Beamforming and Aligned Interference Neutralization Achieve the Degrees of Freedom Region of the $2 \times 2 \times 2$ MIMO Interference Network

(Invited Paper)

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***Abstract*—We study the layered two-hop, two-unicast multi-input, multi-output (MIMO) interference network, which consists of two transmitters, two relays, and two receivers with the first and the second hop networks between transmitters and relays, and between relays and receivers, respectively, both being Gaussian MIMO interference channels. The DoF region is established in the general case, where there are arbitrary numbers of antennas at all terminals. It is shown that the DoF region coincides with the min-cut outer-bound, and is achievable via a scheme involving beamforming and aligned interference neutralization.**

***Index Terms*—Aligned interference neutralization, beamforming, two-hop two-flow networks.**

I. Introduction

THE problems of characterizing the fundamental limits of various interference networks are some of the most sought-after questions in information theory. Capacity approximations with varying degrees of accuracy have been reported lately for the multicast [1]–[3] and multiple unicast settings [4]–[7]. However, such approximations remain largely unknown for the multi-hop multiple unicast wireless networks and are hence among the most important unsolved problems in network information theory.

These problems have recently begun to be addressed, mainly, for the layered two-hop networks originally addressed in [8]–[10], with fundamental advances coming from the degrees of freedom (DoF) characterizations [11]–[13]. Gou et al. [11] proved that over the layered $2\times2\times2$ interference network (IN) of Fig. 1 with M antennas at all terminals (i.e., in the notation of Fig. 1, $M_1 = M_2 = N_1 = N_2 = K_1 = K_2 = M$), the min-cut bound of $2M$ DoF is achievable using the aligned interference neutralization scheme. This scheme involves over the air distributed cancelation of the interference before it reaches the destinations. This result thus shows, at least in the special case, that unlike in single-hop networks, there need not be a loss of DoF over the layered multi-hop networks, even if the sources, relays, and the destinations are not co-located. Motivated by this key development, [14], [15] make significant progress toward the DoF characterizations of the layered multi-hop (more than 2) INs with arbitrary connectivity in each hop. Moving beyond in an another direction, a class of non-layered multi-hop INs (with single-antenna terminals and full connectivity over each hop) has been recently studied in [16], [17] to prove the achievability of the min-cut bound.

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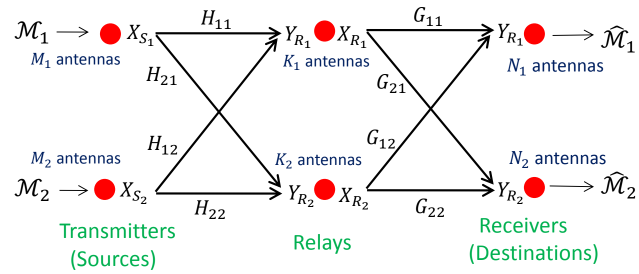


Here, M_i, N_i, and K_i denote the numbers of antennas at the i^{th} source, destination, and relay, respectively.

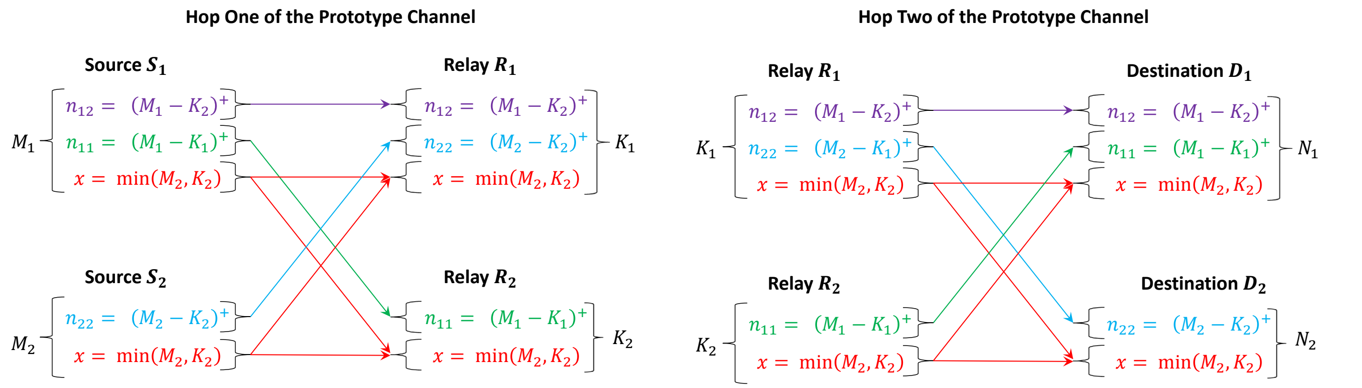
Fig. 1. The $2 \times 2 \times 2$ Interference Network or the layered 2-hop, 2-user Interference Channel

