Phase Transitions in the Achievable Sum-Rate of Symmetric Gaussian Interference Channels

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Abstract -- We consider the rate sum of a symmetric Gaussian interference channel in standard format, given by $Y_1 = X_1 + aX_2 + Z_1$ and $Y_2 = aX_1 + X_2 + Z_2$, with a an interference coefficient, X_1 and X_2 constrained to average power P, and Z_1 and Z_2 distributed as $N(0,1)$. We provide a complete taxonomy of potentially optimal Gaussian signaling strategies in the entire weak and moderate interference parameter space, delimited by $0 < a < 1$ and $P > 0$. The strategies seem to lead to the Han and Kobayashi achievable region for the rate sum. We find that the parameter space comprises eight (potentially) optimal transmission regions formed by four pure mode strategies and four transitional strategies. We characterize the boundaries of these pure mode regions and the associated transitional regions.

***Index Terms*— Gaussian interference channel, Gaussian symmetric interference channel, time division multiplex (TDM), frequency division multiplex (FDM), symmetric superposition, asymmetric superposition, Han and Kobayashi region.**

I. INTRODUCTION

The Gaussian interference channel is a classical model in multiple user interference channel for which the capacity region is still unknown. This model has been analyzed by various researchers [1-18], and the capacity region is known only in the special cases of strong ($a \geq 1$) and very strong ($a^2 \geq 1 + P$) interference. To reduce the dimensionality of the parameter space and simplify the problem, we consider the sum rate of the symmetric version of this model. The aim is to maximize the achievable sum rate for the channel. The model, depicted in Fig.1, is given by $Y_1 = X_1 + aX_2 + Z_1$ and $Y_2 = aX_1 + X_2 + Z_2$, with X_1 and X_2 constrained to average power P, and Z_1 and Z_2 distributed as $N(0,1)$. The constant a plays the role of an interference coefficient.

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This problem has been solved for capacity region in the strong interference regime ($a \geq 1$) [2,6,7]. It has also been solved for sum rate in the very weak interference regime ($2a(1 + a^2P) \leq 1$) [13, 14, 15].

When the interference coefficient a satisfies $0 < a < 1$ we can identify two regions in the (a, P) parameter space, that we call weak and moderate interference regions. When $a^2 + a^4P \leq 1$, the efficient strategies seem to be limited to treating interference as noise (IAN) or avoiding interference all together by time or frequency division multiplexing (TDM/FDM), or a combination of these two strategies, as shown in [17]. We call this the region of weak interference.
The complementary region, delimited by $a^2 + a^4P < 1$ and $a < 1$, is what we call the moderate interference region. In this region, superposition can be a superior strategy to TDM/FDM in some cases. This paper is focused on the tradeoffs between the strategies of superposition (symmetric and asymmetric) and TDM/FDM that occur in this region.

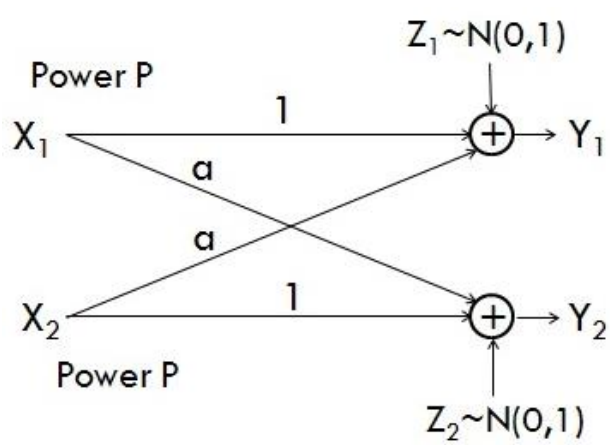

Fig.1. The symmetric Gaussian interference channel.

II. PRELIMINARIES

In an earlier paper [17] we examined the trade-off between the strategies of treating interference as noise

(IAN) and TDM/FDM that occur in the (a,P) region delimited by $a^2(1+a^2P) \leq 1$. In the complementary region, when $a^2(1+a^2P) > 1$, we have seen that superposition can be superior to TDM/FDM. In this paper we take a closer look at the trade-off between the alternatives of symmetric and asymmetric superposition and the alternative of avoiding interference with multiplexing. The various constraints that characterize the sum rate in the Han and Kobayashi region can be simplified to the four inequalities given below:

$$R_1 + R_2 \leq I(X_1; Y_1|U_2, Q) + I(X_2; Y_2|U_1, Q) \quad (1)$$

$$R_1 + R_2 \leq I(X_1; Y_1|U_1, U_2, Q) + I(U_1, X_2; Y_2|Q) \quad (2)$$

$$R_1 + R_2 \leq I(U_2, X_1; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q) \quad (3)$$

$$R_1 + R_2 \leq I(U_2, X_1; Y_1|U_1, Q) + I(U_1, X_2; Y_2|U_2, Q) \quad (4)$$

These inequalities are associated with points in the rate plane (R_1,R_2) below, marked as SR_i, i=1,2,3,4. We note that SR1 is a redundant constraint. The corner points SR2 and SR3 can be combined in a single constraint given by their time shared average. Finally SR4 represents an additional constraint. The combined effect of these constraints can be summarized in two bounds, represented by SR4 (Eq. (4)), and the average of SR2 and SR3 (Eqs. (2) and (3)). If we use Gaussian signals for the private and common messages of both senders, and do not use the time sharing parameter Q for the time being (letting Q=0), we can represent these two bounds by the following inequalities:

