Eulerian numbers revisited : Slices of hypercube

Kingo Kobayashi[*], Hajime Sato [†], Mamoru Hoshi [‡], and Hiroyoshi Morita [§]

Abstract

In this talk, we provide a simple proof on an interesting equality connecting the number of permutations of $1, \ldots, n$ with k runs, i.e., Eulerian numbers to the volumes of slices between $k-1$ and k of the n-dimensional hypercube along the diagonal axis. The proof is simple and elegant, but the detail structures in the problem are left to be unclear. In order to get more information on this problem, we give the second proof relied on the direct calculation of the related numbers and the volumes. By computing conditional probabilities with respect to slices, we can obtain the known recurrence relation on Eulerian numbers.

1 Introduction

It is quite interesting to notice relations between combinatorial concepts and probabilistic ones. We provide a typical example of such cases, that is, a strong relation between the number of permutations with k runs and the probability that the sum of independent uniform random variables taking values between 0 and 1 will be between $k-1$ and k. In the next section, we explain the main theorem and give a simple proof for the theorem. Although the idea of the simple proof is indeed elegant, but by the laborious computations of related values in the section 3, we can have much more information around the problem, and encounter interesting identities through an explicit formula of the number of permutations of $1, \ldots, n$ with k runs. In the section 4, we compute the volume of slice between $k-1$ and k of the n-dimensional hypercube along the diagonal axis. The combined results of section 3 and 4 give another proof of the main theorem.

[*]The University of Electro-Communications, Tokyo, JAPAN, E-mail: kingo@ieee.org
[†]Senshu University,Tokyo, JAPAN, E-mail: h-sato@isc.senshu-u.ac.jp
[‡]The University of Electro-Communications,Tokyo,JAPAN,E-mail:mmrhoshi@gmail.com
[§]The University of Electro-Communications, Tokyo, JAPAN, E-mail: morita@is.uec.ac.jp

2 Number of permutations with specified number of runs and Slices of n dimensional Hypercube

Let $F_{n,k}$ be the number of permutations of $1, \ldots, n$ with k runs, where the run is a consecutive ascent sequence followed by a descent or terminated. For example, 24153 has three runs, and 54321 has five runs.

Next, we consider the sum of independent random variables $U_i (i = 1, \ldots, n)$ with uniform distribution on the interval $[0, 1)$. Thus, we will study the probability:

$$\begin{aligned} G_{n,k} &= \Pr\{k-1 \leq U_1 + \cdots + U_n < k\} && (2.1) \\ &= \int_{\substack{k-1 \leq u_1 + \cdots + u_n < k, \\ 0 \leq u_1 < 1, \ldots, 0 \leq u_n < 1}} du_1 \ldots du_n, && (2.2) \end{aligned}$$

that is, the volume of slice which is defined by a region between $k-1$ and k of the n-dimensional hypercube along the diagonal axis. In what follows, for simplicity we omit the conditions of $0 \leq u_1 < 1, \ldots, 0 \leq u_n < 1$ in the integral.

Then, we have a fascinating equality connecting combinatorics and probability theory:

Theorem 2.1 *It holds that*

$$F_{n,k} = n!\, G_{n,k}.$$

Remark 1 It is quite interesting that this is not an approximate, but exact equality. In [2], p.659, it contains a wrong expression:

$$[z^n \ u^k] \ F(z,u) = \mathbb{P}\left\{\lfloor U_1 + \cdots + U_n \rfloor < k\right\},$$

where $F(z,u)$ is the exponential BGF of the number of permutations with specified number of runs, that is,

$$F(z,u) = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{F_{n,k}}{n!} z^n u^k.$$

In the following sections, we will compute precisely the concerned values $F_{n,k}$ and $G_{n,k}$, and show the equality. But, it is not necessary to evaluate the values only for proving the theorem. It is sufficient to translate related polyhedra to appropriate polyhedra in the hypercube.

