Shannon Meets Walras on Interference Networks

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***Abstract*—In modern wireless communication networks, the layers of the protocol stack close ranks. Technology based layers like the PHY and MAC layer are developed considering assumptions and constraints on the service and application layers. The coexistence of several wireless transmission links operated by different users or operators requires interference coordination on the PHY and MAC. Bilateral agreements or policies consider business aspects and regulatory specifications. In this context, models from multiuser information theory are combined with microeconomic models. There are several connections between both areas: the utility functions and capacities or achievable rates, the strategy spaces and resources or coding schemes, the budget sets and the constraints on powers or rates. In this paper, we focus on one market equilibrium, the Walras equilibrium, and develop a distributed algorithm which finds an efficient operating point for three representative interference channel models: power allocation and single user decoding, beamforming and single user decoding, and rate splitting with successive decoding.**

I. Introduction

Heterogeneous dense wireless networks require the flexible allocation of resources to nodes and network operators. Since simultaneous resource usage results in interference on the physical layer, and the decision for resource sharing is performed at service and network layer, a cross-layer design is required.

One approach to study the conflict situation of coexistence and resource sharing on the physical layer is based on game theory [1]. Depending on the system operation, non-cooperative or cooperative game theory is applied. A coordination mechanism is proposed for the two-user Gaussian interference channel for a simplified Han-Kobayashi scheme in [2]. Recently, methods from microeconomy are also applied to the spectrum allocation and sharing scenario [3]. For hierarchical spectrum sharing, a microeconomic model is developed in [4]. A market model for multi-carrier interference networks is proposed in [5], [6]. Specifically, pricing is proposed to connect service and application oriented and technology oriented layers [7]. Pricing for higher layer interaction between Internet service providers is proposed in [8].

In order to support these developments, we bring multiuser information theoretic models and market models together. After the meeting of Shannon with Nash in [9], we continue with a meeting with Walras. Promising results are reported in [10] for a multiple-input single-output (MISO) interference channel (IC). A more general framework is developed in [11] and the Walras equilibrium is computed for three interference channel scenarios (MISO IC, protected and shared single antenna IC, and multi-carrier IC).

In this paper, we present a market model for the interference channel with a simplified Han-Kobayashi transmission scheme, define goods and budget sets, show the properties of the utility function and compute the Walras equilibrium. First, we perform these steps for the interference channel with rate splitting in detail, then review the steps for the MISO IC with SUD and finally for protected and shared bands. Finally, the main steps are described and discussed. We conclude that the framework can be applied to a number of interference channel scenarios.

II. System and Market Model

A. Mapping of Terms

In Table I, the mapping between the terms and parameters of the competitive market model and the information theoretic model is illustrated. On the left hand side, a selection of terms for market models is listed. On the right side, important terms from data transmission on the physical layer (PHY) and multiple access control (MAC) layer are listed. We show one example mapping between the terms. It is important to stress that this mapping is flexible and should be adapted to the scenario.

Market Model Terms		Wireless Transmission Terms	
Agents	Consumers	Nodes	Transmitters
	Producers		Receivers
Goods			Coding/Decoding
			Precoding and Signal Processing
			Resource Allocation
Budgets			Power Constraints
			Spectral Mask Constraints
			Energy Constraints
Utilities			Achievable Rates
			Capacities
			Estimation Errors
			Error Performances: BER, BLER
Market		Wireless Network	
Equilibrium		Operating point	

TABLE I
Mapping between market model and data transmission (PHY and MAC) model terms.

The active entities in both models act and thereby influence the state of the market or system. Agents either produce or consume goods by supplying them for a certain price or by buying them. Transmitters demand resources (including spectrum and power) when transmitting their coded and processed data to the channel. Receivers observe signals, process and decode the data they are interested in. Consumers are constrained by their budget set, transmitters usually have power or resource constraints as well. The satisfaction of agents is measured by their utility functions or preference relations. Typical utility functions in wireless networks include achievable rates, capacities, estimation or bit error performance. The market equilibrium describes a reasonable outcome of the competitive market and this corresponds to a certain operating point of the wireless interference system.

