Phase Transitions in the Achievable Sum-Rate of Symmetric Gaussian Interference Channels

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Abstract -- We consider the rate sum of a symmetric Gaussian interference channel in standard format, given by \( Y_1 = X_1 + aX_2 + Z_1 \) and \( Y_2 = aX_1 + X_2 + Z_2 \), with \( a \) an interference coefficient, \( X_1 \) and \( X_2 \) constrained to average power \( P \), and \( Z_1 \) and \( Z_2 \) distributed as \( N(0,1) \). We provide a complete taxonomy of potentially optimal Gaussian signaling strategies in the entire weak and moderate interference parameter space, delimited by \( 0 < a < 1 \) and \( P > 0 \). The strategies seem to lead to the Han and Kobayashi achievable region for the rate sum. We find that the parameter space comprises eight (potentially) optimal transmission regions formed by four pure mode strategies and four transitional strategies. We characterize the boundaries of these pure mode regions and the associated transitional regions.

Index Terms— Gaussian interference channel, Gaussian symmetric interference channel, time division multiplex (TDM), frequency division multiplex (FDM), symmetric superposition, asymmetric superposition, Han and Kobayashi region.

I. INTRODUCTION

The Gaussian interference channel is a classical model in multiple user interference channel for which the capacity region is still unknown. This model has been analyzed by various researchers [1-18], and the capacity region is known only in the special cases of strong \((a \geq 1)\) and very strong \((a^2 \geq 1 + P)\) interference. To reduce the dimensionality of the parameter space and simplify the problem, we consider the sum rate of the symmetric version of this model. The aim is to maximize the achievable sum rate for the channel. The model, depicted in Fig.1, is given by \( Y_1 = X_1 + aX_2 + Z_1 \) and \( Y_2 = aX_1 + X_2 + Z_2 \), with \( X_1 \) and \( X_2 \) constrained to average power \( P \), and \( Z_1 \) and \( Z_2 \) distributed as \( N(0,1) \). The constant \( a \) plays the role of an interference coefficient.

This problem has been solved for capacity region in the strong interference regime \((a \geq 1)\) [2,6,7]. It has also been solved for sum rate in the very weak interference regime \((2a(1 + a^2 P) \leq 1)\) [13, 14, 15].

When the interference coefficient \( a \) satisfies \( 0 < a < 1 \) we can identify two regions in the \((a, P)\) parameter space, that we call weak and moderate interference regions. When \( a^2 + a^4 P \leq 1 \), the efficient strategies seem to be limited to treating interference as noise (IAN) or avoiding interference all together by time or frequency division multiplexing (TDM/FDM), or a combination of these two strategies, as shown in [17]. We call this the region of weak interference. The complementary region, delimited by \( a^2 + a^4 P < 1 \) and \( a < 1 \), is what we call the moderate interference region. In this region, superposition can be a superior strategy to TDM/FDM in some cases. This paper is focused on the tradeoffs between the strategies of superposition (symmetric and asymmetric) and TDM/FDM that occur in this region.

II. PRELIMINARIES

In an earlier paper [17] we examined the trade-off between the strategies of treating interference as noise
(IAN) and TDM/FDM that occur in the \((a,P)\) region delimited by \(a^2(1 + a^2 P) \leq 1\). In the complementary region, when \(a^2(1 + a^2 P) > 1\), we have seen that superposition can be superior to TDM/FDM. In this paper we take a closer look at the trade-off between the alternatives of symmetric and asymmetric superposition and the alternative of avoiding interference with multiplexing. The various constraints that characterize the sum rate in the Han and Kobayashi region can be simplified to the four inequalities given below:

\[
R_1 + R_2 \leq I(X_1; Y_1 | U_1, Q) + I(X_2; Y_2 | U_1, Q) \quad (1)
\]

\[
R_1 + R_2 \leq I(X_1; Y_1 | U_1, U_2, Q) + I(U_1, X_2; Y_2 | Q) \quad (2)
\]

\[
R_1 + R_2 \leq I(U_2, X_1; Y_1 | Q) + I(X_2; Y_2 | U_1, U_2, Q) \quad (3)
\]

\[
R_1 + R_2 \leq I(U_2, X_1; Y_1 | U_1, Q) + I(U_1, X_2; Y_2 | U_2, Q) \quad (4)
\]

