A Hybrid Random-Structured Coding Scheme for the Gaussian Two-Terminal Source Coding Problem Under a Covariance Matrix Distortion Constraint

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***Abstract*—This paper focuses on the Gaussian two-terminal source coding problem under a covariance matrix distortion constraint, which subsumes the quadratic Gaussian two-terminal source coding problem with individual distortion constraints, whose complete solution is known. Different from existing schemes which are either random or structured, we propose a new hybrid random-structured scheme with a sum-rate strictly smaller than the quantize-and-bin (QB) upper bound in certain cases. The first layer of our scheme is a QB random coding scheme attempting to achieve an intermediate distortion matrix that is as *symmetric* as possible. The second layer is a structured scheme that targets at reconstructing a weighted difference of the observed sources conditioned on their quantized versions in the first layer. We prove that the gap between the sum-rate of our scheme and its lower bound is no larger than two bits per sample, in particular, this gap decreases to exactly one bit per sample when the source covariance matrix is *symmetrifiable* in the sense that the intermediate covariance matrix can be made purely symmetric.**

I. Introduction

Distributed source coding (DSC) of correlated sources has attracted an increasing amount of attention since Slepian and Wolf's celebrated work [1] in 1973. However, lossy DSC is still an open problem in general, despite of some partial solutions provided for certain special cases. This paper considers a subclass of lossy DSC where two Gaussian sources are separately compressed and jointly reconstructed subject to a covariance matrix distortion constraint. This problem is of particular interest because it is the most general Gaussian two-terminal source coding problem in the sense that its achievable rate region completely determines those of the following problems (as pointed out by Oohama in [2]).

- The quadratic Gaussian two-terminal source coding problem with *individual* MSE distortion constraints whose rate region has been characterized [3].
- The DSC problem of linear functions [4], [5], [6] with a *single* MSE distortion constraint on a *linear function* of the two sources.
- The generalized Gaussian CEO problem [7] with a *sum* MSE distortion constraint on some *remote sources* that are jointly Gaussian with the two observed sources.

For a given source covariance matrix, the target distortion matrix has three degrees of freedom. However, the conventional random coding scheme with a quantize-and-bin (QB) framework can only achieve distortion matrices on a two-dimensional surface parameterized by the two quantization noise variances. Consequently, among those distortion matrices with unit diagonal elements, only one is achievable by a QB scheme, while others require time-sharing between at least two QB schemes. Denote the off-diagonal element of the QB-achievable distortion matrix as $\theta^\star$.

Two lower sum-rate bounds [3], [8] have been given for the covariance matrix constrained two-terminal problem. Specifically, Wagner [3] provided a composite sum-rate lower bound using those for the μ-sum problem and the cooperative two-terminal problem, respectively, and showed that the QB sum-rate upper bound is tight for any distortion matrix with unit diagonal elements and off-diagonal element θ that is no larger than $\theta^\star$. On the other hand, Wagner's composite lower bound degenerates to the cooperative bound, which is relatively loose and diverges from the QB upper bound, as the off-diagonal element increases beyond the threshold $\theta^\star$. Another sum-rate lower bound is provided in [8], which improves the cooperative bound by an unbounded amount when $\theta > \theta^\star$. However, the maximum sum-rate gap between the improved lower bound and the QB upper bound is also unbounded.

The main contribution of this work is a new *hybrid* random-structured scheme that superimposes a distributed difference-forming structured coding component proposed by Krithivasan and Pradhan [4] on a QB random coding scheme, with a minimum sum-rate within two b/s from the lower bound in [8]. Our proposed scheme has a similar structure as Ahlswede and Han's scheme [9, Section VI] which concatenates a QB random coding scheme with Körner and Marton's structured linear coding scheme [10] for encoding the modulo-2 sum of two binary sources (that are asymmetric in general).

The first layer of our hybrid scheme is a QB scheme where both encoders quantize theirs sources using random quantizers, with the resulting quantization indices further compressed by a pair of Slepian-Wolf encoders. Then at each encoder, with the availability of the quantized version of its observed source, a *correlation tuner* forms a tuned source by linearly combining the observed source with its quantized version, in such a way that a certain weighted difference between the resulting tuned sources is independent of the coarsely quantized versions. Next, the second layer comes in by applying the Krithivasan

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and Pradhan's structured scheme [4] on the tuned sources to reconstruct the weighted difference *lossily* at the decoder using two fine lattice quantizers and the *same* coarse lattice channel code. The joint decoder linearly combines the three pieces of reconstructions in an MMSE manner to generate the final estimations of the sources.