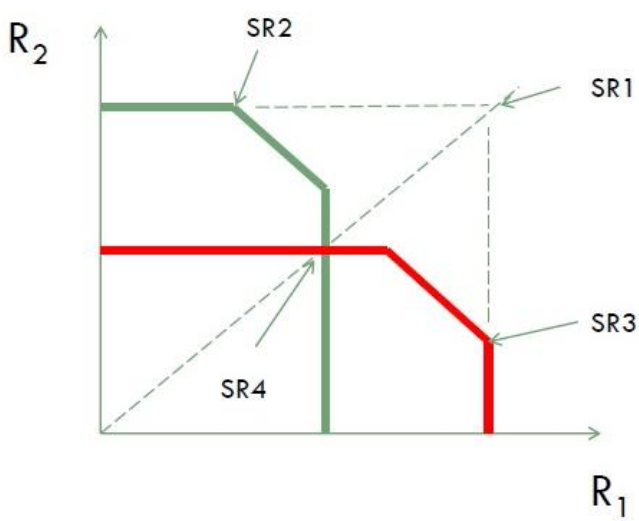

Fig.2. Rate points in the (R_1,R_2) plane that lead to constraints in the sum rate.

$$R_1 + R_2 \leq SR_{23} = \frac{1}{2}\log\left(\frac{(1+\alpha_1 P + a^2\alpha_2 P)}{(1+a^2\alpha_1 P)(1+a^2\alpha_2 P)}\right) + \frac{1}{2}\log(\sqrt{(1+a^2\alpha_1 P + \alpha_2 P)(1+\alpha_1 P + a^2\alpha_2 P)}) \quad (5)$$

$$R_1 + R_2 \leq SR_4 = \frac{1}{2}\log\left(\frac{(1+\alpha_1 P + a^2 P)(1+a^2 P + \alpha_2 P)}{(1+a^2\alpha_1 P)(1+a^2\alpha_2 P)}\right), \quad (6)$$

where we have represented the average of SR_2 and SR_3 by SR_{23}.

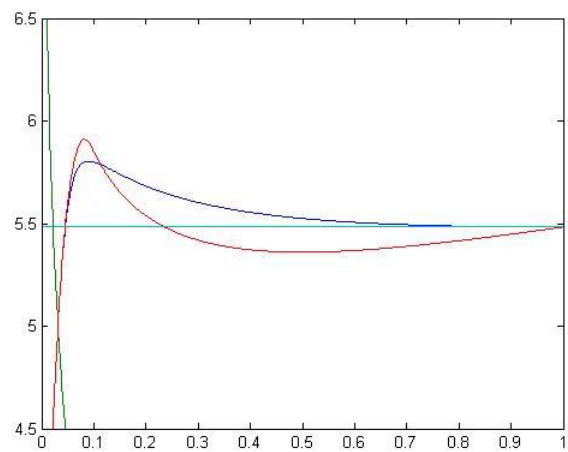

Fig. 3. Achievable sum rates for IAN, TDM/FDM, symmetric and asymmetric superposition for $0 \leq a \leq 1$ and P=1000. The IAN curve is the leftmost one, dropping fast as a increases. The TDM/FDM is the flat level at approximately 5.5. The symmetric superposition is the peaky curve with the peak at a=0.1, and the asymmetric superposition is the last curve, that beats TDM/FDM for and extended set of interference coefficients.

If we plot SR_{23} as a function of α_1 and α_2 we obtain the humming bird function shown in Fig. 4. We note that the true bound on sum rate is the minimum of SR_{23} and SR_4. and SR_4 is plotted in Fig. 5. We call it the shroud function. As P increases, the shroud is lifted to uncover efficient points of the humming bird.

Given these functions we can compare pairwise the various communication alternatives, still restricting to pure modes, i.e., not considering time sharing (or convexification) between modes. Fig. 7 shows a pairwise comparison between asymmetric superposition and TDM/FDM. The region above the curve boundary is where superposition gives higher sum rates.

