulli events, $(i - 1)$ of which have the same success probability y and the remaining $(n - i)$ have success probabilities $q_{i+1}, \ldots, q_n$. An expression for $f_i(y)$ can be easily computed.

Now, to compute $R_i, i \in \{3, \ldots, n\}$, we note that by (7) and (4), $\phi_j(R_i) = q_i, \; j = 2, \ldots, i$, and $\phi_j(R_i) = q_j, \; j = i+1, \ldots, n$. So from the preceding paragraph, with the events $\{p'_j \le R_i\}, j = 2, \ldots, n$ as the $n - 1$ Bernoulli events, and by the definition of $F_{-1}(.)$, we get:

$$F_{-1}(R_i) = f_i(q_i). \tag{10}$$

By (9) *and* (10)*, R_i is unique and is given by:*

$$R_i = c + \frac{(\tilde{p} - c)r}{1 - f_i(q_i)}. \tag{11}$$

2) Computation of $\phi(.)$: Now we compute the function $\{\phi(.) : x \in [\tilde{p}, \nu)\}$ by separately computing it for each interval $[R_{i+1}, R_i)$, $i \in \{2, \ldots, n\}$. If $R_{i+1} = R_i$, then note that the interval $[R_{i+1}, R_i)$ is empty. Now suppose $R_{i+1} < R_i$. For $x \in [R_{i+1}, R_i)$, by (7) and (4):

$$\phi_j(x) = q_j, \; j = i + 1, \ldots, n \tag{12}$$

$$\text{and } \phi_1(x) = \ldots = \phi_i(x) = \phi(x). \tag{13}$$

By definition of the function $f_i(.)$, with the events $\{p'_j \le x\}, j = 2, \ldots, n$ as the $n - 1$ Bernoulli events, by definition of $F_{-1}(x)$ and using $P\{p'_j \le x\} = \phi_j(x)$, (12) and (13):

$$f_i(\phi(x)) = F_{-1}(x), \; R_{i+1} \le x < R_i. \tag{14}$$

Note that $F_{-1}(x)$ is given by (9).

Lemma 1: For each x, (14) has a unique solution $\phi(x)$. The function $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, \nu)$. For $i \in \{2, \ldots, n\}$, $\phi(R_i) = q_i$. Also, $\phi(\tilde{p}) = 0$.

Thus, there is a unique function $\phi(.)$, and by (7)*, unique $\phi_i(.), i = 1, \ldots, n$ that satisfy the conditions in Theorem 1.*

[9] By Property 1, no primary has a jump at any $x \in [\tilde{p}, \nu)$. So $P(p'_{-1} = x) = 0$.

3) Sufficiency:

Theorem 2: The pseudo-price d.f.s $\phi_i(.), i = 1, \ldots, n$ in (7), with $R_1 = R_2 = \nu$, R_i, $i = 3, \ldots, n$ given by (11), and $\phi(.)$ being the solution of (14), constitute the unique NE. The corresponding price d.f.s are $\psi_i(x) = \frac{1}{q_i}\phi_i(x)$, $x \in [c, \nu]$, $i = 1, \ldots, n$.

Thus, in the price competition game at a single location, there is a unique NE that can be computed explicitly. This NE fetches equal expected payoffs for each primary, which by Property 2 is given by:

$$(\tilde{p} - c)r = (\nu - c)(1 - w_1). \tag{15}$$

C. Discussion

The structure of the unique NE identified in Theorems 1 and 2 provides several interesting insights:

1) First, from (1), (4) and the fact that the support set of $\psi_i(.)$ is $[\tilde{p}, R_i]$, it follows that only the primaries with a high bandwidth availability probability (q) play high prices (see Fig. 1). Intuitively this is because all the primaries play low prices (near $\tilde{p}$), so if a primary sets a high price, he is undercut by all the other primaries. But a primary with a high q runs a lower risk of being undercut than one with a low q because of the lower bandwidth availability probabilities of the set of primaries *other than itself.*

2) Second, by Property 1, $\psi_1(.)$ has a jump at v iff $q_1 > q_2$ and is continuous everywhere else, whereas $\psi_2(.), \ldots, \psi_n(.)$ are always continuous on $[c, v]$.

IV. Multiple Locations

We now study the existence, computation and uniqueness of NE in the presence of spatial reuse. Recall that a strategy of a primary now consists of a p.m.f. over I.S. *and* price distributions at individual nodes. Our first observation is that in general, there may be multiple NE in this case. For example, consider the simple setup with two nodes v_1 and v_2 connected by an edge, two primaries and one secondary with probability 1 at each node. It can be easily verified that both of the following strategy profiles constitute NEs: primary 1 offers bandwidth at node v_1 (respectively, v_2) if it has unused bandwidth and primary 2 at node v_2 (respectively, v_1) if it has unused bandwidth, and both primaries set the maximum possible price of ν. The results in games with multiple locations may therefore fundamentally differ from those for a single location.

