Spectrum Pricing Games with Correlated Bandwidth Availabilities and Demands

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***Abstract*—We study price competition among primaries in a Cognitive Radio Network (CRN) with multiple primaries and secondaries located in a large region. In a given slot, some of the primaries have unused bandwidth, which they can potentially lease out to secondaries in exchange for a fee. There is uncertainty in whether a given primary has unused bandwidth in a given slot as well as in the number of secondaries that require bandwidth, with the above random quantities being mutually correlated. Each primary tries to attract secondaries by setting a lower price for its bandwidth than the other primaries. Radio spectrum has the distinctive feature that transmissions at neighboring locations on the same channel interfere with each other, whereas the same channel can be used at far-off locations without mutual interference. So in the above price competition scenario, each primary must jointly select a set of mutually non-interfering locations within the region (which corresponds to an independent set in the conflict graph representing the region) at which to offer bandwidth and the price at each location. In this paper, we analyze this price competition scenario as a game and seek a Nash Equilibrium (NE). We analyze the game at a single location as well as the game at multiple locations. We characterize NE for the cases of (i) symmetric bandwidth availability events of different primaries and (ii) asymmetric bandwidth availability events with a special correlation structure.**

I. Introduction

Cognitive Radio Networks (CRNs) [1] are emerging as a promising solution for the efficient usage of the available radio spectrum. In CRNs, there are two types of spectrum users: (i) *primary* users who lease spectrum bands (channels) directly from the spectrum regulator, and (ii) *secondary* users who lease channels from primaries and can use a channel when it is not in use by the primary. In a region with multiple primaries and multiple secondaries, in every time slot, each primary has unused bandwidth with some probability, which it would like to sell to secondaries. Now, secondaries buy bandwidth from the primaries that offer it at a low price, which results in *price competition* among the primaries. If a primary quotes a low price, it will attract a large number of buyers, but will earn lower profit per sale. Such a scenario commonly arises in an *oligopoly* [7], wherein multiple firms sell a common good to a pool of buyers. Price competition in an oligopoly is naturally modeled using *game theory* [2], and has been extensively studied in economics using, for example, the classic *Bertrand game* [7] and its variants.

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However, a CRN has several distinctive features, which makes the price competition in a CRN very different from that in oligopolies encountered in economics. First, in every slot, each primary may or may not have unused bandwidth available. Second, the number of secondaries who require bandwidth is random and not known apriori to the primaries, since each secondary may be a local spectrum provider or even a user shopping for spectrum in a futuristic scenario, *e.g.*, users at airports, hotspots, etc. Thus, each primary who *has* unused bandwidth is uncertain about the number of primaries from whom it will face competition as well as the demand for bandwidth; it may only have access to imperfect information such as statistical distributions about either. A low price will result in unnecessarily low revenues in the event that very few other primaries have unused bandwidth or several secondaries are shopping for bandwidth, because even with a higher price the primary's bandwidth would have been bought, and vice versa. Third, spectrum is a commodity that allows *spatial reuse*: the same band can be simultaneously used at far-off locations without mutual interference, whereas simultaneous transmissions at neighboring locations on the same band interfere with each other. So when multiple primaries own bandwidth in a large region, each needs to decide on a set of mutually non-interfering locations in the region, which corresponds to an *independent set* in the *conflict graph* representing the region, at which to offer bandwidth. Each primary would like to select a maximum-sized independent set to offer bandwidth at in order to maximize the number of locations from which it potentially gets revenue; but if a lot of primaries offer bandwidth at the same locations, there is intense competition at those locations driving down the prices. So a primary would have benefited by instead offering bandwidth at a smaller independent set and charging high prices at those locations.

Finally, the events as to whether different primaries have unused bandwidth or not in a time slot and the number of secondaries that require bandwidth at different locations in that slot may have an arbitrary joint distribution; in particular, they may be *correlated* across primaries as well as across locations. For example, the probabilities of primaries having unused bandwidth as well as the number of secondaries that require bandwidth in a given region often depend on the subscriber demand in the region, which is similar for different players at a given time: *e.g.*, the demand for bandwidth facing both primary and secondary users would typically be high during working hours in a commercial area or outside of working hours in a residential area, resulting in low probabilities of bandwidth availabilities with all the primaries and a high value of the number of secondaries requiring bandwidth and vice versa.