Our scheme is parameterized by the four quantization noise variances (two in each layer) and the weight factor in the second layer. By assuming the target distortion matrix to have unit diagonal element without loss of generality, we show that the optimal parameter set that achieves the minimum sum-rate is such that

- The first layer and the correlation tuners adjust the correlation between the tuned sources such that the intermediate covariance matrix of the sources given their quantized versions is as *symmetric* as possible.
- If the source covariance matrix is "symmetrifiable" in the sense that the intermediate covariance matrix has equal diagonal elements, the gap between the sum-rate of our hybrid scheme and the lower bound in [8] is exactly one b/s, while this gap is capped at two b/s in general.

An intuitive reason for the first layer to symmetrify the intermediate covariance matrix is that the second structured layer is most efficient in the symmetric case (within one b/s from optimality [4], [5]).

Compared to the QB sum-rate bound, the sum-rate saving in our hybrid scheme is unbounded. However, since the second layer of our scheme has a minimum of one b/s transmission rate, there are cases when the QB sum-rate is strictly smaller. Moreover, to get a convex composite sum-rate upper bound, time-sharing between QB schemes and our scheme is required, with a maximum of three different schemes involved due to Carathéodory's Theorem [11].

II. Problem setup

Let Y_1 and Y_2 be jointly Gaussian with zero mean and covariance matrix

$$\Sigma_Y = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

and $Y_1^n = (Y_{1,1}, Y_{1,2}, ..., Y_{1,n})$, $Y_2^n = (Y_{2,1}, Y_{2,2}, ..., Y_{2,n})$ be two length-n blocks of independent drawings of Y_1 and Y_2, respectively.

Each of the two encoders sends a function of their own source, namely, $\phi_i^{(n)} : \mathbb{R}^n \mapsto \{1, 2, ..., M_i^{(n)}\}$ for $i = 1, 2$. The decoder attempts to reconstruct Y_1^n and Y_2^n using function $\psi^{(n)} : \{1, 2, ..., M_1^{(n)}\} \times \{1, 2, ..., M_2^{(n)}\} \mapsto \mathbb{R}^n \times \mathbb{R}^n$. Denote the reconstructed versions as $\hat{Y}_i^n$, $i = 1, 2$.

For a 2×2 target distortion matrix [1]

$$\boldsymbol{\mathcal{D}} = \begin{bmatrix} 1 & \theta \\ \theta & 1 \end{bmatrix}, \quad (1)$$

[1]Note that without loss of generality, we assume in this paper that the target distortion matrix $\boldsymbol{\mathcal{D}}$ have unit diagonal elements and an off-diagonal element θ, since otherwise one can set $\Sigma_{Y'} = \mu\Sigma_Y\mu^T$ and $\boldsymbol{\mathcal{D}'} = \mu\boldsymbol{\mathcal{D}}\mu^T$ with $\mu = \text{diag}(([\boldsymbol{\mathcal{D}}]_{11})^{-\frac{1}{2}}, ([\boldsymbol{\mathcal{D}}]_{22})^{-\frac{1}{2}})$ to obtain an equivalent problem with $\boldsymbol{\mathcal{D}'}$ taking the form of (1).

such that $\mathbf{0} \prec \boldsymbol{\mathcal{D}} \preceq \Sigma_Y$, we say a triple $(R_1, R_2, \boldsymbol{\mathcal{D}})$ is achievable if there exists a sequence of schemes $(\phi_1^{(n)}, \phi_2^{(n)}, \psi^{(n)})$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 M_i^{(n)} \le R_i, \quad i = 1, 2,$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \text{Cov}(Y_1^n, Y_2^n | W_1, W_2) \preceq \boldsymbol{\mathcal{D}},$$

and denote the *rate-distortion region* $\mathcal{RD}^\star$ as the convex closure of the set of all achievable triple, i.e.,

$$\mathcal{RD}^\star = \text{conv}\Big\{(R_1, R_2, \boldsymbol{\mathcal{D}}) \text{ is achievable}\Big\}.$$

The achievable rate region is defined as

$$\mathcal{R}^\star(\boldsymbol{\mathcal{D}}) \triangleq \Big\{(R_1, R_2) : (R_1, R_2, \boldsymbol{\mathcal{D}}) \in \mathcal{RD}^\star\Big\}.$$

And the *minimum sum-rate* is defined as

$$R^\star(\boldsymbol{\mathcal{D}}) \;=\; \min_{(R_1,R_2)\in\mathcal{R}^\star(\boldsymbol{\mathcal{D}})} R_1 + R_2.$$