In this work, we study the (layered) $2 \times 2 \times 2$ MIMO IN, shown in Fig. 1, with arbitrary numbers of antennas at all six terminals. We prove that even in this general case, the min-cut outer-bound on the DoF region is achievable. To obtain such a result, we initially study a simple *prototype* $2\times2\times2$ IN, which consists of a set of non-interfering point-to-point links and one $2 \times 2 \times 2$ MIMO IN with equal number of antennas at all terminals. For instance, in the special case where $M_1 = N_1$, $M_2 = N_2$, and $M_1 + M_2 = N_1 + N_2 = K_1 + K_2$ (for notation, see Fig. 1), the prototype IN is of the form shown in Fig. 2. Due to the limited connectivity, the DoF region of the prototype IN can be easily shown to be achievable by using point-to-point signaling and aligned interference neutralization. We consider then the general $2\times2\times2$ MIMO IN with full connectivity in each hop; and show that by employing linear beamforming at different terminals, the given $2\times2\times2$ IN can be transformed to the corresponding prototype IN. Since the use of beamforming can not alter the DoF region, we deduce that the DoF region of the given IN is identical to that of the associated prototype IN, and later, show that the two coincide with the min-cut bound. This analysis thus proves that a combination of beamforming and aligned interference neutralization achieves the min-cut bound over the $2 \times 2 \times 2$ IN.

The next section introduces the channel model and states the main result on the DoF region, while the subsequent section presents the proof of the main result.

II. Channel Model and Main Result

In this section, we describe the $2\times2\times2$ MIMO interference network (IN), and state our main result about its DoF region.



$$M_1 = N_1,\ M_2 = N_2, \text{ and } M_1 + M_2 = N_1 + N_2 = K_1 + K_2.\ (a-b)^+ \triangleq \max(0, a-b).$$

Fig. 2. Structure of the Prototype $2 \times 2 \times 2$ MIMO Interference Network.

The $2 \times 2 \times 2$ MIMO interference network consists of two transmitters or sources S_1 and S_2 which must communicate with their respective receivers or destinations D_1 and D_2 with the help of two relays R_1 and R_2. The signals transmitted by the sources are observed only by the relays and the signals transmitted by the relays are observed by the destinations. There are M_1, M_2, N_1, N_2, K_1, and K_2 antennas at sources S_1 and S_2, destinations D_1 and D_2, and relays R_1 and R_2, respectively. The input-output relationships are given by

$$Y_{R_i}(t) = H_{i1}(t)X_{S_1}(t) + H_{i2}(t)X_{S_2}(t) + Z_{R_i}(t),\ \ i = 1, 2,$$
$$Y_{D_i}(t) = G_{i1}(t)X_{R_1}(t) + G_{i2}(t)X_{R_2}(t) + Z_{D_i}(t),\ \ i = 1, 2,$$