Pricing related issues have been extensively studied in the context of wired networks and the Internet; see [12] for an overview. Price competition among spectrum providers in wireless networks has been studied in [13], [14], [15], [16], [17], [18]. Specifically, Niyato *et. al.* analyze price competition among multiple primaries in CRNs [17], [18]. However, neither uncertain bandwidth availability, nor spatial reuse is modeled in any of the above papers. Also, most of these papers do not explicitly find a Nash Equilibrium (NE) (exceptions are [14], [17]). Our model incorporates uncertain bandwidth availability and demands, allowing these random quantities to be mutually correlated, as well as spatial reuse; despite this, we are able to explicitly compute a NE. In the economics literature, the Bertrand game [7] and several of its variants [8], [9], [10], [11], [19] have been used to study price competition.

The closest to our work are [10], [11], which analyze price competition where each seller may be inactive with some probability, as also our prior work [20], [21], [22], [23], [24] in which we analyzed price competition in a CRN. However, all of the above papers assume that the bandwidth (commodity) availabilities and demands of different sellers and buyers are mutually independent. Characterizing the Nash Equilibrium (NE) in a CRN taking into account the uncertainty in availabilities and demands of different players as well as spatial reuse of spectrum, while allowing for correlation of the above random quantities across different players and locations is quite challenging. This is the space where we seek to contribute in this paper.

We consider price competition in a CRN with multiple primaries and multiple secondaries, where in a given time slot, each primary may or may not have unused bandwidth available. Each primary owns bandwidth across multiple locations, which we represent using a conflict graph in which there is an edge between each pair of mutually interfering locations. Each primary must simultaneously select a set of mutually non-interfering locations (independent set) at which to offer bandwidth and the prices at those locations. The number of secondaries at different locations may be random, unequal and unknown to the primaries. Also, the events as to whether different primaries have unused bandwidth or not and the number of secondaries may have an arbitrary joint distribution and in particular, may be mutually correlated. We focus on a class of conflict graphs, referred to as mean valid graphs, which were introduced in our prior work [23] and include the conflict graphs of a large number of topologies that arise in practice. The above general model is described in detail in Section III; for tractability, we analyze two special cases of this model. In the first (Section IV), we assume that the events that different primaries have unused bandwidth in a given slot are symmetric across primaries. In the second (Section V), we allow the above events to be asymmetric, but assume that these events, and the number of secondaries at different locations, are mutually correlated *only through* the load (subscriber demand) in the region. For each of the above two cases, we first consider the single location model in which all the primaries and secondaries are present at a single location and explicitly compute the NE and show its uniqueness. Then, for the model with spatial reuse, we explicitly compute a NE in mean valid graphs and show that it is unique in the class of NE with symmetric independent set selection strategies of the primaries. The structure of the NE in all the above cases is similar to that in our prior work [22], which studied NE in a CRN taking into account uncertainty in bandwidth availabilities and demands and spatial reuse, but did not take into account correlation in bandwidth availabilities and demands across different players and locations (we briefly summarize the results derived in [22] in Section II). This shows that the structural results derived for the simple model in [22] are robust to several generalizations of the model.

Due to space constraints, we prove only some of the analytical results in this paper and relegate the complete proofs to our technical report [25].

II. Model and Background

We describe the basic model, which does not take into account correlation among the bandwidth availability events of primaries and number of secondaries, studied in our prior work [22] in Section II-A. The results obtained for this model in [22] for the single location and multiple locations cases are summarized in Sections II-B and II-C respectively.

A. Model

We consider a scenario with $n \geq 2$ primaries, each of whom owns a channel throughout a large region such as a state or a country. The channels owned by the primaries are mutually orthogonal. In every time slot, each primary independently either uses its channel throughout the region to satisfy its own subscriber demand, or does not use it anywhere in the region. For example, the primaries may be television broadcasters. Let $q_i \in (0, 1)$ be the probability that primary $i \in \{1, \dots, n\}$ does *not* use its channel in a slot (to satisfy its subscriber demand). Without loss of generality, we assume that:

$$q_1 \geq q_2 \geq \dots \geq q_n. \tag{1}$$

Now, the region contains smaller parts (*e.g.*, towns), which we refer to as *locations*. Each secondary is a local spectrum provider, which seeks to lease spectrum bands from primaries to transmit data on an on-demand basis at a location. The number of secondaries seeking to buy bandwidth may be random and unequal at different locations due to user mobility, time varying bandwidth requirements of the secondaries, etc. Thus, the number of secondaries seeking to buy bandwidth (henceforth referred to as the number of secondaries for simplicity) at a location v is a random variable, K_v, with probability mass function (p.m.f.) $Pr(K_v = k) = \gamma_k$. The random variables K_v are independent of the events as to whether primaries have unused bandwidth or not; however, the random variables K_v corresponding to different nodes v are allowed to be correlated. The primaries apriori know only the γ_ks, but not the values of K_v for any given location v. We make the following technical assumptions on the p.m.f. $\{\gamma_k\}$: (i) $\sum_{k=0}^{n-1}\gamma_k > 0$ (i.e., the total number of primaries exceeds the number of secondaries with positive probability, but not necessarily probability 1) (ii) if $\gamma_0 > 0$, then $\gamma_1 > 0$ (if the event that no secondary requires bandwidth has positive probability, then the event that only 1 secondary requires bandwidth also has positive probability).