A. Existing lower and upper bounds on $R^\star(\boldsymbol{\mathcal{D}})$

For any pair of Σ_Y and $\boldsymbol{\mathcal{D}}$, one can always apply the conventional QB scheme that employs independent Gaussian quantization followed by binning (Slepian-Wolf coding) to obtain the following rate-distortion region

$$\begin{aligned}\mathcal{RD}^{QB} = \text{conv}\Big[\bigcup_{(G_1,G_2)\in\mathcal{U}(Y_1,Y_2)} \Big\{&(R_1, R_2, \text{Cov}(Y_1, Y_2 | G_1, G_2)) : \\ &R_1 \ge I(Y_1; G_1 | G_2),\ R_2 \ge I(Y_2; G_2 | G_1), \\ &R_1 + R_2 \ge I(Y_1, Y_2; G_1, G_2)\Big\}\Big].\end{aligned}$$

where $\text{conv}(\cdot)$ means convex closure, and $\mathcal{U}(Y_1, Y_2)$ contains all (G_1, G_2) pairs such that $G_1 = Y_1 + \mathcal{N}(0, \sigma_{q_1}^2)$, $G_2 = Y_2 + \mathcal{N}(0, \sigma_{q_2}^2)$ are independently corrupted versions of Y_1 and Y_2. Consequently, the QB inner rate region and the QB sum-rate upper bound are defined as

$$\mathcal{R}^{QB}(\boldsymbol{\mathcal{D}}) \triangleq \Big\{(R_1, R_2) : (R_1, R_2, \boldsymbol{\mathcal{D}}) \in \mathcal{RD}^{QB}\Big\},$$

$$R^{QB}(\boldsymbol{\mathcal{D}}) \triangleq \min_{(R_1,R_2)\in\mathcal{R}^{QB}(\boldsymbol{\mathcal{D}})} R_1 + R_2,$$

with $\mathcal{R}^{QB}(\boldsymbol{\mathcal{D}}) \subseteq \mathcal{R}^\star(\boldsymbol{\mathcal{D}})$, $R^{QB}(\boldsymbol{\mathcal{D}}) \ge R^\star(\boldsymbol{\mathcal{D}})$.

On the other hand, Wagner *et al.* [3] provided the following composite sum-rate lower bound (which was used to prove the QB sum-rate tightness in the original quadratic Gaussian two-terminal problem) that combines a μ-sum bound with the cooperative bound,

$$R^\star(\boldsymbol{\mathcal{D}}) \ge \underline{R}_W(\boldsymbol{\mathcal{D}}) \triangleq \max\Big\{R_\mu(\boldsymbol{\mathcal{D}}), R_{\text{coop}}(\boldsymbol{\mathcal{D}})\Big\},$$

$$R_\mu(\boldsymbol{\mathcal{D}}) \triangleq \frac{1}{2} \log \frac{((1-\rho^2)\sigma_1^2\sigma_2^2 + 2\rho\sigma_1\sigma_2(1+\theta))}{(1+\theta)^2},$$

$$R_{\text{coop}}(\boldsymbol{\mathcal{D}}) \triangleq \frac{1}{2} \log \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{1-\theta^2}.$$

This lower bound was partially improved in [8] as

$$R^\star(\boldsymbol{\mathcal{D}}) \ge \underline{R}(\boldsymbol{\mathcal{D}}) \triangleq \begin{cases} R_\mu(\boldsymbol{\mathcal{D}}) & \theta \le \theta^\star \\ R_{\text{lb}}(\boldsymbol{\mathcal{D}}) & \theta > \theta^\star \end{cases},$$

where

$$\theta^\star = \frac{\sqrt{(1-\rho^2)^2\sigma_1^2\sigma_2^2 + 4\rho^2} - (1-\rho^2)\sigma_1\sigma_2}{2\rho},$$

$$R_{\rm lb}(\boldsymbol{\mathcal{D}}) \triangleq \frac{1}{2}\log\frac{(1-\rho^2)^2\sigma_1^3\sigma_2^3}{(1-\theta)^2((1-\rho^2)\sigma_1\sigma_2 + 2\rho(1+\theta))}.$$

In addition, $\underline{R}(\boldsymbol{\mathcal{D}})$ is partially tight when $\theta \le \theta^\star$ since

$$\underline{R}(\boldsymbol{\mathcal{D}}) = \underline{R}_W(\boldsymbol{\mathcal{D}}) = R^\star(\boldsymbol{\mathcal{D}}) = R^{QB}(\boldsymbol{\mathcal{D}}) \text{ when } \theta \le \theta^\star.$$

III. The Proposed Hybrid Random-Structured Scheme

We describe a hybrid random-structured coding scheme and give a new inner rate-distortion region $\mathcal{RD}$ for the covariance matrix constrained Gaussian two-terminal problem.