Clearly, Claude Shannon is the founder of information theory [12]. The developments in microeconomics have a longer history. One such development is started by Cournot who has contributed to describe the outcome of a duopoly using the calculus of functions. Walras developed a general microeconomic equilibrium theory and his law states that in equilibrium the excess market demands sum up to zero [13]. In a fictitious discussion between Shannon and Walras[1] the relation between the terms of Table I is a very likely topic.

In the following, we present three example mappings between typical wireless system scenarios and their corresponding market models. We start with a simplified rate splitting model for the interference channel and then briefly review two further models.

B. System Model

Consider the Gaussian single antenna interference channel in standard form [14] with cross channel gains a_{12} for the link between transmitter one and receiver two and a_{21} for the link between transmitter two and receiver one, respectively. We will assume that the system operates in the weak and modest interference regime, i.e., $a_{12} \leq 1, a_{21} \leq 1$. The transmit power constraint at both transmitters is set to P and the noise power is normalized to one.

We assume that the links apply a special case of the scheme which achieves the Han-Kobayashi inner bound [15]: Rate splitting into private messages with rates R_{11} and R_{22} and public messages with rates R_{12} and R_{21} is applied at the transmitters. At the receivers successive decoding is performed in a specific order: Power $\lambda_1 P$ is allocated for the private message part of the codeword and $\bar{\lambda}_1 P = (1-\lambda_1)P$ is allocated for the common message part of the codeword. The common message is decoded by both receivers, the private message only by the intended receiver. The decoding order is as follows: First, the public message of the other link and then, the own public and finally the own private message is decoded. For the Gaussian interference channel, the following conditions [16, Chapter 6] limit the achievable rates for the interference channel

$$R_{11} \leq C\left(\frac{\lambda_1 P}{1+\lambda_2 P a_{21}}\right), \quad R_{22} \leq C\left(\frac{\lambda_2 P}{1+\lambda_1 P a_{12}}\right), \qquad (1)$$

$$R_{12} \leq C\left(\frac{\bar{\lambda}_1 P}{1+\lambda_1 P+\lambda_2 P a_{21}}\right), \qquad (2)$$

$$R_{12} \leq C\left(\frac{\bar{\lambda}_1 a_{12} P}{1+P+\lambda_1 a_{12} P}\right), \qquad (3)$$

$$R_{12} \leq C\left(\frac{\bar{\lambda}_2 P}{1+\lambda_2 P+\lambda_1 P a_{12}}\right), \qquad (4)$$

$$R_{12} \leq C\left(\frac{\bar{\lambda}_2 a_{21} P}{1+P+\lambda_2 a_{21} P}\right), \qquad (5)$$

with $C(x) = \log(1+x)$. The total data rate of link one is the sum of the private and common message rates $R_1 = R_{11} + R_{12}$ and of link two $R_2 = R_{22} + R_{21}$, respectively. Clearly, R_1, R_2 depend on the rate splitting parameter λ_1, λ_2 and we write explicitly

$$R_1(\lambda_1, \lambda_2) = C\left(\frac{\lambda_1 P}{1+\lambda_2 P a_{21}}\right) + \qquad (6)$$
$$\min\left[C\left(\frac{\bar{\lambda}_1 P}{1+\lambda_1 P+\lambda_2 P a_{21}}\right), C\left(\frac{\bar{\lambda}_1 a_{12} P}{1+P+\lambda_1 a_{12} P}\right)\right].$$

The total rate $R_2(\lambda_1, \lambda_2)$ is derived analogue. The rate R_1 is increasing in λ_1 and decreasing in λ_2.

Note that the second term in the minimum in (6) is always smaller than the first term in (6) because

$$\frac{1}{1+\lambda_1 P+\lambda_2 P a_{21}} \geq \frac{a_{12}}{1+P+\lambda_1 a_{12} P}$$
$$\Leftrightarrow \; a_{12} + \lambda_2 P a_{12} a_{21} \leq 1+P, \qquad (7)$$

which is always true for $a_{12} \leq 1$ and $a_{21} \leq 1$. Therefore, the rates expression reduces with $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]$ to

$$R_1(\boldsymbol{\lambda}) = C\left(\frac{\lambda_1 P}{1+\lambda_2 P a_{21}}\right) + C\left(\frac{\bar{\lambda}_1 a_{12} P}{1+P+\lambda_1 a_{12} P}\right). \qquad (8)$$

For $\lambda_1 = \lambda_2 = 1$, the single-user decoding SUD rates are obtained $R_1(1,1) = C\left(\frac{P}{1+Pa_{21}}\right)$ and $R_2(1,1) = C\left(\frac{P}{1+Pa_{12}}\right)$.