A primary that has unused bandwidth in a slot can lease it out to secondaries at a subset of the locations, provided this subset satisfies the *spatial reuse constraints*, which we now describe. The overall region can be represented by an undirected graph [6] $G = (V, E)$, where V is the set of nodes and E is the set of edges, called the *conflict graph*, in which each node represents a location, and there is an edge between two nodes iff transmissions at the corresponding locations interfere with each other. Now, a primary who is not using its channel must offer it to secondaries at a set of mutually non-interfering locations, or equivalently, at an *independent set* [1] (I.S.) of nodes; otherwise secondaries will not be able to successfully transmit simultaneously using the bandwidth they purchase, owing to mutual interference.

A primary i that offers bandwidth at an I.S. I, must also decide for each node $v \in I$, the fee $p_{i,v}$ to be charged to a secondary if the latter leases the bandwidth at node v. We assume that $c \leq p_{i,v} \leq \nu$ for each primary i and each node v, where c and ν are constants such that $\nu > c$. The constants c and ν are known to each primary. Secondaries buy bandwidth from the primaries that offer the lowest price. In particular, the bandwidth of the $\min(Z_v, K_v)$ primaries that offer the lowest prices is bought at node v in a given slot, where Z_v is the total number of primaries that offer bandwidth at the node in the slot. If primary i has unused bandwidth, then its *utility* or *payoff* is defined to be its net revenue. In particular, the utility of a primary i that offers bandwidth at an I.S. I and sets a price of $p_{i,v}$ at node $v \in I$ is given by $\sum(p_{i,v} - c)$, where the summation is over the nodes $v \in I$ at which primary i's bandwidth is bought.

Thus, each primary must jointly select an I.S. at which to offer bandwidth, and the prices to set at the nodes in it. Both the I.S. and price selection may be random; hence a strategy, say ψ_i, of a primary i provides a p.m.f. for selection among the I.S., and the price distribution it uses at each node. The vector $(\psi_1, \ldots, \psi_n)$ of strategies of the primaries is called a *strategy profile* [7]. Let $\psi_{-i} = (\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots, \psi_n)$ denote the vector of strategies of primaries other than i. Let $E\{u_i(\psi_i, \psi_{-i})\}$ denote the expected utility of primary i when it adopts strategy ψ_i and the other primaries adopt ψ_{-i}.

We use the Nash Equilibrium (NE) [7] solution concept, which has been extensively used in game theory. A NE is a strategy profile such that no player can improve its expected utility by unilaterally deviating from its strategy [7]. Thus, $(\psi_1^*, \ldots, \psi_n^*)$ is a NE if for each primary i:

$$E\{u_i(\psi_i^*, \psi_{-i}^*)\} \geq E\{u_i(\widetilde{\psi}_i, \psi_{-i}^*)\}, \ \forall\ \widetilde{\psi}_i \qquad (2)$$

Equation (2) says that when players other than i play ψ_{-i}^*, ψ_i^* maximizes i's expected utility; ψ_i^* is said to be its *best response* [7] to ψ_{-i}^*.

B. Single Location

In this section, we consider the special case of the model described in Section II-A in which all the primaries and secondaries are present at a single location. In our prior work [22], we explicitly computed the NE for this single location model and proved its uniqueness. We briefly summarize these results in this section.

Let the (random) number of secondaries at this location be denoted as K. Note that there are no spatial reuse constraints in this single location model, and the strategy of a primary i is a distribution function (d.f.) [2] $\psi_i(.)$, which it uses to select the price p_i. For convenience, we define the pseudo-price of primary $i \in \{1, \ldots, n\}$, p_i', as the price it selects if it has unused bandwidth and $p_i' = \nu + 1$ otherwise [3]. Also, let $\phi_i(.)$ be the d.f. of p_i'. It is easy to check that $\phi_i(x) = q_i P(p_i \leq x) = q_i\psi_i(x)$ for $c \leq x \leq \nu$. Thus, $\psi_i(.)$ and $\phi_i(.)$ differ only by a constant factor on $[c, \nu]$.