Let Λ_1, Λ_2 and Λ_C be three n-dimensional lattices with normalized second moments q_1, q_2 and q_C, respectively, such that $\Lambda_i \subset \Lambda_C$, $i = 1, 2$. Assume that (Λ_1, Λ_2) are good for source coding, while Λ_C good for channel coding. Also define the dithered quantizer associated with an n-dimensional lattice Λ and a length-n dither sequence T^n as

$$\mathcal{Q}_{\Lambda(T^n)}(x^n) \triangleq \arg\min_{q^n\in\Lambda} \|x^n + T^n - q^n\|^2 - T^n.$$

The block diagram of our proposed two-layer scheme is depicted in Fig. 1. In the first layer of our scheme, the two encoders quantize Y_1^n and Y_2^n using two random quantizers with auxiliary random variables

$$U_i = Y_i + P_i, \quad i = 1, 2, \tag{2}$$

where $P_i \sim \mathcal{N}(0, p_i)$ are independent additive Gaussian noises. The indices of the codewords U_1^n and U_2^n are compressed by a pair of Slepian-Wolf encoders to transmission rates of $R_1^{(1)}$ and $R_2^{(1)}$, respectively.

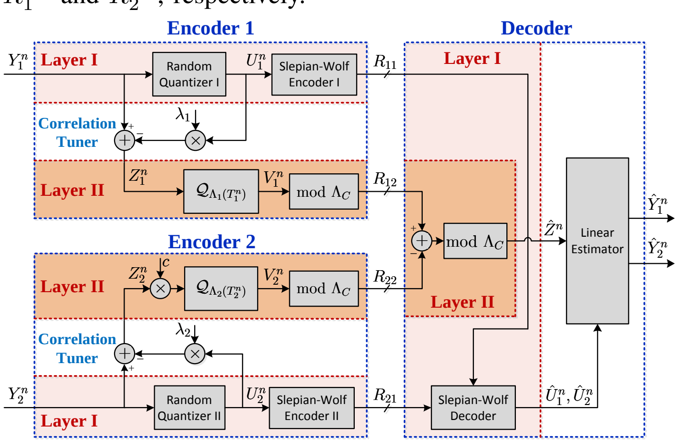

Fig. 1. Block diagram of our proposed scheme.

Then at the each encoder, a *correlation tuner* computes

$$Z_i^n = Y_i^n - \lambda_i U_i^n, \quad i = 1, 2,$$

where

$$\lambda_1 = \frac{(1-\rho^2)\sigma_1^2\sigma_2^2 + p_2(\sigma_1 - c\rho\sigma_2)}{(1-\rho^2)\sigma_1^2\sigma_2^2 + p_1\sigma_2^2 + p_2\sigma_1^2 + p_1p_2}$$

$$\lambda_2 = \frac{(1-\rho^2)c\sigma_1^2\sigma_2^2 + p_1\sigma_2(c\sigma_2 - \rho\sigma_1)}{(1-\rho^2)\sigma_1^2\sigma_2^2 + p_1\sigma_2^2 + p_2\sigma_1^2 + p_1p_2}$$

with $c > 0$ being a positive number. The resulting *tuned sources* (Z_1^n, Z_2^n) are scaled and quantized using another two dithered lattices quantizers as

$$\tilde{V}_1^n = \mathcal{Q}_{\Lambda_1(T_1^n)}(Z_1^n), \quad \tilde{V}_2^n = \mathcal{Q}_{\Lambda_2(T_2^n)}(cZ_2^n),$$

with T_1^n and T_2^n being length-n random dithers uniformly distributed over the Voronoi regions of Λ_1 and Λ_2, respectively. The indices of $S_i^n \triangleq \tilde{V}_i^n \bmod \Lambda_C$, $i = 1, 2$ are sent to the decoder using transmission rates $R_i^{(2)} = \frac{1}{2}\log_2\frac{q_C}{q_i}$, $i = 1, 2$.

The decoder also consists of two layers. The first layer reconstructs $\hat{U}_1^n$ and $\hat{U}_2^n$ using the Slepian-Wolf Decoder, while the second layer receives S_1^n and S_2^n and computes

$$\hat{Z}^n = (S_1^n - S_2^n) \bmod \Lambda_C.$$

The last step at the decoder reconstructs Y_1 and Y_2 as

$$\hat{Y}_i^n \triangleq \alpha_{i_1}\hat{U}_1^n + \alpha_{i_2}\hat{U}_2^n + \alpha_{i_3}\hat{Z}^n, \quad i = 1, 2, \tag{3}$$

where α_{ij}'s are linear MMSE estimation coefficients of (Y_1, Y_2) given (U_1, U_2) in (2) and $Z \triangleq V_1 - V_2$ with

$$V_1 \triangleq Z_1 + Q_1, \quad V_2 \triangleq cZ_2 + Q_2, \quad Z_i \triangleq Y_i - \lambda_i U_i, \tag{4}$$

and Q_1, Q_2 being independent additive Gaussian noises with variances q_1 and q_2, respectively.