where at the t^{th} channel use, $Y_{R_i}(t) \in \mathbb{C}^{K_i \times 1}$, $Y_{D_i}(t) \in \mathbb{C}^{N_i \times 1}$ are the signals received by Relay R_i and Destination D_i, respectively; $X_{S_i}(t) \in \mathbb{C}^{M_i \times 1}$, $X_{R_i}(t) \in \mathbb{C}^{K_i \times 1}$ are the signals transmitted by the i^{th} source and the i^{th} relay, respectively; $H_{ij}(t) \in \mathbb{C}^{K_i \times M_j}$ is the channel coefficient between R_i, the i^{th} relay, and S_j, the j^{th} source, while $G_{ij}(t) \in \mathbb{C}^{N_i \times K_j}$ is the channel coefficient between D_i, the i^{th} destination, and R_j, the j^{th} relay; and finally, $Z_{R_i}(t)$ and $Z_{D_i}(t)$ are respectively the additive noises at R_i and D_i. We impose a power constraint of P on the transmit signals, i.e., $\mathbb{E}|X_{S_i}(t)|^2$, $\mathbb{E}|X_{R_i}(t)|^2 \le P\ \forall\ i,\ t$.

The $2 \times 2 \times 2$ MIMO interference network defined above will henceforth be referred as $\bar{v}$-IN with $\bar{v} = (M_1, M_2, N_1, N_2, K_1, K_2)$. Finally, let

$$H(t) = \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix} \text{ and } G(t) = \begin{bmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{bmatrix}.$$

The coefficients $H(t)$ and $G(t)$ are referred respectively as the first hop and the second hop channels.

We study here the case of additive white Gaussian noise and Rayleigh fading. In particular, the elements of $H_{ij}(t)$, $G_{ij}(t)$, $Z_{R_i}(t)$, $Z_{D_i}(t)$ are independent and identically distributed, across i, j, and t, according to the complex normal distribution with zero-mean and unit-variance.

Throughout this paper, it is assumed that each terminal knows perfectly and instantaneously all channel matrices corresponding to the hop(s) to which it belongs. In particular, the sources know $H(t)$, both relays know $H(t)$ and $G(t)$, and destinations know $G(t)$ perfectly and instantaneously.

Furthermore, the relays are assumed to be full duplex, but not instantaneous. That is, the transmit signal $X_{R_i}(t)$ of Relay R_i can depend on its past received signals, and on the past and present channel matrices, but not on the present received signal $Y_{R_i}(t)$.

Henceforth, we omit the time dependance of various signals and channel matrices, as it will not produce any ambiguity.

The DoF region of the $\bar{v}$-IN is defined in a standard manner, and is denoted by $\mathbf{D}(\bar{v})$. The main result of this paper is the exact characterization of the DoF region.

Theorem 1: The DoF region of the $\bar{v}$-IN with $\bar{v} = (M_1, M_2, N_1, N_2, K_1, K_2)$ is given by

$$\mathbf{D}(\bar{v}) = \mathcal{D}(\bar{v}) \triangleq \Big\{(d_1, d_2)\Big|\ 0 \le d_i \le \min(M_i, N_i, K_1 + K_2),$$
$$i = 1, 2;\ \ d_1 + d_2 \le K_1 + K_2\Big\}.$$

Proof: The converse follows from the min-cut bound [18]. The achievability part is proved in the next section. ■

This result shows that the DoF region depends only on the total number of relay antennas, but not on how these antennas are distributed between the two relays.

III. Proof of the Main Result: Achievability via Beamforming and Aligned Interference Neutralization

We want to prove that for any given $\bar{v}$, $\mathcal{D}(\bar{v})$ is achievable over the $\bar{v}$-IN. A vector $\bar{v} = (M_1, M_2, M_1, M_2, K_1, K_2)$ with $K_1 + K_2 = M_1 + M_2$ is called a *prototype* vector, for reasons that would become apparent later (note here that $N_i = M_i$). Consider the next lemma, which allows us to restrict attention only to prototype vectors, thereby simplifying the proof.

Lemma 1: If $\mathcal{D}(\bar{v})$ is achievable over $2 \times 2 \times 2$ IN for every prototype vector $\bar{v}$, then $\mathcal{D}(\bar{v})$ is achievable for any $\bar{v}$.