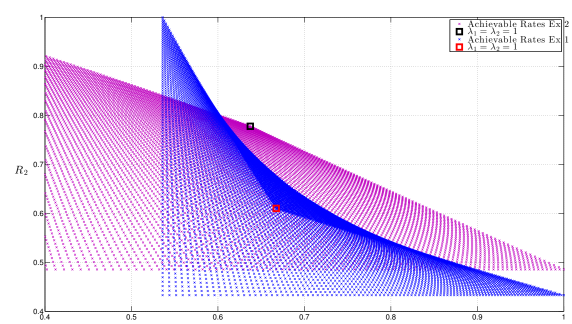

Fig. 1. Achievable rate region for simplified rate splitting in two scenarios: $P = 1, a_{12} = 0.9, a_{21} = 0.7$ and $P = 1, a_{12} = 0.4, a_{21} = 0.8$.

In Figure 1, two examples of achievable rate regions are shown with SUD rate tuple in red. It can be observed that

[1] Such a meeting between Shannon and Walras could not happen because Shannon was born six years later than Walras has died.

depending on the interference power and the transmit power, the SUD rate tuple is close or far from the Pareto boundary.

C. Market Model

For an introduction into Walras models applied to interference channel we refer to [11]. The links are the consumers. We identify the goods of our competitive market to be λ_1 and λ_2. For link one, we define the goods $\lambda_1^{(1)} = \lambda_1$ and $\lambda_2^{(1)} = \bar{\lambda}_2$. For the utility function of link one, we apply the achievable rate $R_1(\boldsymbol{\lambda}^{(1)})$ with $\boldsymbol{\lambda}^{(1)} = [\lambda_1^{(1)}, \lambda_2^{(1)}]$ as follows

$$u_1(\boldsymbol{\lambda}^{(1)}) = C\left(\frac{\lambda_1^{(1)} P}{1 + \bar{\lambda}_2^{(1)} P a_{21}}\right) + C\left(\frac{\bar{\lambda}_1^{(1)} a_{12} P}{1 + P + \lambda_1^{(1)} a_{12} P}\right). \quad (9)$$

Note the following special cases: $u_1(1,0) = \log(1 + P/(1 + Pa_{21}))$ corresponds to SUD, $u_1(1,1) = \log(1+P)$ corresponds to the point to point link, $u_1(0,1) = \log(1 + (Pa_{12})/(1+P))$ corresponds to the multiuser decoding (MUD) and $u_1(0,0)$ corresponds to SUD again. The properties of the utility function in (9) are collected in the following proposition proved in the Appendix A.

Proposition 1. *The utility function u_1 defined in (9) is*
1) *strongly increasing in $\boldsymbol{\lambda}^{(1)}$ and*
2) *quasi-concave in $\boldsymbol{\lambda}^{(1)}$.*

In the proof of Proposition 1 the indifference curves are computed. Therefore, an illustration of the Edgeworth box in Figure 2 shows one example in which the two indifference curves are tangent and the resulting operating point corresponds to a point on the Pareto boundary of the achievable rate region. For further information on the Edgeworth box representation, the interested reader is referred to [11].

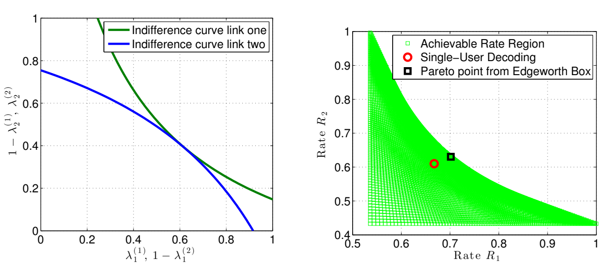

Fig. 2. Tangent indifference curves in Edgeworth box correspond to point on the Pareto boundary in achievable rate region. The rate constants are set to $\gamma_1 = 0.7016$ and $\gamma_2 = 0.6305$.

