Abstract—One of the classic observations in investment theory is that maximizing the expected-log-return of a portfolio results in the greatest long-term growth of wealth. The log-optimal portfolio is both competitively optimal and pathwise dominant. Nevertheless, investment researchers and practitioners don’t all latch on to the log-optimal doctrine, even for theoretical guidance. A common alternative is to use a utility function to evaluate an investment strategy. At first glance it seems that any (non-decreasing) utility function would point to the log-optimal portfolio, at least in the limit. This is known not to be the case. In this work we identify sufficient conditions on a utility function that will produce a happy marriage between utility theory and optimal growth-rate of wealth.\(^1\)

INTRODUCTION

We address the problem of choosing an investment in a setting of compounding growth. To get at the heart of the matter, we consider a memoryless, time-invariant investment setting. Let \(X_i\) be a random vector representing the price relatives of the investment opportunities at time \(i\), and \(b_i\) represent the portion of wealth allocated to each investment opportunity at time \(i\). Then the wealth at the end of time \(n\), which initially started at \(W_0\) is expressed as

\[
W_n = W_0 \prod_{i=1}^{n} b_i^T X_i,
\]

which is a random variable. The question then is which portfolio \(b_i\) to choose each time.

A hefty amount of discussion has centered around the choice of portfolio \(b\). Each choice results in a different random amount of wealth. Choosing between probabilistic outcomes is a matter of preference (so long as one distribution doesn’t stochastically dominate another). In such cases, utility theory is an attractive tool to employ. Under mild consistency assumptions, a person’s body of preferences can be summarized by a utility function\(^2\). The distribution that maximizes the expected utility will be the preferred choice for that individual. In this context, the optimal portfolio \(b\) should be discussed with reference to a particular utility function.

A large body of literature by the likes of Kelly, Thorp, and Cover advocates the use of the so-called log-optimal portfolio, \(b^* = \max_b \mathbb{E} \log b^T X\) (see references in [1]). The log-optimal portfolio maximizes the long-term growth rate of wealth (\(\lim \inf 1/n \log W_n\)). It is also competitively optimal in the sense that it maximizes the probability of beating any other portfolio in even a single time-step [1]. Similarly, \(\mathbb{E} \frac{b^* X}{b^T X} > 1\) for all portfolios \(b\) [2].

The arguments in support of the log-optimal portfolio \(b^*\) are based on objective criteria which do not require knowledge of individual preferences of wealth versus risk. They are based only on the preference that more money is better. It is tempting to believe that any rational decision maker would choose \(b^*\) because of these properties. Yet, utility theory does not agree. There exist non-decreasing utility functions for which the log-optimal portfolio is not optimal even in the long-term limit. And this does not happen only for contrived examples but for even some of the most popular utility functions used in the literature and in practice, such as the power laws: \(u(x) = -x^{-p}\) for some \(p > 0\).

Why might a utility function not prefer the objective claims of the log-optimal portfolio \(b^*\)? It comes down to focussing on the \(\epsilon\)-tails of the wealth distribution rather than the almost-sure set. As long as the tails of the utility function grow in a manageable way, the log-optimal portfolio will be preferred in the limit.

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\(^2\)For alternatives and objections to utility theory, see the reference to prospect theory in [3].
I. RESULTS

Let the set $\mathcal{U}$ be the set of log-optimal utility functions, which is to say that the log-optimal portfolio is utility-optimal in the limit. Here we give a precise definition:

**Definition 1.** The set of log-optimal utility functions $\mathcal{U}$ is the set of function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that for any probability distribution of price-relatives vector $p_X$ and any constant portfolio $b$ we have

$$\lim_{n \to \infty} \inf E u(W_0 \prod_{i=1}^{n} b^{*T} X_i) - E u(W_0 \prod_{i=1}^{n} b^{T} X_i) \geq 0,$$

where $b^{*} = \arg \max_b E \log b^{T} X$.

Let the set $\bar{\mathcal{U}}$ be the set of strongly log-optimal utility functions, which is to say that the ratio between the utility of the log-optimal portfolio and any other portfolio grows unboundedly. Here we give a precise definition:

**Definition 2.** The set of strongly log-optimal utility functions $\bar{\mathcal{U}}$ is the set of function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that for any probability distribution of price-relatives vector $p_X$ and any constant portfolio $b \neq b^{*}$ we have

$$\lim_{n \to \infty} E \frac{u(W_0 \prod_{i=1}^{n} b^{*T} X_i)}{u(W_0 \prod_{i=1}^{n} b^{T} X_i)} = \infty,$$

where $b^{*} = \arg \max_b E \log b^{T} X$.

**Theorem 1.** Any function satisfying $\frac{u'(x)}{u(x)}x \sqrt{\log(x)} \rightarrow 0$ as $x \to \infty$ and as $x \to 0$, as well as growing at least poly-logarithmically with $x$ is in $\mathcal{U}$.

**Theorem 2.** Poly-log functions are in $\bar{\mathcal{U}}$, while polynomials (and inverses of polynomials) are not even in $\mathcal{U}$.

II. SUMMARY

Arguments in favor of the log-optimal portfolio $b^{*}$ are quite convincing. Perhaps this work of identifying the set of log-optimal utility functions should be viewed as a statement about what constitutes a rational utility function (in particular, how the tails should behave).

REFERENCES