Let $u_{i,max}$ be the expected payoff that primary i gets in the NE. Let w_i be the probability of the event that at least K primaries among $\{1, \ldots, n\}\backslash i$ have unused bandwidth. Let r be the probability that $K \geq 1$. Note that $r = 1 - \gamma_0$, and w_i can be easily computed using the p.m.f $\{\gamma_k\}$ and the fact that each primary j independently has unused bandwidth w.p. q_j. Let

$$\tilde{p} = c + \frac{(\nu - c)(1 - w_1)}{r}. \qquad (3)$$

It is easy to check that $c < \tilde{p} < \nu$. We will later see that $\tilde{p}$ is the lower endpoint of the support sets [4] of the NE strategies $\psi_1(\cdot), \ldots, \psi_n(\cdot)$ (see Fig. 1).

For $0 \leq y \leq 1$, let $f_i(y)$ be the probability of K or more successes out of $n-1$ independent Bernoulli events, $(i-1)$ of which have the same success probability y and the remaining $(n-i)$ have success probabilities $q_{i+1}, \ldots, q_n$. An expression for $f_i(y)$ can be easily computed (see [22]). Let $R_1 = R_2 = \nu$,

$$R_i = c + \frac{(\tilde{p} - c)r}{1 - f_i(q_i)}, \ i \in \{3, \ldots, n\} \qquad (4)$$

and $R_{n+1} = \tilde{p}$. It can be checked that:

$$\tilde{p} = R_{n+1} < R_n \leq R_{n-1} \leq \ldots \leq R_1 \leq v. \qquad (5)$$

We will later see that for each $i \in \{1, \ldots, n\}$, R_i is the right endpoint of the support set of $\psi_i(\cdot)$ (see Fig. 1).

Let

$$g(x) = \frac{x - c - (\tilde{p} - c)r}{x - c}, \ x \in [\tilde{p}, \nu). \qquad (6)$$

[1]Recall that an independent set [6] in a graph $G = (V, E)$ is a subset of V such that there is no edge between any pair of nodes in the subset.

[2]Recall that the d.f. of a random variable X is the function $f(x) = P(X \leq x), x \in R$, where R denotes the set of real numbers.

[3]The choice $\nu + 1$ is arbitrary. Any other choice greater than ν also works.

[4]The support set of a d.f. is the smallest closed set such that its complement has probability zero under the d.f. [5].

Consider the equation:

$$f_i(\phi(x)) = g(x),\ R_{i+1} \le x < R_i. \tag{7}$$

where $i \in \{2, \ldots, n\}$. We showed in [22] that for every x, (7) has a unique solution $\phi(x)$. The function $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, v)$. For $i \in \{2, \ldots, n\}$, $\phi(R_i) = q_i$. Also, $\phi(\tilde{p}) = 0$. The function $\phi(\cdot)$ is shown in Fig. 1.

Finally, for each $i \in \{1, \ldots, n\}$, let:

$$\phi_i(x) = \begin{cases} \phi(x), & \tilde{p} \le x < R_i \\ q_i, & x \ge R_i \end{cases} \tag{8}$$

It can be checked that $\phi_2(.), \ldots, \phi_n(.)$ are continuous on $[c, \nu]$ [22]. $\phi_1(.)$ is continuous at every $x \in [c, \nu)$, has a jump [5] of size $q_1 - q_2$ at ν if $q_1 > q_2$ and is continuous at ν if $q_1 = q_2$ [22]. Fig. 1 shows the functions $\phi_1(\cdot), \ldots, \phi_n(\cdot)$.

The following result, which we proved in [22], characterizes the unique NE:

Theorem 1: The pseudo-price selection d.f.s $\phi_i(.), i = 1, \ldots, n$ in (8) constitute the unique NE. The corresponding price selection d.f.s are $\psi_i(x) = \frac{1}{q_i}\phi_i(x),\ x \in [c, v]$, $i = 1, \ldots, n$. The utilities of all the primaries are equal under this NE and are given by:

$$u_{i,max} = (\tilde{p} - c)r = (\nu - c)(1 - w_1),\ i = 1, \ldots, n. \tag{9}$$

Fig. 1 illustrates the structure of the NE characterized in Theorem 1.