Next, we define the budget set of link one as

$$\mathcal{B}_1(\beta) = \{(\lambda_1^{(1)}, \lambda_2^{(1)}) \in [0,1]^2 : \lambda_1^{(1)} + \frac{q_2}{q_1}\lambda_2^{(1)} \leq 1\} \quad (10)$$

with prices q_1 and q_2 for good one $\lambda_1^{(1)}$ and good two $\lambda_2^{(1)}$, respectively. We define the price ratio $\beta = \frac{q_2}{q_1}$. The budget set of link two is defined analogously. The endowment vectors $\boldsymbol{e}$ for the two links are $[1,0]$ and $[0,1]$, because their start point for negotiation is SUD.

In the Walrasian market approach, the two links are identified with two consumers, who can buy goods with their budget described in $\mathcal{B}_k$. The corresponding utility maximization problem (UMP) for consumer one reads

$$\boldsymbol{d}_1(\beta) = \boldsymbol{d}_1(q_1, q_2) = \arg \max_{(\lambda_1^{(1)}, \lambda_2^{(1)}) \in \mathcal{B}_1} u_1(\lambda_1^{(1)}, \lambda_2^{(1)}), \quad (11)$$

with demand vector $\boldsymbol{d}_1 = [d_{1,1}, d_{1,2}]$. Note that the optimum of (11) is achieved by a point on the boundary $\lambda_1^{(1)} + \beta\lambda_2^{(1)} = 1$ of the budget set. Then, the utility function can be expressed over $\lambda_1^{(1)}$ within the range

$$\max(0, 1-\beta) \leq \lambda_1^{(1)} \leq 1.$$

The UMP for link two is similar. Based on the UMPs for both links, the demands $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ as a function of the price ratio β can be computed.

D. Walras Equilibrium and Distributed Algorithm

Walras desccribes in [13] a potential outcome of the competitive market model. In fact, the model is more complicated then our model from the last section because it contains also producers which supply the market with goods. In order to define the Walras equilibrium, we need the excess demand function for both goods [11, Section 2] with prices $\boldsymbol{q} = [q_1, q_2]$

$$z_1(\boldsymbol{q}) = d_{1,1} + d_{2,1} - 1 \text{ and } z_2(\boldsymbol{q}) = d_{1,2} + d_{2,2} - 1. \quad (12)$$

Collect the excess demand functions in one vector $\boldsymbol{z}(\boldsymbol{q}) = [z_1(\boldsymbol{q}), z_2(\boldsymbol{q})]$ and define

Definition 1 (Walras Equilibrium)**.** *A Walrasian equilibrium consists of a price vector $\boldsymbol{q}^* \in \mathbb{R}_{++}^2$ such that $\boldsymbol{z}(\boldsymbol{q}^*) = \mathbf{0}$ and the allocation of the goods to the consumers according to their demand.*

At the Walras equilibrium, the demand equals the supply and the market clears. For our market model described in the last section, this means that $\lambda_2^{(1)} + \lambda_2^{(2)} = 1$ and $\lambda_1^{(1)} + \lambda_1^{(2)} = 1$.

The condition for the existence of a Walras equilibrium are stated in [17, Theorem 5.5]. In our market model, we satisfy the two conditions:

1) The utility functions u_1 and u_2 are continuous, strongly increasing, and strictly quasi-concave as shown in Proposition 1.
2) A quantity of each good is initially possessed by at least one consumer. In our model, link one possess all of good one and link two all of good two.

The uniqueness of a Walras equilibrium can be answered checking the gross substitute property (GSP) as described in [18, Proposition 17.F.3]. The GSP can be written for our competitive market model as

$$\frac{\partial z_1(\boldsymbol{q})}{\partial q_2} > 0 \text{ and } \frac{\partial z_2(\boldsymbol{q})}{\partial q_1} > 0. \quad (13)$$

It says that the excess demand of good one should increase if the price of good two is increased and vice versa. For the UMP and corresponding demand function in (11), we can show in Appendix B that the GSP holds.

Proposition 2. *For the utility functions u_1 and u_2 and the corresponding budget sets $\mathcal{B}_1(\beta)$ and $\mathcal{B}_2(\beta)$ the GSP in (13) holds.*

One important efficiency property of the Walras equilibrium is its Pareto optimality as stated in the following First Welfare Theorem [17, Theorem 5.7].