Proof: Consider any given $\bar{v}$. Set $m_1 = \min(M_1, N_1)$ and $m_2 = \min(M_2, N_2)$. Find integers k_1 and k_2 such that $0 \le k_i \le K_i$ for $i = 1, 2$ and $k_1 + k_2 = \min(m_1 + m_2, K_1 + K_2)$. Define $\bar{v}_1 = (m_1, m_2, m_1, m_2, k_1, k_2)$. It is easy to verify that

$\mathcal{D}(\bar{v}_1) = \mathcal{D}(\bar{v})$. Since every element of $\bar{v}_1$ is less than or equal to the corresponding element of $\bar{v}$, we observe that if $\mathcal{D}(\bar{v}_1)$ is achievable over $\bar{v}_1$-IN, then $\mathcal{D}(\bar{v}) = \mathcal{D}(\bar{v}_1)$ is achievable over $\bar{v}$-IN. Therefore, it is sufficient to prove the achievability of $\mathcal{D}(\bar{v}_1)$ over $\bar{v}_1$-IN.

We need to consider two cases, depending on the relative values of $K_1 + K_2$ and $m_1 + m_2$.

Case a: $K_1 + K_2 \geq m_1 + m_2$: In this case, we have $k_1 + k_2 = m_1 + m_2$, and hence, $\bar{v}_1$ is a prototype vector. Thus, by assumption of this lemma, $\mathcal{D}(\bar{v}_1)$ is achievable over $\bar{v}_1$-IN, which completes the proof.

Case b: $K_1 + K_2 < m_1 + m_2$: Here, $k_1 = K_1$ and $k_2 = K_2$, which yields

$$\mathcal{D}(\bar{v}_1) = \Big\{(d_1, d_2)\Big|\ 0 \leq d_i \leq \min(m_i, K_1 + K_2),\ i = 1, 2, \text{ and } d_1 + d_2 \leq K_1 + K_2\Big\}.$$

Suppose for two real numbers a and b, $(a - b)^+ \triangleq \min(0, a - b)$. Then the two corner points of $\mathcal{D}(\bar{v}_1)$ on lines $d_1 = \min\{K_1 + K_2, m_1\}$ and $d_2 = \min\{K_1 + K_2, m_2\}$ respectively are given by

$$P_1 \equiv \Big(\min\Big\{K_1 + K_2, m_1\Big\}, \min\Big\{m_2, (K_1 + K_2 - m_1)^+\Big\}\Big);$$
$$P_2 \equiv \Big(\min\Big\{m_1, (K_1 + K_2 - m_2)^+\Big\}, \min\Big\{K_1 + K_2, m_2\Big\}\Big).$$

It is sufficient to prove the achievability of P_1 and P_2 over $\bar{v}_1$-IN (the entire region can then be achieved via time sharing), and moreover, by symmetry, we can restrict to just P_1. Set $m_1' \triangleq \min\Big(K_1 + K_2, m_1\Big)$ and $m_2' \triangleq \min\Big\{m_2, (K_1 + K_2 - m_1)^+\Big\}$. Then it is easy to verify that $m_1' + m_2' = K_1 + K_2$. Hence, if we let $\bar{v}_2 = (m_1', m_2', m_1', m_2', K_1, K_2)$, then $\bar{v}_2$ is a prototype vector and $P_1 \in \mathcal{D}(\bar{v}_2)$. Hence, P_1 is achievable over $\bar{v}_2$-IN, and since every element of vector $\bar{v}_2$ is less than or equal to the corresponding element of $\bar{v}_1$, P_1 is achievable over $\bar{v}_1$-IN, as desired. ■

Thus, with the above lemma, we can focus on just prototype vectors $\bar{v}$. Henceforth, we have $M_i = N_i$, $i = 1, 2$, and $M_1 + M_2 = K_1 + K_2$, and assume without loss of generality that $M_1 \geq M_2$ and $K_1 \geq K_2$. Moreover, since $M_1 + M_2 = K_1 + K_2$, we have $M_1 \geq K_2$ and $K_1 \geq M_2$.

It turns out that the case of $M_1 = K_2$ has been addressed before in [11], as claimed by the next lemma.

Lemma 2: For a prototype vector $\bar{v}$ with $M_1 \geq M_2$ and $K_1 \geq K_2$, the DoF region $\mathcal{D}(\bar{v})$ is achievable over the $\bar{v}$-IN, if $M_1 = K_2$.