Note that obtaining the structure of NE in games over graphs is usually extremely challenging. As a result, in many problems of practical importance (*e.g.*, base station deployment games [21]), such characterizations have been done only in small graphs with a few nodes. In spite of this, we will establish the existence of a NE and explicitly compute it for a fairly general class of graphs that we refer to as mean valid graphs. In addition, we will also prove its uniqueness in the class of NEs in which all primaries choose the I.S. they would offer bandwidth at with identical probability mass functions.

A. A Separation Result

We start by providing a separation framework from which the price distributions at individual nodes follow once the I.S. selection p.m.f.s are determined. Let $\mathscr{I}$ be the set of all I.S. in G. For convenience, we assume that the empty I.S. $I_\emptyset \in \mathscr{I}$ and we allow a primary to offer bandwidth at $I_\emptyset$, *i.e.* to not offer bandwidth at any node, with some probability. Consider a NE under which, if primary i has unused bandwidth, it selects I.S. $I \in \mathscr{I}$ w.p. $\beta_i(I)$, where $\sum_{I\in\mathscr{I}} \beta_i(I) = 1$. The probability, say α_v^i, with which primary i offers bandwidth at a node $v \in V$ equals the sum of the probabilities associated with all the I.S. that contain the node:

$$\alpha_v^i = \sum_{I\in\mathscr{I}:v\in I} \beta_i(I). \tag{16}$$

Now, considering that primary i has unused bandwidth w.p. q_i, it offers it at node v w.p. $q_i\alpha_v^i$. *The price selection problem at each node v is now equivalent to that for the single location case, the difference being that primary i offers unused bandwidth w.p. $q_i\alpha_v^i$, instead of q_i, at node v.* Thus:

Lemma 2: Suppose under a NE primary $i \in \{1,\ldots,n\}$ selects node v w.p. α_v^i if it has unused bandwidth. Then under that NE the price distribution of primary i at node v is the d.f. $\psi_i(.)$ in Section III, with $q_1\alpha_v^1,\ldots,q_n\alpha_v^n$ in place of $q_1,\ldots,q_n$ respectively all through.

Thus, the strategy profile of the primaries in an NE is completely specified once the I.S. selection p.m.f.s $\{\beta_i(I) : I\in\mathscr{I}, i\in\{1,\ldots,n\}\}$ (which will in turn provide the α_v^is via (16)) are obtained.

B. Mean Valid Graphs

We now introduce *mean valid graphs*, which model the conflict graphs of several topologies that commonly arise in practice. In the next section, we show that these graphs have a NE, which can be explicitly computed and has a simple form; this NE will also turn out to be unique in a large class of strategy profiles.

1) Definition:

Definition 2 (Valid Distribution): An assignment $\{\alpha_v : v \in V\}$ of probabilities to the nodes is said to be a *valid* distribution if there exists a probability distribution $\{\beta(I) : I \in \mathscr{I}\}$ such that for each $v \in V$, $\alpha_v = \sum_{I\in\mathscr{I}:v\in I} \beta(I)$.

Definition 3 (Mean Valid Graph): We refer to a graph $G = (V,E)$ as *mean valid* if:

1) Its vertex set can be partitioned into d disjoint maximal [10] I.S. for some integer $d \geq 2$: $V = I_1 \cup I_2 \cup \ldots \cup I_d$, where I_j, $j \in \{1,\ldots,d\}$, is a maximal I.S. and $I_j \cap I_m = \emptyset$, $j \neq m$.
   Let $|I_j| = M_j$, $I_j = \{a_{j,l} : l = 1,\ldots,M_j\}$ and:
$$M_1 \geq M_2 \geq \ldots \geq M_d. \tag{17}$$
2) For every valid distribution [11] in which a primary who has unused bandwidth offers it at node $a_{j,l}$ w.p. $\alpha_{j,l}$, $j = 1,\ldots,d$, $l = 1,\ldots,M_j$,
$$\sum_{j=1}^{d} \overline{\alpha}_j \leq 1, \text{ where } \overline{\alpha}_j = \frac{\sum_{l=1}^{M_j}\alpha_{j,l}}{M_j}, j \in \{1,\ldots,d\}. \tag{18}$$