Here, we give a simple proof of Theorem 2.1:

Proof: Let $\mathcal{Q}$ be the n-dimensional hypercube, that is,

$$\mathcal{Q} = \{\mathbf{u} = (u_1, \ldots, u_n) \ : \ 0 \leq u_1 < 1, \ldots, 0 \leq u_n < 1\}.$$

Define the permutation $\sigma = (\sigma_1, \ldots, \sigma_n) = \sigma(\mathbf{u})$ for the point $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathcal{Q}$ by

$$(u_1, u_2, \ldots, u_n) \to u_{\sigma_1} > u_{\sigma_2} > \cdots > u_{\sigma_n}.$$

Next, we introduce a transformation T from the point $\mathbf{u}$ in $\mathcal{Q}$ to the point $\mathbf{v}$ in $\mathcal{Q}$ as follows:

$$T(\mathbf{u}) = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ u_2 - u_3 \\ \vdots \\ u_{n-1} - u_n \\ -u_1 \end{pmatrix} + \mathbf{c_u},$$

where

$$\mathbf{c_u} = (c_i) \ : \ c_i = \mathbb{1}[u_{i+1} > u_i] \quad \text{for } i = 1, \ldots, n-1, \text{ and} \quad c_n = 1.$$

Then, if the number of runs in the permutation $\sigma(\mathbf{u})$ is k, then it holds that

$$\begin{gathered} 0 < v_1 < 1, \\ \vdots \\ 0 < v_n < 1, \end{gathered}$$

and

$$k - 1 < v_1 + v_2 + \cdots + v_n < k.$$

Here, we note that the number of 1 in $\mathbf{c_u}$ is the number of descents plus one, or the number of runs in the permutation $\sigma(\mathbf{u})$.

Thus, the $\mathbf{u}$, $\sigma(\mathbf{u})$ of which has k runs, is transformed into the slice $k - 1 \leq v_1 + \cdots + v_n < k$ as $\mathbf{v} = T(\mathbf{u})$. Now, let $\mathbf{U}$ be the random vector distributed uniformly in $\mathcal{Q}$, and $\mathbf{V} = (V_1, \ldots, V_n) = T(\mathbf{U})$. Then, it is clear that $\mathbf{V}$ is also distributed uniformly in $\mathcal{Q}$, and it holds that

$$\Pr\{\ \sigma(\mathbf{U}) \text{ has } k \text{ runs }\} = \Pr\{k - 1 \leq V_1 + \cdots + V_n < k\}.$$

□

Remark 2 Figure 1 depicts the situation of $n = 3$ showing how the transformation of polyhedra is performed.

3 Evaluation of Number of permutations with specified number of runs and Eulerian numbers

We start by the explicit expression of BGF $F(z, u)$:

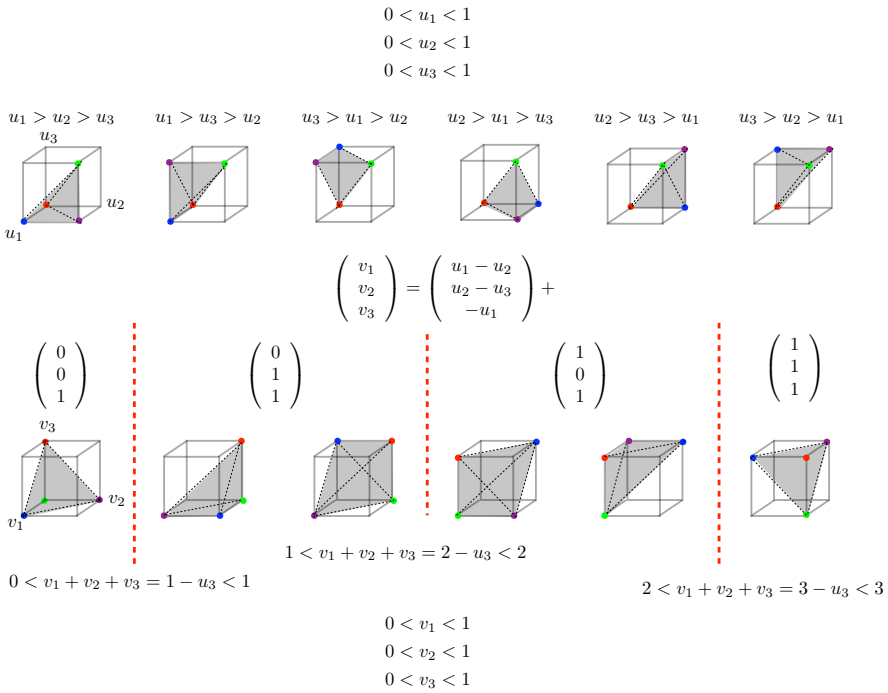

Figure 1: Transform of polyhedra in the cube

Proposition 3.1

$$F(z,u) = \frac{u(1-u)}{e^{(u-1)z} - u}. \tag{3.1}$$

Proof: Omitted. □

From this BGF $F(z,u)$, we can get the formula on the number of permutations of $1,\ldots,n$ with k runs. Thus, we can show the following theorem.