Proof: When $M_1 = K_2$, then all terminals have equal number of antennas, and the result of [11] applies. ■

Therefore, we need to consider the case of $M_1 > K_2$, which also implies that $K_1 > M_2$.

In summary, it is now sufficient to prove that DoF region $\mathcal{D}(\bar{v})$ is achievable over $\bar{v}$-IN for any given prototype vector $\bar{v}$ for which the following conditions hold:

$$M_1 = N_1,\ M_2 = N_2,\ M_1 \geq M_2,\ K_1 \geq K_2,$$
$$M_1 + M_2 = K_1 + K_2,\ M_1 > K_2, \text{ and } K_1 > M_2. \quad (1)$$

These conditions are assumed to be true henceforth in this section.

For a prototype vector $\bar{v}$, we now define a prototype $2 \times 2 \times 2$ interference network (IN), for which the DoF region can be easily shown to be equal to $\mathcal{D}(\bar{v})$. In fact, we define the prototype IN so that it has a structure that is shown in Fig. 2. Later, we will prove that for any prototype vector $\bar{v}$, the DoF region of the $\bar{v}$-IN is equal to that of the corresponding prototype $2 \times 2 \times 2$ IN, which completes the proof.

To define a prototype IN for a prototype vector $\bar{v}$, let us define some integers (recall conditions in (1 are true):

$$n_{12} = M_1 - K_2 > 0, \quad n_{11} = (M_1 - K_1)^+,$$
$$n_{22} = (M_2 - K_2)^+, \quad x = \min(M_2, K_2) > 0.$$

It is easy to verify that $n_{12} + n_{11} + x = M_1$, $n_{12} + n_{22} + x = K_1$, $n_{22} + x = M_2$, and $n_{11} + x = K_2$. Now, we want the prototype IN to have the structure illustrated in Fig. 2. With this motivation, consider a matrix

$$H_p = \begin{array}{c} \\ n_{12} \\ n_{22} \\ x \\ n_{11} \\ x \end{array} \begin{array}{c} \begin{array}{ccc|cc} n_{12} & n_{11} & x & n_{22} & x \end{array} \\ \left[\begin{array}{ccc|cc} W_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & W_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W_3 & \mathbf{0} & W_4 \\ \hline \mathbf{0} & W_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W_6 & \mathbf{0} & W_7 \end{array}\right] \end{array}, \quad (2)$$

where the numbers at the top and on the left denote the sizes of the respective block (for instance, W_1 is $n_{12} \times n_{12}$, while the second block in the first row, namely, $\mathbf{0}$ denotes the all-zero matrix of size $n_{12} \times n_{11}$), all $\{W_i\}$'s are invertible, matrices W_3, W_4, W_6, and W_7 follow a continuous distribution, H_p is invertible with probability 1, and $\mathbf{0}$ denotes the all-zero matrix of an appropriate size (this notation is used in the rest of the paper). In an analogous fashion, let

$$G_p = \begin{array}{c} \\ n_{12} \\ n_{11} \\ x \\ n_{22} \\ x \end{array} \begin{array}{c} \begin{array}{ccc|cc} n_{12} & n_{22} & x & n_{11} & x \end{array} \\ \left[\begin{array}{ccc|cc} W_1' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & W_2' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W_3' & \mathbf{0} & W_4' \\ \hline \mathbf{0} & W_5' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W_6' & \mathbf{0} & W_7' \end{array}\right] \end{array}. \quad (3)$$

Note that G_p and the transpose of H_p have the same structure.

For a prototype vector $\bar{v}$ (conditions in (1) hold in this discussion), a $\bar{v}$-IN is said to be a prototype IN if the channel matrices H and G admit the structures of H_p and G_p, respectively. A prototype IN is represented symbolically in Fig. 2.