Remarks:

- The *correlation tuners* between the two layers of the encoders select λ_1 and λ_2 such that
  $$Z_1 - cZ_2 = (Y_1 - cY_2) - E\big[Y_1 - cY_2 | U_1, U_2\big],$$
  i.e., they completely remove the portion of $Y_1^n - cY_2^n$ that is correlated to (U_1^n, U_2^n), which can be added back at the decoder once (U_1^n, U_2^n) is recovered.
- There is no loss of generality by using a single weight factor c on Y_2, since reconstructing $Y_1 - cY_2$ as $\hat{Z}$ is equivalent to restoring $Y_2 - \frac{1}{c}Y_1$ as $-\frac{\hat{Z}}{c}$.

A. Performance of our proposed scheme

Our proposed scheme provides the following inner rate-distortion region, with detailed proof given in Section IV.

Theorem 1: It holds that

$$\mathcal{RD} \subseteq \mathcal{RD}^\star,$$

where $\mathcal{RD}$ contains all triples $(R_1, R_2, \boldsymbol{\mathcal{D}})$ satisfying

$$\begin{aligned} R_1 &\ge I(Y_1; U_1|U_2) + I(Z; Y_1, V_2|U_1, U_2) \\ R_2 &\ge I(Y_2; U_2|U_1) + I(Z; V_1, Y_2|U_1, U_2) \\ R_1 + R_2 &\ge I(Y_1, Y_2; U_1, U_2) + I(Z; Y_1, V_2|U_1, U_2) \\ &\quad + I(Z; V_1, Y_2|U_1, U_2) \end{aligned}$$

for some (U_1, U_2, Z, V_1, V_2) defined in (2) and (4) such that

$$\mathrm{Cov}(Y_1, Y_2|U_1, U_2, Z) \preceq \boldsymbol{\mathcal{D}}.$$

A natural corollary of Theorem 1 is that the convex closure of the QB rate-distortion region $\mathcal{RD}^{QB}$ and the rate-distortion region $\mathcal{RD}$ of our new scheme is a subset of $\mathcal{RD}^\star$.

Corollary 1: It holds that

$$\mathrm{conv}\Big(\mathcal{RD} \cup \mathcal{RD}^{QB}\Big) \subseteq \mathcal{RD}^{\star}. \tag{5}$$

On the other hand, the rate-distortion region $\mathcal{RD}$ of our new scheme gives a sum-rate upper bound stated in the following theorem. The proof is also outlined in Section IV.

Theorem 2: For a $\boldsymbol{\mathcal{D}}$ in (1) with $d_1^2 = d_2^2 = 1$ and $\theta \geq \theta^{\star}$, it holds that

$$R(\boldsymbol{\mathcal{D}}) \leq \overline{R}(\boldsymbol{\mathcal{D}}) \tag{6}$$

$$\triangleq \begin{cases} \frac{1}{2}\log_2 \frac{4\sigma_1^3\sigma_2^3(1-\rho^2)^2}{(1-\theta)^2((1-\rho^2)\sigma_1\sigma_2+2\rho(1+\theta))} & \delta_1, \delta_2 \geq 0 \\ \frac{1}{2}\log_2 \frac{4\sigma_1^4\sigma_2^4(1-\rho^2)^2((1-\rho^2)\sigma_2^2-(1-\theta^2))}{(1-\theta^2)^2((1-\rho^2)\sigma_1^2\sigma_2^2-\sigma_1^2-\rho\sigma_2^2+2\theta\rho\sigma_1\sigma_2)} & \delta_1 < 0, \delta_2 \geq 0 \\ \frac{1}{2}\log_2 \frac{4\sigma_1^4\sigma_2^4(1-\rho^2)^2((1-\rho^2)\sigma_1^2-(1-\theta^2))}{(1-\theta^2)^2((1-\rho^2)\sigma_1^2\sigma_2^2-\sigma_2^2-\rho\sigma_1^2+2\theta\rho\sigma_1\sigma_2)} & \delta_1 \geq 0, \delta_2 < 0 \end{cases}$$

where for $i, j \in \{1, 2\}, i \neq j$,

$$\delta_i \triangleq \frac{\sigma_i^2\sigma_j(1-\rho^2)(1+\theta)}{(1-\rho^2)\sigma_i^2\sigma_j - (\sigma_j - \rho\sigma_i)(1+\theta)}.$$

Comparing our new sum-rate upper bound with the lower bound $\underline{R}(\boldsymbol{\mathcal{D}})$, we obtained the following theorem, which states that the sum-rate gap between $\overline{R}(\boldsymbol{\mathcal{D}})$ and $\underline{R}(\boldsymbol{\mathcal{D}})$ is no larger than two b/s, sandwiching the sum-rate-distortion function $R(\boldsymbol{\mathcal{D}})$ in a constant and relatively small range. The detailed proof is omitted.