Theorem 1 (First Welfare Theorem)**.** *Consider an exchange economy with utilities u_1, u_2 and initial endowments $\boldsymbol{e}_1, \boldsymbol{e}_2$. If each consumer's utility function u_i is strictly increasing, then every Walrasian equilibrium allocation is Pareto efficient.*

The final question remains how to compute or achieve the Walrasian equilibrium. The corresponding problem is to compute the prices such that the excess demand is zero for all goods. An iterative algorithm called tâtonnement process adjusts the prices according to the excess demand. Here, we consider the discrete update variant provided in [19] with the following update rule

$$q_i^{(t+1)} = \left[q_i^{(t)} + a_i z_i(\boldsymbol{q}^{(t)}) \right]_0, \qquad (14)$$

for all i and some step size parameter $a_i > 0$. It is shown in [19] that the updates in (14) are globally convergent if the aggregate excess demand satsfies the GSP in (13).

For our market model, we have shown in Proposition 2 that the GSP is satisfied for all goods and thereby the aggregate excess demand function also possesses the GSP. The update rule (14) can be applied to compute the Walrasian prices.

In practice, the update algorithm in (14) can be implemented in two variants: An arbitrator sends the prices to the two links, receives the demand and updates to new prices according to (14). Another implementation is to collect the channel gains a_{12} and a_{21} and the transmit power P and compute the Walras equilibrium offline at some arbitrator's processor. It depends on the number of iterations required to converge to the Walrasian equilibrium to determine which implementation has lower complexity.

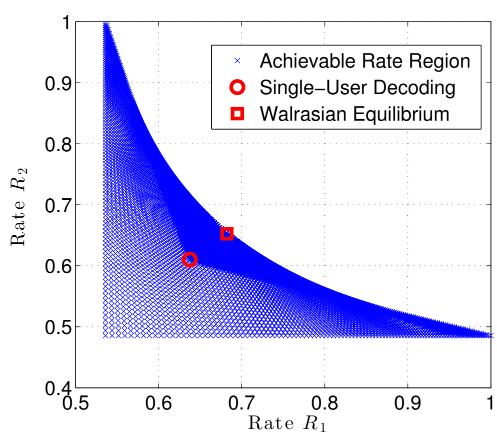

Fig. 3. The Walrasian equilibrium achieves a point on the Pareto boundary. The tâtonnement process needs two iterations to converge.

In Figure 3, we show the simulation of the Walras equilibrium in the achievable rate region. We start from the SUD point and move in two iterations straight to the Pareto boundary. For the rate-splitting interference channel model, we conclude that the market model and the computation of the corresponding Walras equilibrium leads to the development of an iterative distributed algorithm to compute an efficient operating point in the achievable rate region.

III. General Framework and Two Special Cases

The interaction between two competing links in a wireless interference network can be modelled as a competitive market. In this section, we describe a general framework including a recipe how to apply it to your own interference network model.

Our simple exchange economy market model consists of a set of consumers $\mathcal{K}$ with utility function $u_k, k \in \mathcal{K}$, a set of goods $\mathcal{X} = \{x_1, ..., x_N\}$ and some initial endowment of consumer k given by $\boldsymbol{e}_k, k \in \mathcal{K}$. The benefit of owning good n is expressed by a utility function $u_k(\boldsymbol{x})$ which maps from the set of goods to the non-negative real numbers.

When designing the goods and utility function, the following properties should be considered:

1) The utility function should be continuous in $\boldsymbol{x}$ strictly increasing in the goods, i.e., $\frac{\partial u_k(\boldsymbol{x})}{\partial x_n} > 0$ for all $k \in \mathcal{K}$ and $n \in [1, ..., N]$.
2) The utility function should be quasi-concave with respect to $\boldsymbol{x}$, i.e.,
$$u_k(t\boldsymbol{x}_1 + \bar{t}\boldsymbol{x}_2) > \min(u_k(\boldsymbol{x}_1), u_k(\boldsymbol{x}_2)).$$
For an exchange economy with two goods, this requirement is equivalent to the convexity of the indifference curves in the Edgeworth box (as in the proof of Proposition 1).
3) The utility function should satisfy the gross substitute property, i.e., the excess demand of good n should increase with increasing price of any other good.