The first condition in Definition 3 says that G is a d-partite graph [12] and has the additional property that each of $I_1,\ldots,I_d$ is a *maximal* I.S.. Next, let $\{\alpha_{j,l} : j = 1,\ldots,d; l = 1,\ldots,M_j\}$ be an arbitrary valid distribution. Consider the distribution $\alpha'_{j,l} = \overline{\alpha}_j$, with $\overline{\alpha}_j$ as in (18), *i.e.* for each j and $l = 1,\ldots,M_j$, $\alpha'_{j,l}$ is set equal to the *mean* of $\alpha_{j,m}, m = 1,\ldots,M_j$. If (18) is true, then this distribution of means is a valid distribution because it corresponds to the I.S. distribution $\{\beta(I_j) = \overline{\alpha}_j, j = 1,\ldots,d; \beta(I_\emptyset) = 1 - \sum_{j=1}^{d}\overline{\alpha}_j; \beta(I) = 0, I \neq I_1,\ldots,I_d,I_\emptyset\}$. Thus, Condition 2 in Definition 3 says that in G, the distribution of means corresponding to every valid distribution is valid– a fact that we extensively use in the proofs of the characterization of a NE in Section IV-C.

2) Examples:

Technical as Definition 3 may seem, it turns out that several conflict graphs that commonly arise in practice are mean valid. For example, consider the following graphs:

1) Let $\mathcal{G}_m$ denote a graph that is a linear arrangement of $m \geq 2$ nodes as shown in part (a) of Fig. 2, with an edge between each pair of adjacent nodes. As an example, this would be the conflict graph for locations along a highway or a row of roadside shops.
2) We consider two types of $m\times m$ grid graphs, denoted by $\mathcal{G}_{m,m}$ (see part (b) of Fig. 2) and $\mathcal{H}_{m,m}$ (see part (a) of Fig. 3). In both these graphs, m^2 nodes (locations) are arranged in a square grid. In $\mathcal{G}_{m,m}$, there is an edge only between each pair of adjacent nodes in the same row or column. In $\mathcal{H}_{m,m}$, in addition to these edges, there are also edges between nodes that are neighbors along a diagonal as shown in part (a) of Fig. 3. For example, $\mathcal{G}_{m,m}$ or $\mathcal{H}_{m,m}$ may represent a shopping complex, with the nodes corresponding to the locations of shops with WiFi Access Points (AP) for Internet access. Depending on the proximity of the shops to each other and the transmission ranges of the APs, the conflict graph could be $\mathcal{G}_{m,m}$ or $\mathcal{H}_{m,m}$. $\mathcal{H}_{m,m}$ is also the conflict graph of a cellular network with square cells as shown in part (b) of Fig. 3.
3) Let $\mathcal{T}_{m,m,m}$ be a three-dimensional grid graph (see Fig. 4), which may, for example, be the conflict graph for offices in a corporate building or rooms in a hotel.
4) The conflict graph (Fig. 6) of a cellular network with hexagonal cells (Fig. 5).
5) Consider a clique [13] of size e, where $e \geq 1$ is any integer. This is the conflict graph for any set of e locations that are close to each other.

All of the above are mean valid graphs:

[10]Recall that an I.S. I is said to be maximal if for each node $v \notin I$, $I \cup \{v\}$ is not an I.S. [3].

[11]Note that we write $\alpha_{j,l}$ in place of $\alpha_{a_{j,l}}$ to simplify the notation.

[12]Recall that a graph $G = (V,E)$ is said to be d-partite if V can be partitioned into d disjoint I.S. $I_1,\ldots,I_d$ [3]. For example, when $d = 2$, G is a bipartite graph.

[13]Recall that a *clique* or a complete graph of size e is a graph with e nodes and an edge between every pair of nodes [3].

Theorem 3: The following graphs are mean valid, with d, the number of disjoint maximal I.S., indicated in each case:

1) a clique of size $e \geq 1$ ($d = e$),
2) a line graph $\mathcal{G}_m$ ($d = 2$),
3) a two-dimensional grid graph $\mathcal{G}_{m,m}$ ($d = 2$),
4) a two-dimensional grid graph $\mathcal{H}_{m,m}$ ($d = 4$),
5) a three-dimensional grid graph $\mathcal{T}_{m,m,m}$ ($d = 8$).
6) a cellular network with hexagonal cells ($d = 3$)[14].