Note that the prototype IN consists of some non-interfering paths and one $2 \times 2 \times 2$ MIMO interference network with x antennas at all terminals. Over this $2 \times 2 \times 2$ IN, applying the result of [11], we know that aligned interference neutralization scheme achieves x DoF for each transmit-receive pair[1]. Hence,

[1] [11] mainly considers the SISO $2 \times 2 \times 2$ IN. However, that result can be easily extended to the MIMO $2 \times 2 \times 2$ IN with an equal number of antennas at all terminals, provided all channel coefficients are generated from a continuous distribution. See Section 4 of [11] and also the proof of Corollary 3 of [19].

for a prototype vector $\bar{v}$, we can achieve a DoF pair $\big(n_{12} + n_{11} + x, n_{22} + x\big) \equiv (M_1, M_2)$ over the prototype IN, and therefore, the region $\mathcal{D}(\bar{v})$ is achievable over the prototype IN.

Our goal now is to prove that the DoF region of $\bar{v}$-IN, where $\bar{v}$ is a prototype vector, is equal to that of the prototype IN. To accomplish this, we show that if appropriate beamforming is employed at different terminals, then it is possible to transform the given $\bar{v}$-IN to the prototype form. Since beamforming can not alter the DoF region, we get the desired result.

We will first prove in the following lemma that beamforming does not alter the DoF region and then discuss the transformation a given $\bar{v}$-IN to the prototype form.

Lemma 3: Consider two $\bar{v}$-INs, the first with channel matrices H and G, and the second with channel matrices H' and G'. Suppose there exist matrices $L_{h,i}$, $R_{g,i}$, $R_{h,i}$, and $L_{g,i}$, where $i = 1, 2$, such that they are invertible with probability 1, and

$$\begin{aligned} H_p &= \begin{bmatrix} L_{h,1} & 0 \\ 0 & L_{h,2} \end{bmatrix} H \begin{bmatrix} R_{h,1} & 0 \\ 0 & R_{h,2} \end{bmatrix} \quad \text{and} \\ G_p &= \begin{bmatrix} L_{g,1} & 0 \\ 0 & L_{g,2} \end{bmatrix} G \begin{bmatrix} R_{g,1} & 0 \\ 0 & R_{g,2}, \end{bmatrix} \end{aligned}$$

where $L_{h,i}, R_{g,i} \in \mathbb{C}^{K_i \times K_i}$, $R_{h,i} \in \mathbb{C}^{M_i \times M_i}$, and $L_{g,i} \in \mathbb{C}^{M_i \times M_i}$. Then the two $\bar{v}$-INs have an identical DoF region.

Proof: It is sufficient to prove that a DoF pair achievable over one IN is also achievable over the other. By symmetry, it is sufficient to prove that if a pair is achievable over the second, then it is achievable over the first. Suppose now that a DoF pair (d_1, d_2) is achievable over the second IN using some achievability scheme that involves transmitting signals X'_{S_1} and X'_{S_2} from the sources and X'_{R_1} and X'_{R_2} from the relays, and receiving signals Y'_{R_1} and Y'_{R_2} at the relays and Y'_{D_1} and Y'_{D_2} at the destinations. This scheme can be easily transformed over the first channel by setting $X_{S_i} = R_{h,i} X'_{S_i}$ and $X_{R_i} = R_{g,i} X'_{R_i}$, where $i = 1, 2$, and by making relays and destinations compute $\hat{Y}_{R_i} = L_{h,i} Y_{R_i}$ and $\hat{Y}_{D_i} = L_{g,i} Y_{D_i}$ for i=1,2. Note that signal $\hat{Y}_{R_i}$ ($\hat{Y}_{D_i}$, respectively) is identically distributed as Y'_{R_i} (Y'_{D_i}, respectively), except for the additive noise term, which can not change a DoF result. Hence, the DoF pair (d_1, d_2) is achievable over the first channel, as desired. ■

Note here that matrices $R_{h,i}$ and $R_{g,i}$ can be considered as the transmit beamforming matrices at S_i and R_i, whereas $L_{h,i}$ and $L_{g,i}$ can be regarded as the receive beamforming matrices at S_i and D_i.

Thus, it now only remains to show that by appropriate beamforming, any given $\bar{v}$-IN can be transformed to the prototype form, and as discussed above, this implies the achievability of $\mathcal{D}(\bar{v})$ over the given $\bar{v}$-IN, which would complete the proof. In light of the above lemma, it is sufficient to prove the existence of matrices $L_{h,i}$, $L_{g,i}$, $R_{h,i}$, and $R_{g,i}$, where $i \in \{1, 2\}$.