If the properties described above are fulfilled by the designed exchange economy, then the Walrasian Equilibrium exists, is unique and can be computed via the tâtonnement process. In order to show that these requirements can be met naturally, we present two examples below.

A. MISO interference channel

In this section, we present a comprehensive summary of the system model and market model for the MISO interference channel with single-user decoding. The properties of the utility function are reviewed. It is shown that all results regarding the existence of the Walras equilibrium and its computation can be applied. For a complete derivation including proofs, the interested reader is referred to [10].

We consider the two user MISO interference channel with SUD, beamforming vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ under transmit power constraints $||\boldsymbol{w}_1||^2 \leq 1$, $||\boldsymbol{w}_2||^2 \leq 1$, quasi-static flat-fading vector channels $\boldsymbol{h}_{kl}$ from transmitter k to receiver l. The utility function is the achievable rate with SUD. Since the beamforming vectors cannot be represented as goods, we exploit the parameterization derived in [20] and write the utility function (SINR) of link (consumer) one as [11, Eq. (1.40)]

$$u_1(\lambda_1^{(1)}, \lambda_2^{(1)}) = \frac{\left(\sqrt{\lambda_1^{(1)} g_1} + \sqrt{\bar{\lambda}_1^{(1)} \hat{g}_1}\right)^2}{1 + \lambda_2^{MRT} g_{21} - \lambda_2^{(1)} g_{21}}, \tag{15}$$

where $g_1 = ||\Pi_{\boldsymbol{h}_{12}} \boldsymbol{h}_{11}||^2$, $\hat{g}_1 = ||\Pi^{\perp}_{\boldsymbol{h}_{12}} \boldsymbol{h}_{11}||^2$, and $g_{21} = ||\boldsymbol{h}_{21}||^2$. λ_2^{MRT} is the parameter which corresponds to the maximum ratio beamformer (MRT). And the utility function for the second link is defined analogously.

In the market model, the goods are the parameters for the beamforming vectors $0 \leq \lambda_1^{(1)} \leq \lambda_1^{MRT}$ and $0 \leq \lambda_2^{(1)} \leq \lambda_2^{MRT}$. First, we check the properties from the list above and review the following results.

Theorem 2 (Theorem 1 in [10])**.** *The utility function defined in (15) is continuous, strongly increasing, and strictly quasi-concave.*

Furthermore, the GSP also holds for the market model described above. The excess demand for the market is computed in [11, Eq. (1.86)]. It is obvious that increasing the price q_2 increases the aggregate excess demand of good one and vice versa. Thus the iterative algorithm described above converges to a Pareto efficient outcome on the Pareto boundary of the achievable SINR or rate region.

B. SISO interference channel with protected and shared band

In this section, we summarize the system and market model for the SISO interference with protected and shared bands. The properties of the utility function again allow to apply the results on the existence and computation of the Walras equilibrium. For a complete derivation including proofs, the interested reader is referred to [11].

We consider a two user interference channel in which both users have a separate set of bands (protected bands) and a shared band. The channels in the protected band are denoted by $\bar{h}_1$ and $\bar{h}_2$. The interference channel of the shared band in denoted by h_{kl} from transmitter k to receiver l. The utility function is given by a linear approximation of the average SINR

$$\begin{aligned} u_1(\lambda_1^{(1)}, \lambda_2^{(1)}) &= P_1 \bar{h}_1 + \lambda_1^{(1)} \left[\frac{h_{11}}{1 + P_2 h_{21}} - \bar{h}_1\right] \\ &\quad + \lambda_1^{(1)} \lambda_2^{(1)} \frac{h_{11} h_{21}}{1 + P_2 h_{21}}. \end{aligned} \tag{16}$$

The goods are defined by the power allocated to the shared band for link one and one minus the power allocated to the shared band by link two. The analysis of monotonicity and quasi-concavity depends on the channel realizations: If $\bar{h}_1 \leq \frac{h_{11}}{1+P_2 h_{21}}$ then u_1 in (16) is strongly increasing and strictly quasi-concave [11, Lemma 1.5]. If $h_{11} \geq \bar{h}_1 > \frac{h_{11}}{1+P_2 h_{21}}$, the utility u_1 in (16) is neither increasing nor quasi-concave. However, the UMP can be formulated and solved in closed form. For $h_{11} < \bar{h}_1$, the utility function u_1 in (16) is strongly decreasing in $\lambda_1^{(1)}$, by a change of variables a modified strongly increasing and strictly quasi-concave utility function can be developed.