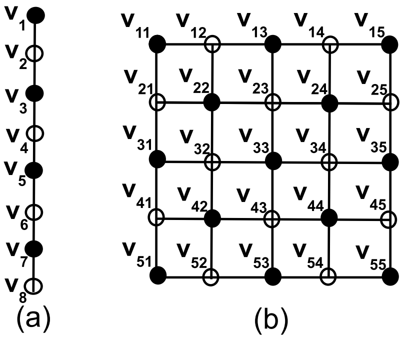

Fig. 2. Part (a) shows a linear graph, $\mathcal{G}_m$, with $m = 8$ and part (b) shows a grid graph, $\mathcal{G}_{m,m}$, with $m = 5$. Both graphs are mean valid with $d = 2$ and I_1 and I_2 being disjoint maximal I.S. (in the notation of Definition 3), where the darkened and un-darkened nodes constitute I_1 and I_2 respectively.

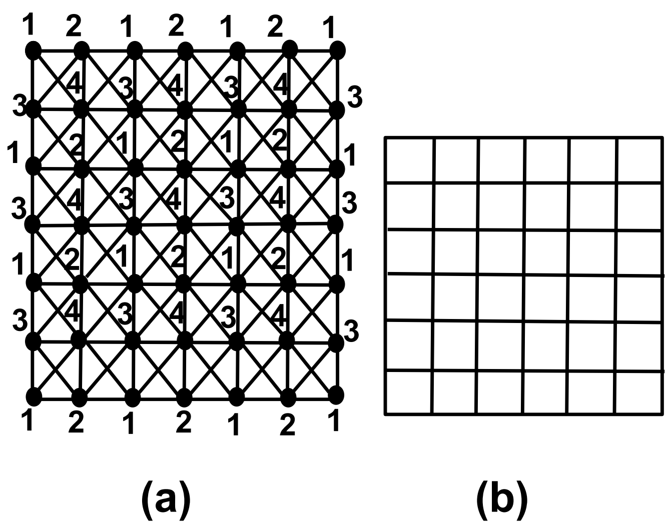

Fig. 3. Part (a) shows a grid graph $\mathcal{H}_{m,m}$ with $m = 7$. It is mean valid with $d = 4$ and the disjoint maximal I.S. $I_1, \ldots, I_4$ (in the notation of Definition 3), where the nodes labelled j, $j \in \{1, 2, 3, 4\}$, constitute I.S. I_j. Part (b) shows a tiling of a plane with squares, *e.g.* cells in a cellular network. Transmissions at neighboring cells interfere with each other. The corresponding conflict graph is $\mathcal{H}_{6,6}$.

C. Existence and computation of a NE in Mean Valid Graphs

Let G be a mean valid graph with d disjoint maximal I.S. $I_1, \ldots, I_d$. We start by considering a class of simple strategy

[14]This holds under the following assumption that eliminates problems arising due to boundary effects: There are an even number of rows of nodes, each containing 3δ nodes, for some integer $\delta \geq 1$.

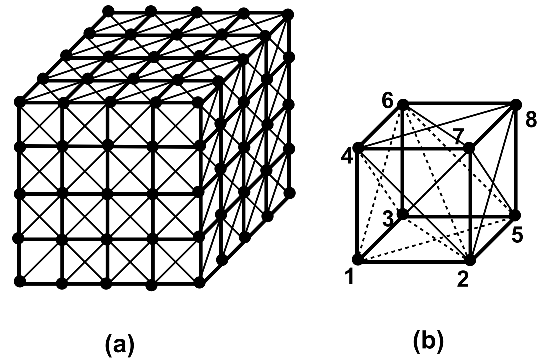

Fig. 4. Part (a) shows a three-dimensional grid graph $\mathcal{T}_{m,m,m}$ for $m = 5$. It consists of periodic repetitions of the graph shown in part (b). $\mathcal{T}_{m,m,m}$ is mean valid with $d = 8$ and disjoint maximal I.S. $I_1, \ldots, I_8$ (in the notation of Definition 3). In part (b), the node labels show the I.S. the nodes are in, *i.e.* a node with the label j is part of the I.S. I_j, $j \in \{1, \ldots, 8\}$.

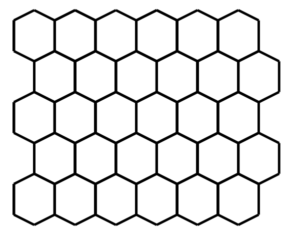

Fig. 5. The figure shows a tiling of a plane with hexagons, *e.g.* cells in a cellular network. Transmissions at neighboring cells interfere with each other.