These inequalities are associated with points in the rate plane \((R_1, R_2)\) below, marked as \(SR_i\), \(i=1,2,3,4\). We note that \(SR_1\) is a redundant constraint. The corner points \(SR_2\) and \(SR_3\) can be combined in a single constraint given by their time shared average. Finally \(SR_4\) represents an additional constraint. The combined effect of these constraints can be summarized in two bounds, represented by \(SR_4\) (Eq. (4)), and the average of \(SR_2\) and \(SR_3\) (Eqs. (2) and (3)). If we use Gaussian signals for the private and common messages of both senders, and do not use the time sharing parameter \(Q\) for the time being (letting \(Q=0\)), we can represent these two bounds by the following inequalities:

\[
R_1 + R_2 \leq SR_{23} = \frac{1}{2} \log \left( \frac{(1 + \alpha_1 P + a^2 \alpha_2 P)}{(1 + a^2 \alpha_1 P)(1 + a^2 \alpha_2 P)} \right) + \frac{1}{2} \log \left( \sqrt{(1 + a^2 \alpha_1 P + \alpha_2 P)(1 + a_1 P + a^2 \alpha_2 P)} \right) \quad (5)
\]

\[
R_1 + R_2 \leq SR_4 = \frac{1}{2} \log \left( \frac{(1+\alpha_1 P+a^2 P)(1+a_1 P+a^2 \alpha_2 P)}{(1+a^2 \alpha_1 P)(1+a^2 \alpha_2 P)} \right), \quad (6)
\]

where we have represented the average of \(SR_2\) and \(SR_3\) by \(SR_{23}\).

**Fig. 2.** Rate points in the \((R_1, R_2)\) plane that lead to constraints in the sum rate.

**Fig. 3.** Achievable sum rates for IAN, TDM/FDM, symmetric and asymmetric superposition for \(0 \leq a \leq 1\) and \(P=1000\). The IAN curve is the leftmost one, dropping fast as \(a\) increases. The TDM/FDM is the flat level at approximately 5.5. The symmetric superposition is the peaky curve with the peak at \(a=0.1\), and the asymmetric superposition is the last curve, that beats TDM/FDM for an extended set of interference coefficients.

If we plot \(SR_{23}\) as a function of \(\alpha_1\) and \(\alpha_2\) we obtain the humming bird function shown in Fig. 4. We note that the true bound on sum rate is the minimum of \(SR_{23}\) and \(SR_4\), and \(SR_4\) is plotted in Fig. 5. We call it the shroud function. As \(P\) increases, the shroud is lifted to uncover efficient points of the humming bird.

Given these functions we can compare pairwise the various communication alternatives, still restricting to pure modes, i.e., not considering time sharing (or convexification) between modes. Fig. 7 shows a pairwise comparison between asymmetric superposition and TDM/FDM. The region above the curve boundary is where superposition gives higher sum rates.
Fig. 4: The humming bird function shows the rate sum for symmetric superposition ($\alpha_1=\alpha_2$) and asymmetric superposition ($\alpha_1$ variable and $\alpha_2=0$, or $\alpha_1=0$ and $\alpha_2$ variable).

Fig. 5: The shroud function shows the rate sum bounded by $SR_4$. Note that symmetric superposition corresponds to the diagonal ($\alpha_1=\alpha_2$) and asymmetric superposition ($\alpha_1$ variable and $\alpha_2=0$, or $\alpha_1=0$ and $\alpha_2$ variable) corresponds to the ordinate edges.

Extending these pairwise comparisons to the other pairs of strategies, we obtain a full taxonomy of leading schemes to operate in the interference channel. This is shown in Fig. 8, where all the pairwise boundaries are shown. A simplified diagram of strategies, including only the winning method for each region, is shown in Fig. 9. The asymptotic boundary between asymmetric and symmetric superposition happens at approximately $\alpha^2 = 0.087$.

Fig. 6. This curves shows the true bound on sum rate, giver by the $\min(SR_{23}, SR_4)$. The chosen value of $P$ is just before the point where symmetric superposition loses to asymmetric superposition, moving the maximum from the head to the wing of the humming bird.

Fig. 7. Pairwise comparison between asymmetric superposition and TDM/FDM.

Fig 10 shows the phase transitions in water for different points in the temperature x pressure plane. It interesting to note a pleasant resemblance between the two boundaries, both of which include a triple point.

In all these comparisons we have not considered time sharing between the pure modes. As this is included in the analysis, the frontiers between these modes will become fuzzy, to
Fig. 8. Boundaries produced by pairwise comparisons of the two superposition schemes (symmetric and asymmetric) and TDM/FDM. Also shown is the Sason’s band of symmetric superposition [17].

Fig. 9. Leading strategies for the maximizing the sum rate in the moderate interference region ($a^2(1 + a^2P) \geq 1$).

include the transitions caused by convexification, just as we achieved an intermediate mixed mode region between IAN and TDM/FDM in [17]. Fig. 11 shows the typical extent of the transition region between modes. When these transitions are considered, the complete space of weak and moderate interferences delimited by $0 \leq a \leq 1$ and $P \geq 0$, is divided in eight regions, four pure mode areas and four transitional areas, corresponding to mixings of IAN and TDM/FDM, TDM/FDM and symmetric superposition, TDM/FDM and asymmetric superposition, and finally the one between the two modes of superposition.

Note: We have been recently pointed to the findings of [18] that also explore the boundaries of asymmetric and symmetric superposition and orthogonal signaling, and that obtain similar results.

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REFERENCES