Theorem 3: For a $\boldsymbol{\mathcal{D}}$ in (1) with $d_1^2 = d_2^2 = 1$ and $\theta \geq \theta^{\star}$, it holds that

$$\overline{R}(\boldsymbol{\mathcal{D}}) - \underline{R}(\boldsymbol{\mathcal{D}}) \leq 2. \tag{7}$$

Furthermore, the following theorem states that the gap between $\overline{R}(\boldsymbol{\mathcal{D}})$ and $\underline{R}(\boldsymbol{\mathcal{D}})$ is exactly one b/s in a special "symmetrifiable" case.

Theorem 4: If $\boldsymbol{\mathcal{D}}$ takes the form in (1) with $d_1^2 = d_2^2 = 1$ and $\theta \geq \theta^{\star}$, and Σ_Y is *symmetrifiable* in the sense that there exists $p_1, p_2 \geq 0$ such that

$$[\Sigma_Z(p_1, p_2)]_{11} = [\Sigma_Z(p_1, p_2)]_{22} \tag{8}$$

with $\Sigma_Z(p_1, p_2) \triangleq \left[\Sigma_Y^{-1} + \mathrm{diag}(\frac{1}{p_1}, \frac{1}{p_2})\right]^{-1}$, then

$$\overline{R}(\boldsymbol{\mathcal{D}}) - \underline{R}(\boldsymbol{\mathcal{D}}) = 1. \tag{9}$$

IV. Proof of Main Results

This section contains proof outlines of Theorems 1 and 2.

A. Proof of Theorem 1

Proof: Before proving Theorem 1, we states two lemmas with detailed proofs omitted.

Lemma 1: There exist sequences of lattices $\{\Lambda_i(n)\}_{n\in\mathbb{N}^+}$, $i, j = 1, 2$ that are good for source coding in the sense that

$$\tilde{V}_i^n \overset{n\to\infty}{\rightsquigarrow} V_i^n,\ i = 1, 2,$$
$$\hat{Z}^n = \tilde{V}_1^n - \tilde{V}_2^n \overset{n\to\infty}{\rightsquigarrow} Z^n,$$

with (Z, V_1, V_2) defined in (4), and $X^n \overset{n\to\infty}{\rightsquigarrow} Y^n$ means X^n converges to Y^n in the Kullback-Leibler divergence sense as $n \to \infty$.

Lemma 2: There exist a sequence of lattices $\{\Lambda_C(n)\}_{n\in\mathbb{N}^+}$ that is good for channel coding in the sense that $q_C = \mathrm{Var}(Z)$, and

$$\Pr(\hat{U}_i^n \neq U_i^n) \overset{n\to\infty}{\rightarrow} 0,\ i = 1, 2,$$
$$\Pr(\hat{Z}^n \neq \tilde{V}_1^n - \tilde{V}_2^n) \overset{n\to\infty}{\rightarrow} 0.$$

Consequently, for $\hat{Y}_i^n$'s defined in (3),

$$\mathrm{Cov}(Y_1^n - \hat{Y}_1^n, Y_2^n - \hat{Y}_2^n) \overset{n\to\infty}{\rightarrow} \mathrm{Cov}(Y_1, Y_2 | U_1, U_2, Z). \tag{10}$$

To prove Theorem 1, we can simply apply Lemmas 1 and 2 to show achievability of $(R_1^{\dagger}, R_2^{\dagger}, \boldsymbol{\mathcal{D}})$ with

$$R_1^{\dagger} \geq I(Y_1; U_1) + I(Z; Y_1, V_2 | U_1, U_2)$$
$$R_2^{\dagger} \geq I(Y_2; U_2 | U_1) + I(Z; V_1, Y_2 | U_1, U_2).$$

Then using the same technique, we can prove that the dual scheme can achieve $(R_1^{\ddagger}, R_2^{\ddagger}, \boldsymbol{\mathcal{D}})$ with

$$R_1^{\ddagger} \geq I(Y_1; U_1 | U_2) + I(Z; Y_1, V_2 | U_1, U_2)$$
$$R_2^{\ddagger} \geq I(Y_2; U_2) + I(Z; V_1, Y_2 | U_1, U_2),$$

and Theorem 1 follows by taking the convex closure of $(R_1^{\dagger}, R_2^{\dagger}, \boldsymbol{\mathcal{D}})$ and $(R_1^{\ddagger}, R_2^{\ddagger}, \boldsymbol{\mathcal{D}})$ triples. ■