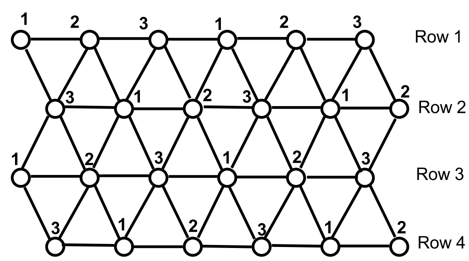

Fig. 6. The figure shows the conflict graph of a hexagonal tiling of a plane. It is mean valid with $d = 3$ and the disjoint maximal I.S. I_1, I_2, I_3 (in the notation of Definition 3), where the nodes labelled j, $j \in \{1, 2, 3\}$, constitute I.S. I_j. There are four rows of nodes.

profiles. Every primary selects I.S. I_j with probability t_j where $\{t_j : j = 1, \ldots, d\}$ represents a p.m.f., i.e, $\sum_{j=1}^{d} t_j = 1$ and $t_j \geq 0$ for each j. Interestingly enough, it turns out that a NE strategy profile belongs in this class, and furthermore, the corresponding p.m.f $\{t_j : j = 1, \ldots, d\}$ constitutes the unique solution of a set of equations that we provide, and can therefore be explicitly computed by solving them.

We first evaluate the expected payoff of a primary under an NE in the above class. We introduce some notations towards that end. Since primary i has unused bandwidth w.p. q_i and offers it at node $v \in I_j$ w.p. t_j, it offers bandwidth at node $v \in I_j$ w.p. $q_i t_j$. Analogous to the w_js that we introduced in Section III-A, we introduce $w_i(t_j)$ that represents the probability that K_v or more out of primaries $\{1, \ldots, n\} \backslash i$ offer bandwidth at a given node $v \in I_j$ under the above I.S. p.m.f. $\{t_j : j = 1, \ldots, d\}$. Under this p.m.f, by Lemma 2, and similar to (15) in the single location case, the primaries choose the price at each node in I_j as per the single-node NE strategy with $q_1 t_j, \ldots, q_n t_j$ in place of $q_1, \ldots, q_n$ respectively throughout, and each primary obtains an expected payoff of $W(t_j)$ at that node, where

$$W(x) = (1 - w_1(x))(\nu - c).$$

Now, for simplicity, we normalize $\nu - c = 1$. Then:

$$W(x) = (1 - w_1(x)). \tag{19}$$

Since I.S. I_j has M_j nodes, each primary receives a total expected payoff of $M_j W(t_j)$ if it chooses I_j.

We now state the main result of this section, which establishes the existence of a NE and also shows how it can be explicitly computed.

Theorem 4: In a mean valid graph, the following strategy profile constitutes a NE: each primary who has unused bandwidth selects I.S. I_j, $j \in \{1, \ldots, d\}$, w.p. t_j, where $(t_1, \ldots, t_d)$ is the unique distribution satisfying the following conditions. There exists an integer d' such that $1 \leq d' \leq d$ and[15]

$$t_j = 0 \text{ if } j > d', \text{ and} \tag{20}$$

$$M_1 W(t_1) = \ldots = M_{d'} W(t_{d'}) > M_{d'+1} r. \tag{21}$$

Also, $t_1 \geq t_2 \ldots \geq t_d$.

We first explain the result: (20) states that under the above NE, each primary selects with positive probability only some or all I.S. out of the I.S. $I_1, I_2, \ldots, I_d$. Since the total number of I.S. is exponential in the number of nodes in most graphs, it is surprising that an NE exists in which primaries offer bandwidth at only a small number of I.S. with positive probability. In addition, note that among $I_1, \ldots, I_d$, primaries do not select $I_{d'+1}, \ldots I_d$. Recall that by (17), $I_1, \ldots, I_d$ are in decreasing order of size. So primaries do not choose I.S. smaller than a certain size (out of $I_1, \ldots, I_d$). Similarly, the fact that $t_1 \geq t_2 \ldots \geq t_d$ is consistent with the intuition that primaries offer bandwidth at the larger I.S. with a larger probability. Next, since $\nu - c = 1$ and at each location, there exists at least one secondary w.p. r, whenever a primary offers bandwidth at a location, its expected payoff at that location is r or less. Thus, by (17), if it would have selected an I.S. in $I_{d'+1}, \ldots I_d$, it would have earned a payoff of at most $M_{d'+1} r$. As discussed above, a primary earns an expected payoff of $M_j W(t_j)$ if it selects I_j. Thus, (21) states that a primary earns equal expected payoffs by choosing I.S. in $I_1, I_2, \ldots, I_{d'}$ and this payoff exceeds the maximum payoff it could have earned by selecting an I.S. in $I_{d'+1}, \ldots I_d$– hence it never opts for the latter choice. Interestingly, although different primaries have different bandwidth availability probabilities, there exists at least one NE where all use the same I.S. selection p.m.f. They will however use different price distributions at the same node: primary i selects the d.f. $\psi_i(.)$ in Section III, with $q_1 t_j, \ldots, q_n t_j$ in place of $q_1, \ldots, q_n$ throughout at each node in I_j (Lemma 2).