To transform a given $\bar{v}$-IN to the prototype form, we consider three cases. Of these, only the first one turns out to be important, and based on its solution, the other two cases can be handled.

- Case I: $n_{11} = 0$ and $n_{22} > 0$
- Case II: $n_{11} = 0$ and $n_{22} = 0$
- Case III: $n_{11} > 0 \Rightarrow n_{22} = 0$.

We start below with Case I.

Case I: $n_{11} =$ and $n_{22} > 0$: We will first transform the channel matrix H to the form H_p. In this particular case, H can be partitioned as

$$H = \begin{array}{c} \\ n_{12} \\ n_{22} \\ x \\ x \end{array} \begin{array}{c} \begin{array}{cc|cc} n_{12} & x & n_{22} & x \end{array} \\ \left[\begin{array}{cc|cc} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ \hline D_1 & D_2 & D_3 & D_4 \end{array}\right] \end{array},$$

and needs to be transformed to the form of H_p shown in (2). This transformation will be done using a series of steps.

Step I: Find unitary matrices U_1 and U_2 via singular-value decompositions of $\begin{bmatrix} D_1 & D_2 \end{bmatrix}$ and $\begin{bmatrix} D_3 & D_4 \end{bmatrix}$ such that

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} U_1 = \begin{bmatrix} \mathbf{0} & D_{2,1} \end{bmatrix} \text{ and } \begin{bmatrix} D_3 & D_4 \end{bmatrix} U_2 = \begin{bmatrix} \mathbf{0} & D_{4,1} \end{bmatrix}$$

for some full-rank $D_{2,1}$ and $D_{4,1}$. Note that U_1 and U_2 are $M_1 \times M_1$ and $M_2 \times M_2$, respectively. Since H is i.i.d. Rayleigh faded, we have

$$H \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \sim H_1 \triangleq \begin{array}{c} \\ n_{12} \\ n_{22} \\ x \\ x \end{array} \begin{array}{c} \begin{array}{cc|cc} n_{12} & x & n_{22} & x \end{array} \\ \left[\begin{array}{cc|cc} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ \hline \mathbf{0} & D_{2,1} & \mathbf{0} & D_{4,1} \end{array}\right] \end{array},$$

where $a \sim b$ indicates that a and b have identical distribution. Note that the above operation, pre-multiplication by a block-diagonal matrix, corresponds to beamforming at two sources. Also, it is now sufficient to transform H_1 to the form of H_p (since H_p is defined only in terms of its distribution).

Step II: Compute a unitary matrix U_3 such that

$$U_3 \begin{bmatrix} A_1 & A_3 \\ B_1 & B_3 \\ C_1 & C_3 \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

for some invertible $(n_{12} + n_{22}) \times (n_{12} + n_{22})$ matrix $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$. Therefore, since U_3 is independent of C_2 and C_4,

$$\begin{array}{c} \\ n_{12}{+}n_{22} \\ x \\ x \end{array} \begin{array}{c} \begin{array}{cc|c} n_{12}{+}n_{22} & x & x \end{array} \\ \left[\begin{array}{cc|c} M^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_x & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_x \end{array}\right] \end{array} \times \begin{bmatrix} U_3 & 0 \\ 0 & I_x \end{bmatrix} \times H_1$$

$$\sim H_2 \triangleq \begin{array}{c} \\ n_{12} \\ n_{22} \\ x \\ x \end{array} \begin{array}{c} \begin{array}{cc|cc} n_{12} & x & n_{22} & x \end{array} \\ \left[\begin{array}{cc|cc} I_{n_{12}} & A_{2,1} & \mathbf{0} & A_{4,1} \\ \mathbf{0} & B_{2,1} & I_{n_{22}} & B_{4,1} \\ \mathbf{0} & C_2 & \mathbf{0} & C_4 \\ \hline \mathbf{0} & D_{2,1} & \mathbf{0} & D_{4,1} \end{array}\right] \end{array}$$

for some $A_{2,1}$, $A_{4,1}$, $B_{2,1}$, and $B_{4,1}$. Note that the above operation, post-multiplication by a block-diagonal matrix, corresponds to beamforming at relays; in fact, here only the first relay performs non-trivial beamforming.