B. Proof of Theorem 2

Proof: With Theorem 1, we only need to provide proper (c, p_1, p_2, q_1, q_2) parameters for the three different cases. When $\delta_1, \delta_2 \geq 0$, let $c^{(1)} = \frac{\sigma_2}{\sigma_1}$, $(p_1^{(1)}, p_2^{(1)}) = (\delta_1, \delta_2)$,

$$q_i^{(1)} = \frac{\sigma_1\sigma_2(1-\rho^2)(1-\theta^2)}{2(\theta\sigma_1\sigma_2(1-\rho^2) - \rho(1-\theta^2))}, i = 1, 2.$$

When $\delta_1 < 0$ and $\delta_2 \geq 0$, let $p_1^{(2)} = \infty$,

$$c^{(2)} = \frac{\theta\sigma_1\sigma_2(1-\rho^2) - \rho(1-\theta^2)}{\sigma_1^2(1-\rho^2) - (1-\theta^2)},$$
$$p_2^{(2)} = \frac{\sigma_2^2((1-\rho^2)\sigma_1^2 - (1-\theta^2))}{(\sigma_1^2-1)(\sigma_2^2-1) - \rho^2\sigma_1^2\sigma_2^2 + \theta(2\rho\sigma_1\sigma_2 - \theta)},$$
$$q_i^{(2)} = \frac{\sigma_1^2(1-\rho^2)(1-\theta^2)}{2((1-\rho^2)\sigma_1^2 - (1-\theta^2))},\ i = 1, 2.$$

When $\delta_2 < 0$ and $\delta_1 \geq 0$, let $p_2^{(3)} = \infty$,

$$c^{(3)} = \frac{(1-\rho^2)\sigma_2^2 - \rho(1-\theta^2)}{\theta\sigma_1\sigma_2(1-\rho^2) - \rho(1-\theta^2)},$$
$$p_1^{(3)} = \frac{\sigma_1^2((1-\rho^2)\sigma_2^2 - (1-\theta^2))}{(\sigma_1^2-1)(\sigma_2^2-1) - \rho^2\sigma_1^2\sigma_2^2 + \theta(2\rho\sigma_1\sigma_2 - \theta)},$$
$$q_i^{(3)} = \frac{\sigma_1^2(1-\rho^2)(1-\theta^2)((1-\rho^2)\sigma_2^2 - (1-\theta^2))}{2(\theta\sigma_1\sigma_2(1-\rho^2) - \rho(1-\theta^2))^2},\ i = 1, 2.$$

Then verifications of (6) is straightforward. ■

V. Numerical Examples

Fig. 2 compares the sum-rate lower and upper bounds $\underline{R}(\boldsymbol{\mathcal{D}})$, $\underline{R}_W(\boldsymbol{\mathcal{D}})$, $R^{QB}(\boldsymbol{\mathcal{D}})$, $\overline{R}(\boldsymbol{\mathcal{D}})$, and $\overline{R}^{TS}(\boldsymbol{\mathcal{D}})$, respectively, with $\sigma_1^2 = 20$, $\sigma_2^2 = 10$, $\rho = 0.9$, $d_1^2 = d_2^2 = 1$, and

$$\overline{R}^{TS}(\boldsymbol{\mathcal{D}}) \triangleq \min\left\{R_1+R_2 : (R_1, R_2, \boldsymbol{\mathcal{D}}) \in \text{conv}\left(\mathcal{RD} \cup \mathcal{RD}^{QB}\right)\right\}.$$

We observe that for this case, $\overline{R}(\boldsymbol{\mathcal{D}})$ does not directly improves $R^{QB}(\boldsymbol{\mathcal{D}})$. However, by time-sharing between our proposed two-layer scheme and the QB scheme, $\overline{R}^{TS}(\boldsymbol{\mathcal{D}})$ is strictly smaller than $R^{QB}(\boldsymbol{\mathcal{D}})$ for all $\theta > \theta^\star$. In addition, due to Theorem 3, the gap between $\overline{R}(\boldsymbol{\mathcal{D}})$ and $\underline{R}(\boldsymbol{\mathcal{D}})$ is no larger than two b/s.

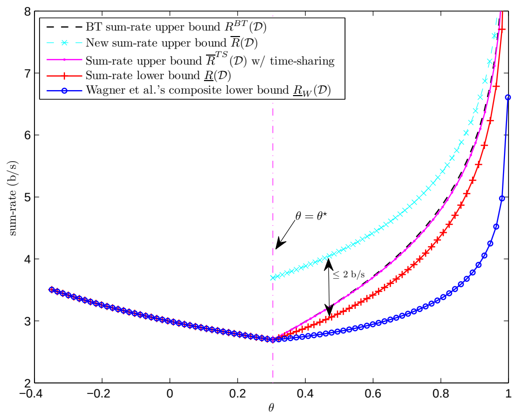

Fig. 2. A comparison among different lower and upper sum-rate bounds when $\sigma_1^2 = 20$, $\sigma_2^2 = 10$, $\rho = 0.9$, and $d_1^2 = d_2^2 = 1$.