Theorem 4 implies that every mean valid graph has a NE, which can be explicitly computed by solving the system of equations (20) and (21). Note that this is a system of non-linear equations in the variables $t_1, \ldots, t_d$ and d'. It can be solved using a standard solver for non-linear equations (e.g., *fsolve* in Matlab) in combination with a search procedure to find d'.

Since there is only one probability distribution $(t_1, \ldots, t_d)$ that satisfies (20) and (21), and $t_1 \geq \ldots \geq t_d$, it follows that:

$$t_i = t_j \text{ if } M_i = M_j. \tag{22}$$

We now illustrate the NE in Theorem 4 using an example.

Example: Suppose there are $n = 2$ primaries with probabilities of having unused bandwidth q_1 and q_2, where $q_1 \geq q_2$, and $K_v = 1$ secondary w.p. 1 at every node v. Consider a grid graph $\mathcal{H}_{m,m}$, which was introduced in Section IV-B2, with $m = 7$ (see part (a) of Fig. 3). By part 4 of Theorem 3, this is a mean valid graph and, in the notation of Definition 3, $d = 4$, the I.S. I_1, I_2, I_3 and I_4 are as shown in part (a) of Fig. 3, and $M_1 = 16$, $M_2 = M_3 = 12$, $M_4 = 9$. In the NE characterized in Theorem 4, it turns out that d', t_1, t_2, t_3 and t_4 are independent [16] of q_1, and their values for different $q_2 \in (0, 1)$ are as follows:

1) For $0 < q_2 < \frac{1}{4}$, $d' = 1$, $t_1 = 1$, $t_2 = t_3 = t_4 = 0$.
2) For $\frac{1}{4} \leq q_2 < \frac{15}{16}$, $d' = 3$, $t_1 = \frac{1}{11}\left(3 + \frac{2}{q_2}\right)$, $t_2 = t_3 = \frac{1}{11}\left(4 - \frac{1}{q_2}\right)$ $t_4 = 0$.
3) For $\frac{15}{16} \leq q_2 < 1$, $d' = 4$, $t_1 = \frac{1}{49}\left(9 + \frac{13}{q_2}\right)$, $t_2 = t_3 = \frac{1}{49}\left(\frac{1}{q_2} + 12\right)$ $t_4 = \frac{1}{49}\left(16 - \frac{15}{q_2}\right)$.

Note that $t_1 \geq t_2 \geq t_3 \geq t_4$ for each value of q_2, consistent with Theorem 4. Also, $t_2 = t_3$ for all q_2, which is consistent with (22). Fig. 7 plots t_1, t_2 and t_4 versus q_2. For small q_2, primaries offer bandwidth at the largest I.S. I_1 with probability 1; but as q_2 increases, the competition at I_1 increases, inducing the primaries to shift probability mass from I_1 to the other I.S. So t_1 decreases in q_2. However, note that for all values of q_2, $t_1 \geq t_2 \geq t_4$ and t_4 is very small (less than 0.02).

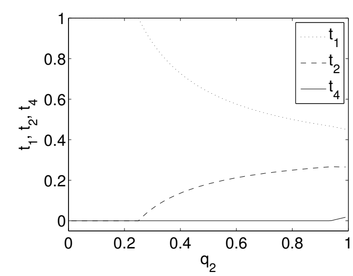

Fig. 7. The figure shows the NE probabilities t_1, t_2 and t_4 for the example in Section IV-C.

Finally, at the beginning of this section we showed that a system with multiple locations may have multiple NE. In fact, the example chosen was one where the conflict graph is linear, and is therefore mean valid by part 2 of Theorem 3.

[15]For notational simplicity, let $M_j = 0$ if $j > d$.

[16]This, in fact, holds in general because $d', t_1, \ldots, t_d$ are the solution of (20) and (21), which contain terms in the function $W(\alpha) = 1 - w_1(\alpha)$ and $w_1(.)$ is independent of q_1 by definition. However, the price distributions in the NE do depend on q_1.

Nevertheless, the NE in Theorem 4 turns out to be the unique one in a large class of strategy profiles, which we define next.

Definition 4: Let $\mathcal{S}$ be the class of strategy profiles in which every primary uses the same distribution (p.m.f.) to select the independent set at which to offer bandwidth.

Lemma 3: The NE characterized in Theorem 4 is unique in class $\mathcal{S}$.