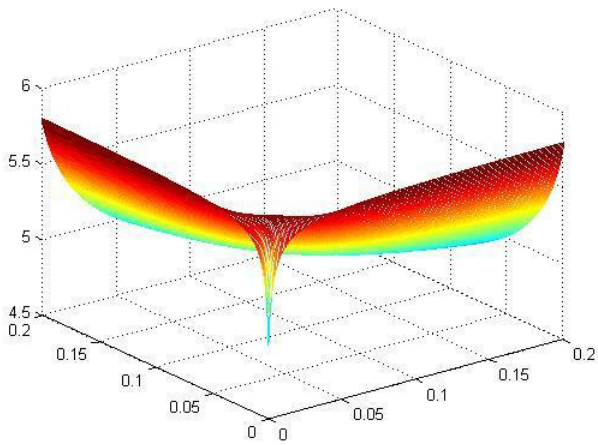

Fig. 4: The humming bird function shows the rate sum for symmetric superposition (α_1=α_2) and asymmetric superposition ((α_1 variable and α_2=0, or α_1=0 and α_2 variable).

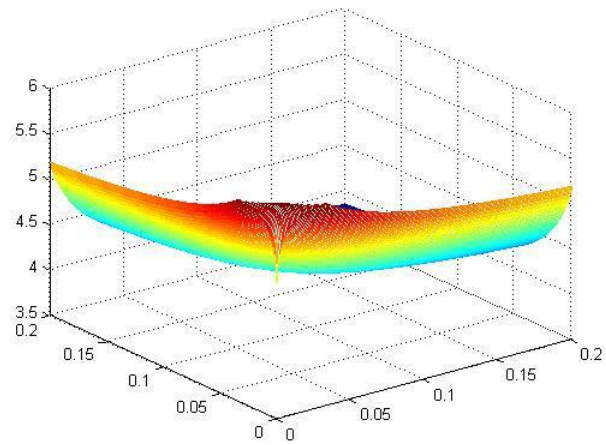

Fig. 6. This curves shows the true bound on sum rate, giver by the min(SR_{23}, SR_4). The chosen value of P is just before the point where symmetric superposition loses to asymmetric superposition, moving the maximum from the head to the wing of the humming bird.

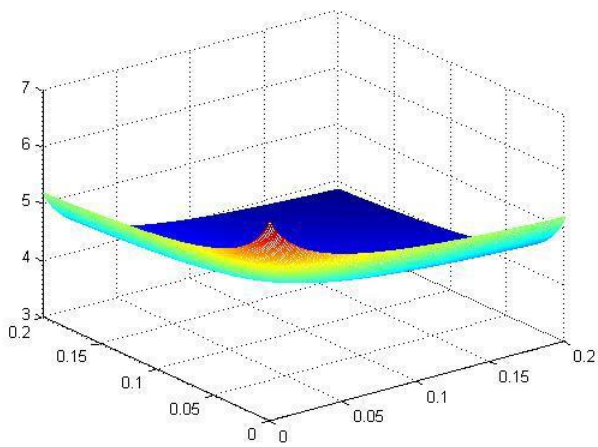

Fig. 5: The shroud function shows the rate sum bounded by SR_4. Note that symmetric superposition corresponds to the diagonal (α_1=α_2) and asymmetric superposition ((α_1 variable and α_2=0, or α_1=0 and α_2 variable) corresponds to the ordinate edges.

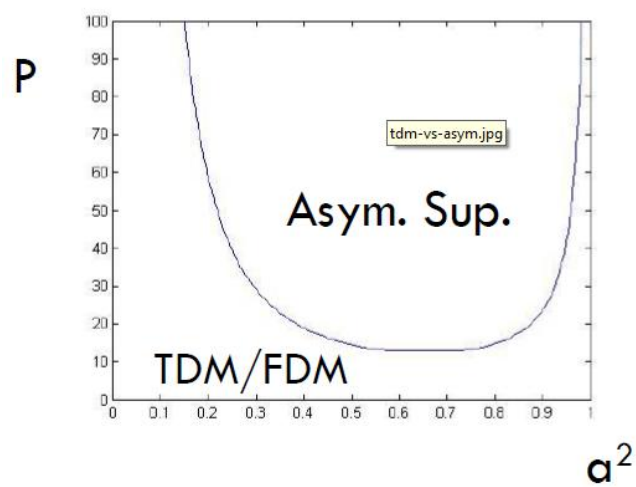

Fig. 7. Pairwise comparison between asymmetric superposition and TDM/FDM.

Extending these pairwise comparisons to the other pairs of strategies, we obtain a full taxonomy of leading schemes to operate in the interference channel. This is shown in Fig. 8, where all the pairwise boundaries are shown. A simplified diagram of strategies, including only the winning method for each region, is shown in Fig. 9. The asymptotic boundary between asymmetric and symmetric superposition happens at approximately $a^2 = 0.087$.