Theorem 3.1 *The coefficient of $z^n u^k$ $(n \geq 1)$ of the generating function $F(z,u)$ is give by*

$$[z^n u^k]F(z,u) = \frac{1}{n!}\sum_{i=0}^{k-1}(-1)^i \binom{n+1}{i}(k-i)^n. \tag{3.2}$$

Proof:

By the argument by David and Barton([1]), we will proceed as follows:

$$\begin{aligned} A(z,u) &= \frac{1-u}{e^{(u-1)z} - u} \\ &= \frac{(1-u)e^{(1-u)z}}{1 - e^{(1-u)z}} \\ &= (1-u)\sum_{k=0}^{\infty} u^k e^{(k+1)(1-u)z} \\ &= (1-u)\sum_{k=0}^{\infty} u^k \sum_{n=0}^{\infty} \frac{1}{n!}(k+1)^n (1-u)^n z^n \end{aligned}$$

Thus, we have

$$\begin{aligned}
[z^n]\,A(z,u) &= (1-u)\sum_{k=0}^{\infty} u^k \frac{1}{n!}(k+1)^n(1-u)^n \\
&= \frac{(1-u)^{n+1}}{n!}\sum_{k=0}^{\infty}(k+1)^n u^k \\
&= \frac{1}{n!}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}u^j\sum_{k=0}^{\infty}(k+1)^n u^k \\
&= \frac{1}{n!}\sum_{s=0}^{\infty}u^s\sum_{j=0}^{s}(-1)^j\binom{n+1}{j}(s-j+1)^n \\
&= \frac{1}{n!}\sum_{s=0}^{n-1}u^s\sum_{j=0}^{s}(-1)^j\binom{n+1}{j}(s-j+1)^n.
\end{aligned}$$

where we need at the last equality the fact that for $s \geq n$, it holds

$$\sum_{j=0}^{s}(-1)^j\binom{n+1}{j}(s-j+1)^n = 0,$$

that is deduced from the proposition 3.2 at the end of this section.

Therefore, we can conclude

$$[z^n u^k]F(z,u) = [z^n u^k]uA(z,u) = \frac{1}{n!}\sum_{i=0}^{k-1}(-1)^i\binom{n+1}{i}(k-i)^n.$$

Note that in David and Barton([1]), there is not the explicit statement equivalent to the proposition 3.2, and don't mention about the term of Eulerian numbers.

□

Remark 3 Note that $F_{n,k} = n!\,[z^n u^k]F(z,u)$ is commonly expressed as follows:

$$F_{n,k} = \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle, \tag{3.3}$$

where

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{i=0}^{k}(-1)^i\binom{n+1}{i}(k+1-i)^n$$

is called as Eulerian numbers.

Remark 4 Moreover, $F_{n,k}$ of (3.3) is the number of permutations with k runs among $n!$ permutations of $1,\ldots,n$. Thus, the value of (3.2) gives the probability

of occurence of permutation with k runs when permutations of $1, \ldots, n$ are equally produced.

From the exponential BGF $F(z, u)$, we can derive the mean and the variance of the number of runs in permutations of $1, \ldots, n$ as $\mu_n = \frac{n+1}{2}, \sigma_n^2 = \frac{n+1}{12}$. And its distribution asymptotically obeys the Gaussian law(refer to Proposition IX.9 of [2]). Compare this result with the binomial distribution case that has the BGF $\frac{2}{2-z(1+u)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} z^n u^k$, from which the mean and the variance are $\mu_n = \frac{n}{2}, \sigma_n^2 = \frac{n}{4}$, respectively.

Proposition 3.2 *For all $n = 1, 2, \ldots,$ and any real α, we have for integer c satisfying $0 \leq c \leq n-1$,*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (i+\alpha)^c = 0. \tag{3.4}$$

Moreover, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (i+\alpha)^n = (-1)^n n!. \tag{3.5}$$

Proof: We can prove by the induction on n, but will omit the proof due to the restriction of space. But we would like to mention that it might be difficult to solve only the equation (3.4), or the equation (3.5). It is effective to solve jointly both of equations by induction.

□

4 The Volumes of Slices of Hypercube and their Recursion

Now, we consider the volume $G_{n,k}$ of slice which is defined by a region between $k-1$ and k of the n-dimensional hypercube along the diagonal axis.