Note that in a strategy profile in class $\mathcal{S}$, primaries may choose I.S. other than $I_1, \ldots, I_d$. The above lemma rules out the choice of any such I.S. under an NE.

D. Threshold behavior

We first define the *efficiency*, η, of a NE as $\eta = \frac{R_{\text{NE}}}{R_{\text{OPT}}}$, where R_{NE} is the expected sum of payoffs of the n primaries at the NE and R_{OPT} is the maximum possible (optimal) expected sum of payoffs, attained when all primaries jointly select the independent sets and prices to maximize their aggregate revenue. Clearly, $\eta \leq 1$ quantifies the loss in aggregate revenue incurred owing to lack of cooperation among primaries. Also, since the above NE is unique (overall for the single location game and in class $\mathcal{S}$ for multiple locations), η quantifies fundamental limits on the performance of NE in the respective categories.

Let $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{q_i}{n} = q$ for some $q \in (0,1)$. Here, q represents the "average" bandwidth availability probability of the primaries. For simplicity, we assume that each secondary from a given pool independently seeks bandwidth, and let k_n be the expected number of secondaries at any given location[17]. Then, the NE structure exhibits interesting threshold behavior as $n \to \infty$; in particular, η switches from 1 to 0 depending on the relations between nq (availability) and k_n (demand).

Lemma 4: When [18] there are n primaries, let $\tilde{p}_{jn}$ denote the common lower endpoint of the price distributions of the primaries who have unused bandwidth in the NE at nodes in I.S. I_j (if they select I.S. I_j). Also, let d'_n and t_{mn}, $m \in \{1, \ldots, d\}$, denote d' and t_m respectively in Theorem 4.

1) If there exists an $\epsilon > 0$ such that for all large n, $q < k_n/(n-1) - \epsilon$, then $\eta \to 1$ as $n \to \infty$. Also, for all large n, $d'_n = 1$, $t_{1n} = 1, t_{2n} = t_{3n} = \ldots t_{dn} = 0$, $\tilde{p}_{1n} \to \nu$.
2) Let $l < d$. If there exists an $\epsilon > 0$ such that for all large n, $lk_n/(n-1) + \epsilon < q < (l+1)k_n/(n-1) - \epsilon$, then for all large n, $d'_n \geq l+1$, and $t_{jn} q \to k_n/(n-1)$ for all $j \leq l$.
3) If there exists an $\epsilon > 0$ such that for all large n, $q > k_n d/(n-1) + \epsilon$, then $\eta \to 0$ as $n \to \infty$. Also, for all large n, $d'_n = d$ and $\tilde{p}_{jn} \to c$, $j = 1, \ldots, d$.

Intuitively, if availability is less than demand, then owing to limited competition, primaries with available bandwidth select only the maximum-sized I.S. among $I_1, \ldots, I_d$, and choose prices in a neighborhood of ν. Thus, $\eta \to 1$, since no other strategy can enhance any primary's payoff. As availability increases, under NE, primaries diversify their choices among the I.S. $I_1, \ldots, I_d$ and are more likely to select low prices as well (the lower limits of the price distributions hover around c once availability exceeds demand), thereby drastically reducing the efficiency of the NE.

V. Numerical Studies

In this section, we describe numerical computations that are directed towards assessing the impact of price competition among the primaries on the aggregate revenue of the primaries and the affordability of spectrum for the secondaries. We consider the specific case of a grid graph $\mathcal{H}_{m,m}$ (see Section IV-B2). By part 4 of Theorem 3, this is a mean valid graph and, in the notation of Definition 3, $d = 4$ and the I.S. I_1, I_2, I_3 and I_4 are as shown in part (a) of Fig. 3. Throughout, we use the parameter values $\nu = 1$ and $c = 0$, and a constant number of secondaries k at each node. Also, $q_1, \ldots, q_n$ are uniformly spaced in $[q_L, q_H]$ for some parameters q_L and q_H. Let $q = \frac{q_L + q_H}{2}$ be the mean bandwidth availability probability of the primaries.