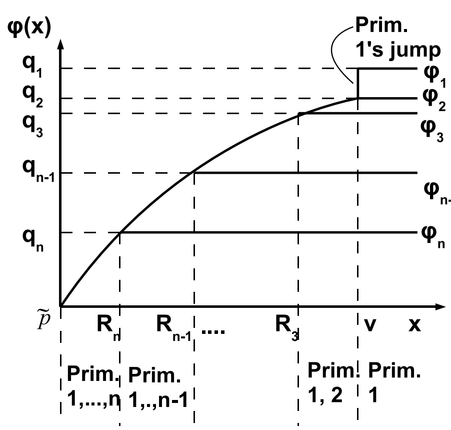

Fig. 1. The figure shows the structure of the NE described in Theorem 1. The horizontal axis shows prices in the range $x \in [\tilde{p}, \nu]$ and the vertical axis shows the functions $\phi(.)$ and $\phi_1(.), \ldots, \phi_n(.)$.

Next, for later use, we summarize some facts that were used in [22] to compute the NE in Theorem 1. Let p'_{-i} be the K'th smallest pseudo-price out of the pseudo-prices, $\{p'_l : l \in \{1, \ldots, n\}, l \neq i\}$, of the primaries other than i (with $p'_{-i} = 0$ if $K = 0$ and $p'_{-i} = \nu + 2$ if $K > n - 1$). Also, let $F_{-i}(x)$ denote the d.f. of p'_{-i}. Since there are K secondaries, if primary 1 has unused bandwidth and sets $p_1 = x \in [\tilde{p}, \nu)$, its bandwidth is bought iff $p'_{-1} > x$, which happens w.p. $1 - F_{-1}(x)$. Note that primary 1's payoff is $(x - c)$ if its bandwidth is bought and 0 otherwise. So, letting $E\{u_i(x, \psi_{-i})\}$ denote the expected payoff of primary i if it sets a price x and the other primaries use the strategy profile ψ_{-i}, we have:

$$E\{u_1(x, \psi_{-1})\} = (x - c)(1 - F_{-1}(x)) = (\tilde{p} - c)r,\ x \in [\tilde{p}, \nu) \tag{10}$$

where the second equality follows from the facts that each $x \in [\tilde{p}, \nu)$ is a best response for primary 1 in the NE [6] and $u_{1,max} = (\tilde{p} - c)r$ by (9). By (10), we get:

$$F_{-1}(x) = g(x),\ x \in [\tilde{p}, \nu), \tag{11}$$

where $g(\cdot)$ is given by (6).

C. Multiple Locations

We now consider the model with spatial reuse constraints described in Section II-A. In our prior work [22], we studied the existence, computation and uniqueness of NE in this model. We briefly summarize these results in this section.

Let $\mathscr{I}$ be the set of all I.S. in the conflict graph G. It is convenient to assume that the empty I.S. $I_\emptyset \in \mathscr{I}$ and that a primary may offer bandwidth at $I_\emptyset$, *i.e.* may not offer bandwidth at any node, with some probability. Consider a NE under which, if primary i has unused bandwidth, it selects I.S. $I \in \mathscr{I}$ w.p. $\beta_i(I)$, where $\sum_{I \in \mathscr{I}} \beta_i(I) = 1$. Then the probability, say α_v^i, with which primary i offers bandwidth at a node $v \in V$ is given by:

$$\alpha_v^i = \sum_{I \in \mathscr{I} : v \in I} \beta_i(I). \tag{12}$$

Since primary i has unused bandwidth w.p. q_i, it offers it at node v w.p. $q_i \alpha_v^i$. Thus, the price selection problem at each node v is equivalent to that for the single location case considered in Section II-B, the difference being that primary $i \in \{1, \ldots, n\}$ offers unused bandwidth w.p. $q_i \alpha_v^i$, instead of q_i, at node v. Hence, we get the following result [22]:

Lemma 1: Suppose under a NE, primary $i \in \{1, \ldots, n\}$ selects node v w.p. α_v^i if it has unused bandwidth. Then under that NE the price distribution of primary i at node v is the d.f. $\psi_i(.)$ in Section II-B, with $q_1\alpha_v^1, \ldots, q_n\alpha_v^n$ in place of $q_1, \ldots, q_n$ respectively all through.

Thus, the strategy profile of the primaries in an NE is completely specified once the I.S. selection p.m.f.s $\{\beta_i(I) : I \in \mathscr{I}, i \in \{1, \ldots, n\}\}$ (and thereby the node selection p.m.f.s α_v^is from (12)) are obtained. In [22], the I.S. selection p.m.f.s were characterized for a class of graphs called mean valid graphs, which are defined next.