Another comparison when $\sigma_1^2 = \frac{100}{9}$, $\sigma_2^2 = 10$, $\rho = 0.99$, and $d_1^2 = d_2^2 = 1$. We observe that for this case, $\overline{R}(\boldsymbol{\mathcal{D}})$ itself already improves $R^{QB}(\boldsymbol{\mathcal{D}})$. And one can show that $(\rho, \boldsymbol{\mathcal{D}})$ is always symmetrifiable for any $\theta > \theta^\star$, hence due to Theorem 4, the gap between $\overline{R}(\boldsymbol{\mathcal{D}})$ and $\underline{R}(\boldsymbol{\mathcal{D}})$ is exact one b/s.

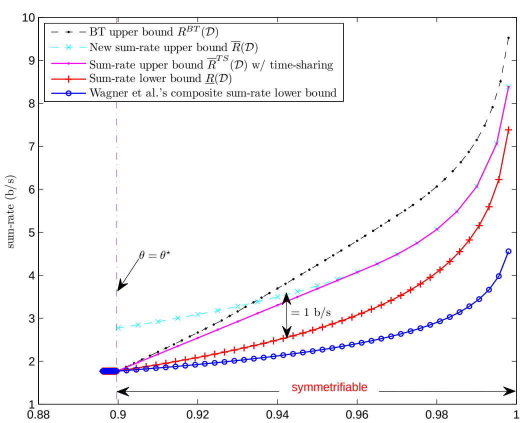

Fig. 3. A comparison among different lower and upper sum-rate bounds when $\sigma_1^2 = \frac{100}{9}$, $\sigma_2^2 = 10$, $\rho = 0.99$, and $d_1^2 = d_2^2 = 1$.

Fig. 4 shows the $(\frac{1}{\sigma_1}, \frac{1}{\sigma_2})$ region where $\overline{R}(\boldsymbol{\mathcal{D}}) < R^{QB}(\boldsymbol{\mathcal{D}})$ for some θ when ρ is fixed. Analytical forms of the boundaries are very hard to compute hence the plot is obtained through numerical comparisons between $\overline{R}(\boldsymbol{\mathcal{D}})$ and $R^{QB}(\boldsymbol{\mathcal{D}})$. We observe that in order for $\overline{R}(\boldsymbol{\mathcal{D}}) < R^{QB}(\boldsymbol{\mathcal{D}})$ to hold, ρ must be at least 0.6.

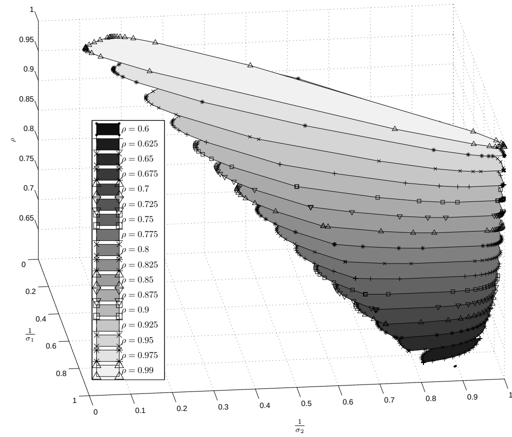

Fig. 4. $(\frac{1}{\sigma_1}, \frac{1}{\sigma_2})$ region where $\overline{R}(\boldsymbol{\mathcal{D}}) < R^{QB}(\boldsymbol{\mathcal{D}})$ for some θ when ρ is fixed.

One can show that as ρ goes to one, the maximum sum-rate saving between $\overline{R}(\boldsymbol{\mathcal{D}})$ and $R^{QB}(\boldsymbol{\mathcal{D}})$ is unbounded, with detailed proof omitted. It is worth noting that $\overline{R}^{TS}(\boldsymbol{\mathcal{D}})$ can be achieved by at most three different schemes due to Carathéodory's Theorem [11].

VI. Conclusions

We have proposed a two-layer structured scheme for the covariance matrix constrained Gaussian two-terminal source coding problem. The first layer is a QB component that adjusts the source correlation while the second layer directly aims at reconstructing a weighted difference of the sources given the information sent from the first layer to the decoder. By choosing appropriate parameters, the proposed scheme can strictly improve the QB rate-distortion region, with a sum-rate that is within two b/s from optimality. Conceptually, our proposed hybrid scheme brings together the two different worlds of random QB coding and structured lattice coding.

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