In $\mathcal{H}_{m,m}$, the NE is of the form in Theorem 4 and the plot on the left in Fig. 8 reveals, as expected, that price competition significantly reduces the aggregate revenue of the primaries under this NE relative to OPT, the optimal scheme in which the primaries collaborate to attain R_{OPT}, the maximum aggregate revenue of the primaries (Note that under OPT, the I.S. $I_1, \ldots, I_4$ are selected in order of size and all the primaries always select the highest price ν). Also, overall, the efficiency (η) decreases as q increases since the competition increases. The plot on the right in Fig. 8 shows that the trends are similar for a larger topology (larger m). The plot on the left in Fig. 9 shows that η improves as k increases. This is because, for small values of k, demand for bandwidth is scarce at each node. Under the NE, bandwidth is wasted at several nodes since $k+1$ or more primaries offer bandwidth at those nodes, resulting in a shortage of bandwidth at other nodes. On the other hand, since all primaries cooperate in OPT, it judiciously supplies bandwidth precisely where it is needed. So OPT outperforms the NE by a large margin for small values of k. For large values of k, the demand is high and so is the tolerable margin of error in assigning the primaries to I.S.; and hence the performance of the NE improves relative to OPT. The plot on the right in Fig. 9 shows that η increases as m increases, which is because the four I.S. $I_1, \ldots, I_4$ become closer to each other in size as m increases and hence the loss in revenue resulting from choosing a smaller I.S. is lower.

Fig. 10 shows that under price competition, the expected price per unit of bandwidth is lower at the nodes in the larger I.S. This is because primaries prefer larger I.S. and hence the competition is more intense there, driving down the prices.

VI. Conclusions

We analyzed price competition among primaries in a CRN with a random number of secondaries taking into account bandwidth uncertainty and spatial reuse. For the game at a single location, we explicitly computed a NE and showed its uniqueness in the class of all NE. Also, for the game with spatial reuse, we computed a NE in mean valid graphs and showed its uniqueness in the class of NE with symmetric

[17] We allow (but do not require) the number (rather statistics) of the secondaries to scale with increase in n.

[18] For simplicity, we state this lemma under the assumption that $M_1, \ldots, M_d$ are distinct. Our technical report [23] provides the lemma with this assumption relaxed.

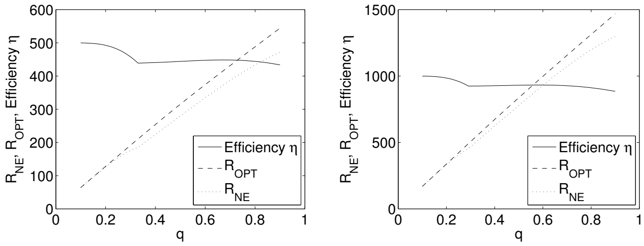

Fig. 8. Both figures plot the aggregate revenues of the primaries, R_{NE} and R_{OPT}, under the NE and OPT respectively, and the efficiency of the NE, $\eta = \frac{R_{NE}}{R_{OPT}}$, versus q. In both figures, $n = 10$, $k = 5$ and $q_H - q_L = 0.2$ are used. Also, $m = 15$ (respectively, $m = 25$) for the figure on the left (respectively, right). η is scaled by a factor of 500 (respectively, 1000) in the figure on the left (respectively, right) in order to show it on the same figure as the other plots.

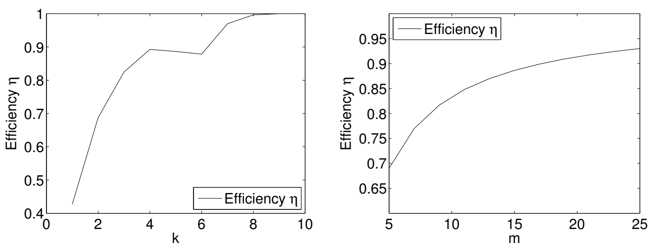

Fig. 9. The figure on the left (respectively, right) plots the efficiency η of the NE versus k (respectively, m). For both figures, $n = 10$, $q_L = 0$ and $q_H = 1$ are used. Also, $m = 15$ for the figure on the left and $k = 5$ for the figure on the right.

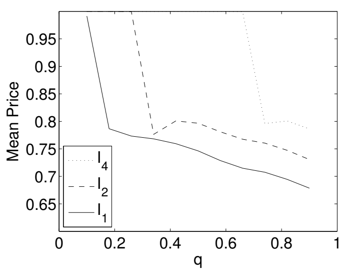

Fig. 10. The figure shows the mean price of bandwidth quoted by primary 1, given that it is offered, at a (fixed) node in each of I_1, I_2 and I_4 under the NE vs q. Note that since $|I_3| = |I_2|$, the mean price of bandwidth at nodes in I_3 is the same as that at nodes in I_2. The parameter values used are $m = 15$, $n = 8$ and $k = 3$. Also, $q_H - q_L = 0.2$.

independent set selection strategies of the primaries. Our analysis provides several insights, *e.g.*, there is randomization in the selection of prices by the primaries in the NE, and there exists a NE of simple form in mean valid graphs, in which primaries select only a small number of independent sets with positive probability. An open problem for future research is to investigate the existence, computation and uniqueness of NE in general graphs, *i.e.* graphs that need not be mean valid.

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