Fig 10 shows the phase transitions in water for different points in the temperature x pressure plane. It interesting to note a pleasant resemplance between the two boundaries, both of which include a triple point.

In all these comparisons we have not considered time sharing between the pure modes. As this is included in the analysis, the frontiers between these modes will become fuzzy, to

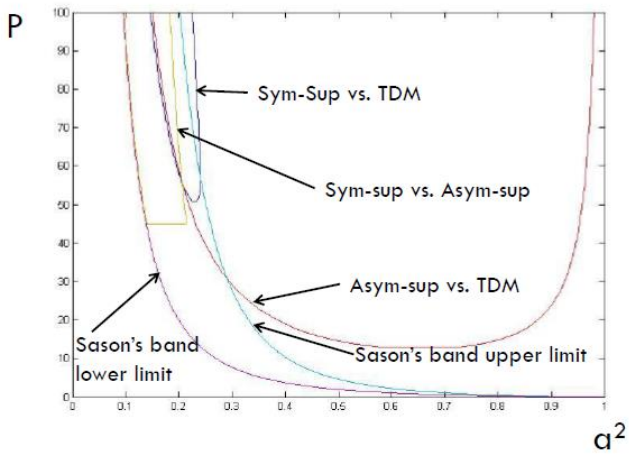

Fig. 8. Boundaries produced by pairwise comparisons of the two superposition schemes (symmetric and asymmetric) and TDM/FDM. Also shown is the Sason's band of symmetric superposition [17].

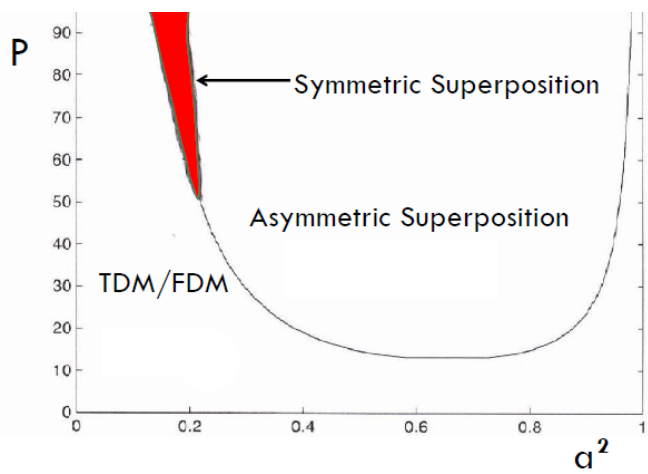

Fig. 9. Leading strategies for the maximizing the sum rate in the moderate interference region ($a^2(1 + a^2P) \geq 1$).

include the transitions caused by convexification, just as we achieved an intermediate mixed mode region between IAN and TDM/FDM in [17]. Fig. 11 shows the typical extent of the transition region between modes. When these transitions are considered, the complete space of weak and moderate interferences delimited by $0 \leq a \leq 1$ and $P \geq 0$, is divided in eight regions, four pure mode areas and four transitional areas, corresponding to mixings of IAN and TDM/FDM, TDM/FDM and symmetric superposition, TDM/FDM and asymmetric superposition, and finally the one between the two modes of superposition.

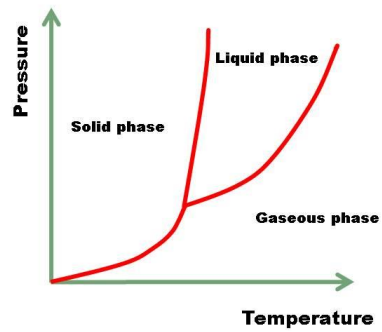

Fig. 10. Phase transitions in water.

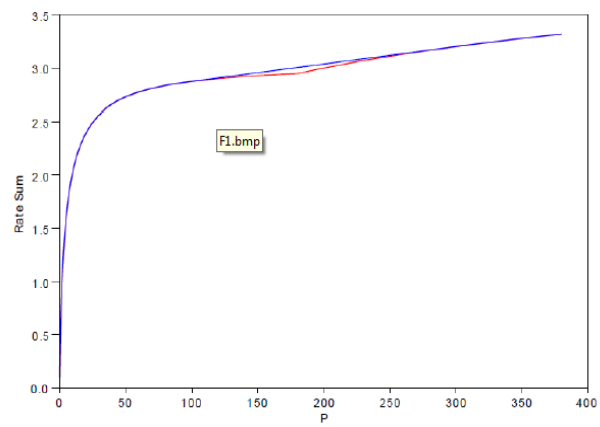

Fig. 11. Typical convex envelopes between pure modes of combating interference.

Note: We have been recently pointed to the findings of [18] that also explore the boundaries of asymmetric and symmetric superposition and orthogonal signaling, and that obtain similar results.

III. Acknowledgement

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