The uniqueness of the Walrasian equilibrium for the market model described above is shown in

Theorem 3 (Theorem 1.10 in [11])**.** *The Walrasian equilibrium is unique for*

$$\bar{h}_1 \leq \frac{h_{11}}{1 + P_2 h_{21}} \quad \textit{and} \quad \bar{h}_2 \leq \frac{h_{22}}{1 + P_1 h_{12}}.$$

For these cases, the Walrasian prices can be computed in closed form. However, by [11, Lemma 1.12] it follows that the GSP is not fulfilled for

$$\bar{h}_1 > h_{11} \quad \text{or} \quad \bar{h}_2 > h_{22}.$$

IV. Conclusion

In this paper, we described the discussion between an information theorist working on interference channels and a microeconomist working on exchange economies. In an interference channel with rate splitting, the rate splitting parameter is identified with goods and we define a utility function and budget sets for which the Walrasian equilibrium exists, is unique and can be found by an iterative price update process. The advantage from an information theoretic point is the Pareto efficiency of the Walras outcome. Two further examples are reviewed for which the exchange economy market model can be applied. In conclusion, there are many more scenarios and application for which the market model and the Walrasian equilibrium is a suitable solution concept.

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Appendix A
Proof of Proposition 1

Monotonicity of u_1 with respect to $\boldsymbol{\lambda}^{(1)}$ follows from

$$\begin{aligned} u_1^{(1)} &= \frac{\partial u_1(\lambda_1^{(1)}, \lambda_2^{(1)})}{\partial \lambda_1^{(1)}} \\ &= \frac{P(1 - a_{12}) + P^2(1 - \bar{\lambda}_2^{(1)} a_{12} a_{21})}{(1 + \lambda_1^{(1)} P + \bar{\lambda}_2^{(1)} P a_{21})(1 + P + \lambda_1^{(1)} P a_{12})} > 0 \\ u_1^{(2)} &= \frac{\partial u_1(\lambda_2^{(1)}, \lambda_2^{(1)})}{\partial \lambda_2^{(1)}} \\ &= -\frac{P a_{21}}{1 + \lambda_1^{(1)} P + \bar{\lambda}_2^{(1)} P a_{21}} + \frac{P a_{21}}{1 + \bar{\lambda}_2^{(1)} P a_{21}} > 0, \end{aligned}$$

where the second inequality is strict for all $\lambda_1^{(1)} \neq 0$. The proof for u_1 strongly increasing is similar to [10, Appendix B]. It is sufficient to show that $u_1^{(1)}$ and $u_1^{(2)}$ are strictly positive. The directional derivative in direction $\boldsymbol{z} = [z_1, z_2]$ with $z_i \geq 0$ is equal to $u_1^{(1)} z_1 + u_1^{(2)} z_2 > 0$ except for the case where both $z_1 = 0$ and $\lambda_1^{(1)} = 0$. The second derivatives of u_1 with respect to $\lambda_2^{(1)}$ and $\lambda_1^{(1)}$ are given by

$$\begin{aligned} u_1^{(1,2)} &= \frac{\partial^2 u_1}{\partial \lambda_2^{(1)} \partial \lambda_1^{(1)}} = \frac{P^2 a_{21}}{(1+\lambda_1^{(1)} P + \bar{\lambda}_2^{(1)} P a_{21})^2} \geq 0 \\ u_1^{(1,1)} &= \frac{\partial^2 u_1}{\partial^2 \lambda_1^{(1)}} = -\frac{P^2}{(1+\lambda_1^{(1)} P + \bar{\lambda}_2^{(1)} P a_{21})^2} \\ &\quad + \frac{a_{12}^2 P^2}{(1+P+\lambda_1^{(1)} p a_{12})^2} \leq 0 \\ u_1^{(2,2)} &= \frac{\partial^2 u_1}{\partial^2 \lambda_2^{(1)}} = -\frac{P^2 a_{21}^2}{(1+\lambda_1^{(1)} P + \bar{\lambda}_2^{(1)} P a_{21})^2} \\ &\quad + \frac{P^2 a_{21}^2}{(1+\bar{\lambda}_2^{(1)} P a_{21})^2} \geq 0. \end{aligned}$$