We start by establishing the convolutional formula for uniform distribution. It is convenient to introduce a function called the right-half power function:

$$x_+^m = \begin{cases} x^m, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{4.1}$$

Proposition 4.1 *The n-th convolution $g_n(x)$ of the uniform distribution $\mathbf{1}(x) = \mathbb{1}[0 \leq x < 1]$ on the interval $[0, 1)$ is given by the function:*

$$g_n(x) = \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)_+^{n-1}, \tag{4.2}$$

where $\mathbb{1}[\cdot]$ is the indicator function taking the value 1 if the condition in brackets is true, otherwise 0.

Proof: We will prove this theorem by the induction on n with the help of the following useful relation: for any x and $n \geq 1$,

$$\int_0^1 (x-t)_+^{n-1} dt = \frac{1}{n}\left(x_+^n - (x-1)_+^n\right). \tag{4.3}$$

For $n = 1$, we have

$$g_1(x) = x_+^0 - (x-1)_+^0 = \mathbf{1}(x). \tag{4.4}$$

Thus, the formula (4.2) is obviously true in this case.

Now we assume that the formula (4.2) is true for up to $n = m$.

The $(m+1)$-th convolution $g_{m+1}(x)$ of the uniform distribution $\mathbf{1}(x)$ is the convolution of g_m with $\mathbf{1}$ that is computed as follows:

$$\begin{aligned}
g_{m+1}(x) &= g_m \circ \mathbf{1}(x) \\
&= \int_0^1 g_m(x-y)dy \\
&= \frac{1}{(m-1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\int_0^1 (x-y-i)_+^{m-1}dy \\
&= \frac{1}{(m-1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\frac{1}{m}\left((x-i)_+^m - (x-i-1)_+^m\right) \qquad (4.5) \\
&= \frac{1}{m!}\left(\sum_{i=0}^{m}(-1)^i\binom{m}{i}(x-i)_+^m + \sum_{i=1}^{m+1}(-1)^i\binom{m}{i-1}(x-i)_+^m\right) \\
&= \frac{1}{m!}\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(x-i)_+^m,
\end{aligned}$$

where we use (4.3) at the equality (4.5).

Therefore, for $n = m+1$, we have shown that the formula (4.2) is valid.

□

Remark 5 This function $g_n(x)$ is actually a piecewise-polynomial function that is expressed as follows: for $k-1 \leq x < k$ $(k = 1, \ldots, n)$

$$g_n(x) = f_{n,k}(x) \stackrel{\text{def}}{=} \frac{1}{(n-1)!}\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}(x-i)^{n-1}. \tag{4.6}$$

Moreover, the function $g_n(x)$ is actually identical to the scaling function of B-spline wavelet.

By using $f_{n,k}(x)$ we can provide the explicit formula of the volume $G_{n,k}$ of slice of n-dimensional hypercube, and reestablish the main theorem 2.1, $F_{n,k} = n!\,G_{n,k}$.

The second proof of Theorem 2.1:

$$
\begin{aligned}
G_{n,k} &= \int_{k-1}^{k} f_{n,k}(x)dx \\
&= \int_{k-1}^{k} \frac{1}{(n-1)!} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (x-i)^{n-1} dx \\
&= \frac{1}{(n-1)!} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} \int_{k-1}^{k} (x-i)^{n-1} dx \\
&= \frac{1}{n!} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} \{(k-i)^n - (k-1-i)^n\} \\
&= \frac{1}{n!} \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n \qquad (4.7) \\
&= \frac{1}{n!} \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle,
\end{aligned}
$$

that is identical to $F_{n,k}/n!$.

□

Moreover, with respect to the conditional probability, we have a simple formula as follows:

Proposition 4.2 *It holds the identity,*

$$
\begin{aligned}
&\Pr\{k-1 \le U_1 + \cdots + U_n < k \mid k-1 \le U_1 + \cdots + U_{n-1} < k\} \\
&\quad = \frac{\Pr\{k-1 \le U_1 + \cdots + U_n < k, k-1 \le U_1 + \cdots + U_{n-1} < k\}}{\Pr\{k-1 \le U_1 + \cdots + U_{n-1} < k\}} \\
&\quad = \frac{k}{n}.
\end{aligned}
$$