Definition 1 (Mean Valid Graph): An assignment $\{\alpha_v : v \in V\}$ of probabilities to the nodes is said to be a *valid* distribution if there exists a probability distribution $\{\beta(I) : I \in \mathscr{I}\}$ such that for each $v \in V$, $\alpha_v = \sum_{I \in \mathscr{I} : v \in I} \beta(I)$. We refer to a graph $G = (V, E)$ as *mean valid* if:

[5] A d.f. $f(x)$ is said to have a *jump* (discontinuity) of size $b > 0$ at $x = a$ if $f(a) - f(a-) = b$, where $f(a-) = \lim_{x \uparrow a} f(x)$.

[6] This follows from the facts that $\phi_1(x) = \phi(x)$ for all $x \in [\tilde{p}, \nu)$ by (8), and $\phi(\cdot)$ is strictly increasing on $[\tilde{p}, \nu)$.

1) Its vertex set can be partitioned into d disjoint maximal [7] I.S. for some integer $d \geq 2$: $V = I_1 \cup I_2 \cup \ldots \cup I_d$, where I_j, $j \in \{1, \ldots, d\}$, is a maximal I.S. and $I_j \cap I_m = \emptyset$, $j \neq m$.
   Let $|I_j| = M_j$, $I_j = \{a_{j,l} : l = 1, \ldots, M_j\}$ and:
$$M_1 \geq M_2 \geq \ldots \geq M_d. \qquad (13)$$
2) For every valid distribution [8] in which a primary who has unused bandwidth offers it at node $a_{j,l}$ w.p. $\alpha_{j,l}$, $j = 1, \ldots, d$, $l = 1, \ldots, M_j$,
$$\sum_{j=1}^{d} \overline{\alpha}_j \leq 1, \text{ where } \overline{\alpha}_j = \frac{\sum_{l=1}^{M_j} \alpha_{j,l}}{M_j}, j \in \{1, \ldots, d\}. \qquad (14)$$

The following graphs, which commonly arise in practice, are mean valid: line graphs, two and three dimensional grid graphs, the conflict graph of a cellular network with hexagonal cells and a clique of size $e \geq 1$ [23].

Let G be a mean valid graph with d disjoint maximal I.S. $I_1, \ldots, I_d$. Consider the class of strategy profiles in which every primary selects I.S. I_j with probability t_j, where $\{t_j : j = 1, \ldots, d\}$ represents a p.m.f., i.e, $\sum_{j=1}^{d} t_j = 1$ and $t_j \geq 0$ for each j. It was shown in [22] that there exists a NE strategy profile in this class; this NE is characterized in Theorem 2 below. Similar to the w_is that were defined in Section II-B, let $w_i(t_j)$ be the probability that K_v or more out of primaries $\{1, \ldots, n\} \backslash i$ offer bandwidth at a given node $v \in I_j$ under the above I.S. p.m.f. $\{t_j : j = 1, \ldots, d\}$. It is easy to check that under this p.m.f, each primary obtains an expected payoff of $W(t_j)$ at that node [22], where:
$$W(x) = (1 - w_1(x))(\nu - c). \qquad (15)$$

Since I.S. I_j has M_j nodes, each primary receives a total expected payoff of $M_j W(t_j)$ if it chooses I_j. The following result, which was proved in [22], characterizes a NE:

Theorem 2: In a mean valid graph, the following strategy profile constitutes a NE: each primary who has unused bandwidth selects I.S. I_j, $j \in \{1, \ldots, d\}$, w.p. t_j, where $(t_1, \ldots, t_d)$ is the unique distribution satisfying the following conditions. There exists an integer d' such that $1 \leq d' \leq d$ and[9]
$$t_j = 0 \text{ if } j > d', \text{ and} \qquad (16)$$
$$M_1 W(t_1) = \ldots = M_{d'} W(t_{d'}) > M_{d'+1} r(\nu - c). \qquad (17)$$

Also, $t_1 \geq t_2 \ldots \geq t_d$.

Finally, the above NE is unique in class $\mathcal{S}$, which is the class of strategy profiles in which every primary uses the same distribution (p.m.f.) to select the independent set at which to offer bandwidth.

It was shown in [22] that for the model with spatial reuse, there are multiple NE even for the simple scenario with two locations, two primaries and one secondary at each node. Nevertheless, Theorem 2 characterizes a NE that is unique in the class of NE with symmetric I.S. selection strategies of the primaries in mean valid graphs.