Studying the Hessian shows that the function u_1 is not jointly concave in $\boldsymbol{\lambda}^{(1)}$. In order to show quasi-concavity, we study the upper contour sets of the function u_1 and show that they are strictly convex [21]. Note that the boundary of the contour sets correspond to the indifference curves [11, Figure 1.3]. For fixed feasible utility γ_1 of link one, we have

$$\begin{aligned} & u_1(\boldsymbol{\lambda}^{(1)}) = \gamma_1 \qquad (17) \\ =\; & \log\left(\frac{(1+\lambda_1^{(1)} P + \bar{\lambda}_2^{(1)} P a_{21})(1+P+P a_{12})}{(1+\bar{\lambda}_2^{(1)} P a_{21})(1+P+\lambda_1^{(1)} P a_{12})}\right). \end{aligned}$$

Solving (17) for $\lambda_2^{(1)}$ with $z = \frac{2^\gamma}{1+P(1+a_{12})}$ gives

$$\lambda_2^{(1)}(\lambda_1^{(1)}) = 1 - \frac{\lambda_1^{(1)}}{z(1+P+\lambda_1^{(1)} P a_{12})} \frac{1}{a_{21}} + \frac{1}{P a_{21}}. \qquad (18)$$

Clearly the function $\lambda_2^{(1)}(\lambda_1^{(1)})$ in (18) is strictly convex and decreasing. The upper contour set is given in the $\lambda_1^{(1)}, \lambda_2^{(1)}$ plane by the set

$$\Lambda = \{[\lambda_1^{(1)}, \lambda_2^{(1)}] \in [0,1]^2 : \lambda_2^{(1)} \geq \lambda_2^{(1)}(\lambda_1^{(1)})\}.$$

Since $\lambda_2^{(1)}(\lambda_1^{(1)})$ is strictly convex and decreasing the set Λ is strictly convex. Therefore, the function u_1 is strictly quasi-concave in $\boldsymbol{\lambda}^{(1)}$. □

APPENDIX B
PROOF OF PROPOSITION 2

Consider the GSP for the excess demand function of good one d_1. The corresponding UMP reads

$$\max_{\boldsymbol{\lambda}^{(1)} \in \mathcal{B}_1(\beta)} u_1(\boldsymbol{\lambda}^{(1)}) \qquad (19)$$

where the price ratio β impacts only the budget set. An increase in price q_2 corresponds to an increase in β whereas an increase in price q_1 corresponds to a decrease in β. The utility function in (19) can be rewritten as

$$\begin{aligned} \max_{[1-\beta]^+ \leq \lambda_1^{(1)} \leq 1} & \underbrace{\log\left(1 + \frac{\lambda_1^{(1)} P}{1 + (1-\frac{1}{\beta}\bar{\lambda}_1^{(1)}) P a_{21}}\right)}_{\phi_1(\lambda_1^{(1)}, \beta)} \\ & + \underbrace{\log\left(1 + \frac{\bar{\lambda}_1^{(1)} P a_{12}}{1 + P + \lambda_1^{(1)} a_{12} P}\right)}_{\phi_2(\lambda_1^{(1)})}. \qquad (20) \end{aligned}$$

The first function ϕ_1 is monotonic increasing in $\lambda_1^{(1)}$ whereas the second function ϕ_2 is monotonic decreasing in $\lambda_1^{(1)}$. In the optimum of (19), it holds

$$\frac{\partial \phi_1(\lambda_1^{(1)}, \beta)}{\partial \lambda_1^{(1)}} = -\frac{\partial \phi_2(\lambda_1^{(1)})}{\partial \lambda_1^{(1)}} \qquad (21)$$

or the optimum is achieved at the boundary. When β is increased, $\phi_2(\lambda_1^{(1)})$ is unchanged but $\phi(\lambda_1^{(1)}, \beta)$ is growing slower in $\lambda_1^{(1)}$ and therefore, the optimum $\lambda_1^{(1)}$ which solves (20) and (21) gets larger. This shows that an increase in q_2 (and thereby an increase in β) leads to an increased demand of good one $\lambda_1^{(1)}$ of consumer one. Similar arguments show that the demand of good one at consumer two is also increased. □

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