Proof: First of all, we note that it holds

$$
\frac{\Pr\{k-1 \le U_1 + \cdots + U_n < k, k-1 \le U_1 + \cdots + U_{n-1} < k\}}{\Pr\{k-1 \le U_1 + \cdots + U_{n-1} < k\}} = \frac{\displaystyle\int_0^1 \int_{k-1}^{k-\alpha} f_{n-1,k}(x)dxd\alpha}{\displaystyle\int_{k-1}^{k} f_{n-1,k}(x)dx}.
$$

For the denominator, we have by the previous equation (4.7),

$$
\int_{k-1}^{k} f_{n-1,k}(x)dx = \frac{1}{(n-1)!} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1}. \qquad (4.8)
$$

Next, the numerator can be deduced as follows:

$$\begin{aligned}
\int_0^1 \int_{k-1}^{k-\alpha} f_{n-1,k}(x)dxd\alpha &= \int_0^1 \int_{k-1}^{k-\alpha} \frac{1}{(n-2)!} \sum_{i=0}^{k-1} (-1)^i \binom{n-1}{i} (x-i)^{n-2} dxd\alpha \\
&= \frac{1}{(n-1)!} \sum_{i=0}^{k-1} (-1)^i \binom{n-1}{i} \int_0^1 \left\{ (k-\alpha-i)^{n-1} - (k-1-i)^{n-1} \right\} d\alpha \\
&= \frac{1}{n} \cdot \frac{1}{(n-1)!} \sum_{i=0}^{k-1} (-1)^i \binom{n-1}{i} \left\{ (k-i)^n - (k-i-1)^n - n(k-1-i)^{n-1} \right\} \\
&= \frac{1}{n} \cdot \frac{1}{(n-1)!} \left\{ \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^n - n \sum_{i=0}^{k-1} (-1)^i \binom{n-1}{i} (k-i-1)^{n-1} \right\} \\
&= \frac{1}{n} \cdot \frac{1}{(n-1)!} \left\{ k \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1} - \sum_{i=0}^{k-1} (-1)^i i \binom{n}{i} (k-i)^{n-1} \right. \\
&\qquad \left. - n \sum_{i=0}^{k-1} (-1)^i \binom{n-1}{i} (k-i-1)^{n-1} \right\} \\
&= \frac{k}{n} \cdot \frac{1}{(n-1)!} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{n-1}. \qquad (4.9)
\end{aligned}$$

Summarizing (4.8) and (4.9) , we have established the proposition.

□

From Proposition 4.2, we can deduce the recursion of the slices of hypercube:

Corollary 4.1

$$G_{n,k} = \frac{n-k+1}{n} G_{n-1,k-1} + \frac{k}{n} G_{n-1,k}.$$

In other words by Eulerian numbers,

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (n-k) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle + (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle.$$

Remark 6 The meaning of this Corollary is depicted in the Figure 2.

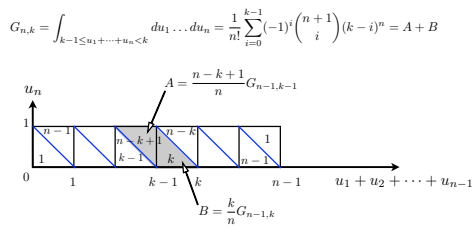

Figure 2: The meaning of recurrence

Here, we show Euler's triangle:

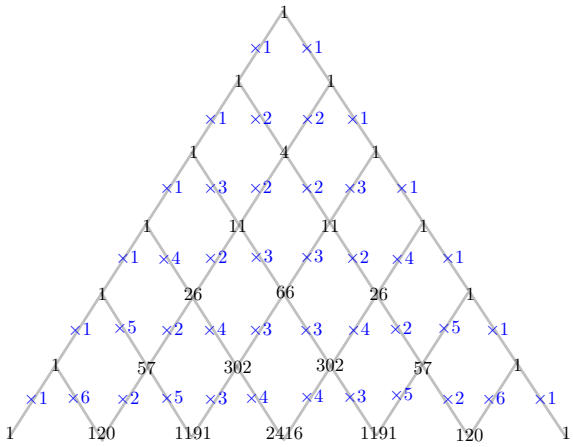

Figure 3: Euler's triangle

Acknowledgment

One of the authors would like to thank National Insititute of Information and Communications Technology (NICT) of Japan for the financial support to participate in ITA2014.

References

[1] F.N. David and D.E.Barton, *Combinatorial Chance*, Charles Griffin & Company Limited, London, 1962.

[2] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.