III. Correlated Events

In the model described in Section II-A, it was assumed that the events that different primaries have unused bandwidth and the number of secondaries are independent of each other. However, in practice, these events may be correlated since there are often several common factors influencing the bandwidth requirements of all the users at any given time and place. For example, the primary as well as secondary users in a region may be providers of Internet and/ or cellular telephone service. In this case, during times of peak subscriber demand (which typically occur during working hours in commercial areas and outside of working hours in residential areas), the probabilities of having unused bandwidth would be low for all the primaries and the number of secondaries would be high and vice versa. Similarly, the hosting of special events (*e.g.*, festivals, sport and cultural events) in a region would impact the bandwidth requirements of all the primaries and secondaries in the region. To account for such phenomena, in this section, we generalize the model described in Section II-A by allowing the events that different primaries have unused bandwidth and the number of secondaries to be mutually correlated.

First, we describe the model at a single location. There are n primaries and K secondaries. Let E_i be the event that primary i has unused bandwidth in a given slot. The events $E_1, \ldots, E_n$ and the random variable K have an arbitrary joint distribution; in particular, they are allowed to be correlated among themselves. Let $P(E, K = k)$ be the probability that primaries in the set E have unused bandwidth, the others do not and $K = k$.

For the case when primaries own bandwidth at multiple locations in a region, we consider the model described in Section II-A with the change that the events E_i, $i \in \{1, \ldots, n\}$ (defined as in the preceding paragraph) and the random variables K_v, $v \in V$ have an arbitrary joint distribution. Let $P(E, K_v = k_v : v \in V)$ be the probability that primaries in the set E have unused bandwidth, the others do not and $K_v = k_v$, $v \in V$. For tractability, we assume that the secondaries at different nodes are symmetrically distributed; more precisely:

Assumption 1: The distribution $P(E, K_v = k_v : v \in V)$ is *symmetric across all nodes* $v \in V$; in particular, note that for all E and $\{k_v : v \in V\}$, $P(E, K_v = k_v : v \in V) = P(E, K_v = \tilde{k}_v : v \in V)$ for every permutation $\{\tilde{k}_v : v \in V\}$ of $\{k_v : v \in V\}$.

In Sections IV and V, we study NE in two special cases of the above general model that are important in practice.

IV. Symmetric Bandwidth Availability Events

In this section, we analyze the special case of the model described in Section III in which the events that different primaries have unused bandwidth in a given slot are symmetric

[7] Recall that an I.S. I is said to be maximal if for each node $v \notin I$, $I \cup \{v\}$ is not an I.S. [6].

[8] Note that we write $\alpha_{j,l}$ in place of $\alpha_{a_{j,l}}$ to simplify the notation.

[9] For notational simplicity, let $M_j = 0$ if $j > d$.

across primaries. The single location and multiple locations cases are considered in Sections IV-A and IV-B respectively. In Section IV-C, we prove some of the analytical results provided in Section IV-A.

A. Single Location

Consider the single location model described in Section III under the following assumption:

Assumption 2: The distribution $P(E, K = k)$ is *symmetric across different primaries*; in particular, note that $P(E, K = k)$ is completely determined by [10] $|E|$, *i.e.*, $P(E_1, K = k) = P(E_2, K = k)$ if $|E_1| = |E_2|$.

Let $\gamma_k = P(K = k|E_i)$, $k = 0, 1, 2, \ldots$. Note that γ_k is independent of i by Assumption 2. We assume that the technical conditions stated in the second paragraph of Section II-A hold for the above p.m.f. $\{\gamma_k\}$.

We will show that a unique NE exists, whose structure is similar to that in Section II-B (see Theorem 1); we now state the differences. The pseudo-price is defined as in the second paragraph of Section II-B. The functions $\phi_i(\cdot)$ as defined in the second paragraph of Section II-B are not useful in the current context. So we work with only [11] the price d.f.s $\psi_i(\cdot)$, $i = 1, \ldots, n$. We modify the definition of w_i as follows: it is now the probability of the event that at least K primaries among $\{1, \ldots, n\}\backslash i$ have unused bandwidth *given* E_i. Let [12] $r = P(K \geq 1|E_i)$. By Assumption 2, $w_1 = \ldots = w_n$. As in the NE in Section II-B, $\tilde{p}$ in (3) is the lower endpoint of the support sets of the NE strategies $\psi_1(\cdot), \ldots, \psi_n(\cdot)$.

Recall that in the NE in Theorem 1, $\psi_1(.), \ldots, \psi_n(.)$ are continuous on $[c, \nu]$ in the symmetric case $q_1 = \ldots = q_n$. This property generalizes to the current context as stated in the following lemma, which also provides some additional necessary conditions:

Lemma 2: The following are necessary conditions for strategies $\psi_1(.), \ldots, \psi_n(.)$ to constitute a NE:

1) $\psi_1(\cdot), \ldots, \psi_n(\cdot)$ are continuous on $[c, v]$.
2) $$\psi_1(x) = \ldots = \psi_n(x) = \psi(x) \text{ (say)}, \ \tilde{p} \leq x \leq \nu. \quad (18)$$
3) Every point in $[\tilde{p}, \nu]$ is a best response for each primary and it plays every sub-interval in $[\tilde{p}, \nu]$ with positive probability.