Step III: Consider now the row operations, which correspond to beamforming at the relays:

$$\begin{array}{c|ccc|c} & n_{12} & n_{22} & x & x \\ n_{12} & I_{n_{12}} & \mathbf{0} & -A_{4,1}C_4^{-1} & \mathbf{0} \\ n_{22} & \mathbf{0} & I_{n_{22}} & -B_{2,1}C_2^{-1} & \mathbf{0} \\ x & \mathbf{0} & \mathbf{0} & I_x & \mathbf{0} \\ \hline x & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_x \end{array} H_2$$

$$\sim H_3 \triangleq \begin{array}{c|cc|cc} & n_{12} & x & n_{22} & x \\ n_{12} & I_{n_{12}} & A_{2,2} & \mathbf{0} & \mathbf{0} \\ n_{22} & \mathbf{0} & \mathbf{0} & I_{n_{22}} & B_{4,2} \\ x & \mathbf{0} & C_2 & \mathbf{0} & C_4 \\ \hline x & \mathbf{0} & D_{2,1} & \mathbf{0} & D_{4,1} \end{array}$$

for some $A_{2,2}$ and $B_{4,2}$.

Step IV: Now with beamforming at sources, we can reduce H_3 to the required form as follows:

$$H_3 \times \begin{array}{c|cc|cc} & n_{12} & x & n_{22} & x \\ n_{12} & I_{n_{12}} & -A_{2,2} & \mathbf{0} & \mathbf{0} \\ x & \mathbf{0} & I_{n_{22}} & \mathbf{0} & \mathbf{0} \\ \hline n_{22} & \mathbf{0} & \mathbf{0} & I_{n_{22}} & -B_{4,2} \\ x & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_x \end{array} =$$

$$\begin{array}{c|cc|cc} & n_{12} & x & n_{22} & x \\ n_{12} & I_{n_{12}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ n_{22} & \mathbf{0} & \mathbf{0} & I_{n_{22}} & \mathbf{0} \\ x & \mathbf{0} & C_2 & \mathbf{0} & C_4 \\ \hline x & \mathbf{0} & D_{2,1} & \mathbf{0} & D_{4,1} \end{array}.$$

It is easy to verify that $D_{2,1}$, $D_{4,1}$, C_2, and C_4 follow a continuous distribution and are invertible with probability 1; and therefore, we have reduced H to the form of H_p with just beamforming at various terminals.

Consider now G, which needs to be transformed to form G_p. G can be represented in the following form:

$$G = \begin{array}{c|ccc|c} & n_{12} & n_{22} & x & x \\ n_{12} & A'_1 & A'_2 & A'_3 & A'_4 \\ x & B'_1 & B'_2 & B'_3 & B'_4 \\ \hline n_{22} & C'_1 & C'_2 & C'_3 & C'_4 \\ x & D'_1 & D'_2 & D'_3 & D'_4 \end{array},$$

while G_p is of the form shown in (3). Note that G and G_p have the same structures as the transposes of H and H_p, respectively. Hence, the solution developed earlier can be used with appropriate modifications.

Case II: $n_{11} = 0$ and $n_{22} = 0$: This case is identical to the previous one, except that all blocks, which in the previous case had either n_{22} rows and/or n_{22} columns, are now absent. Thus, the solution of the previous case can still be used.

Case III: $n_{11} > 0$ and $n_{22} = 0$: It can be shown that H and H_p under this case have the same structure that the transposes of G and G_p have under Case I. Hence, the solution developed for Case I applies. Similarly, we can handle the transformation of G to G_p under this case.

In short, for any prototype vector $\bar{v}$, the given IN can be transformed to the prototype form via beamforming at different terminals, which completes the proof.

IV. Conclusion

We studied the $2 \times 2 \times 2$ MIMO IN with arbitrary numbers of antennas at all terminals. In this general case, we prove that a concatenation of channel decomposition beamforming and aligned interference neutralization achieves the min-cut bound on its DoF region.

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