The second part of the above result says that primaries must use symmetric strategies in a NE, which intuitively is consistent with the fact that the game is symmetric by Assumption 2.

Now, let p'_{-i} be as defined in the last paragraph of Section II-B and let $F_{-i}(x) = P(p'_{-i} \leq x|E_i)$. We now let $E\{u_i(x, \psi_{-i})\}$ denote the expected payoff of primary i if it sets price x and each of the other primaries uses the price d.f. $\psi(\cdot)$ *conditioned on* E_i. Equations (10), (11) and (6) go through without change, and in fact, hold on the closed interval $x \in [\tilde{p}, \nu]$.

Next, in place of the function $f_i(y)$ defined in Section II-B, we have the following. For $0 \leq y \leq 1$, let $f(y)$ be the probability, conditioned on E_1, that the pseudo-prices of K or more primaries out of primaries $2, \ldots, n$ are $\leq x$ when $\psi(x) = y$. An expression for $f(\cdot)$ can be easily computed, using which we show in Section IV-C that:

Lemma 3: $f(\cdot)$ is a continuous and strictly increasing function.

We also have the following result, proved in Section IV-C:

Lemma 4: For every $x \in [\tilde{p}, \nu]$, the equation $f(\psi(x)) = g(x)$ has a unique solution $\psi(x)$. The function $\psi(\cdot)$ is strictly increasing and continuous on $[\tilde{p}, \nu]$. Also, $\psi(\tilde{p}) = 0$ and $\psi(\nu) = 1$.

Finally, the following result, proved in Section IV-C, characterizes the NE:

Theorem 3: The strategy profile in which each primary uses the d.f. $\psi(\cdot)$ (the solution of the equation $f(\psi(x)) = g(x)$) to select its price constitutes the unique NE.

Thus, a unique NE exists and has a structure similar to that in Section II-B (with $q_1 = \ldots = q_n$).

B. Multiple Locations

Consider the model with spatial reuse described in Section III under the following assumption, which generalizes Assumption 2:

Assumption 3: The distribution $P(E, K_v = k_v : v \in V)$ is *symmetric across different primaries*.

Let [13] $\gamma_k = P(K_v = k_v|E_i)$; we assume that the technical conditions stated in the second paragraph of Section II-A hold for the above p.m.f. $\{\gamma_k\}$.

We will show that results similar to those in Section II-C hold; we state the differences. Let $\beta_i(I)$, $I \in \mathcal{I}$ and α_v^i be as in the second paragraph of Section II-C. Note that a primary i that has unused bandwidth offers it w.p. α_v^i at node $v \in V$. However, from the viewpoint of primaries other than i, this is equivalent to primary i selecting node v w.p. α_v^i irrespective of whether it has unused bandwidth and offering bandwidth at node v if it *has* unused bandwidth. For convenience, we adopt this latter viewpoint. Let $F_i^v(\alpha_v^i)$ be a Bernoulli event with success probability α_v^i, which is independent of E_i and corresponds to primary i selecting node v. Also, let $\tilde{E}_i(\alpha_v^i) = E_i \cap F_i^v(\alpha_v^i)$ be the event that primary i has unused bandwidth and offers it at node v. When all primaries use the same I.S. selection p.m.f. $\beta_1(I) = \ldots = \beta_n(I)$, $I \in \mathcal{I}$, then $\alpha_v^1 = \ldots = \alpha_v^n = \alpha_v$ (say), and the price selection problem at each node v is equivalent to that for the single location case analyzed in Section IV-A with $\tilde{E}_i(\alpha_v)$ in place of E_i for $i = 1, \ldots, n$. In particular, similar to Lemma 1, we get the following:

[10] $|A|$ denotes the cardinality of set A.

[11] Recall that in Section II-B, the functions $\psi_i(\cdot)$ and $\phi_i(\cdot)$ differ only by a constant factor. Also, several of the results in Section II-B were stated in terms of $\phi_i(\cdot)$. Wherever applicable in the rest of this section, we simply state that the corresponding results go through in the current context without reiterating the fact that $\phi_i(\cdot)$ should be replaced with $\psi_i(\cdot)$, $i = 1, \ldots, n$.

[12] Note that the quantity $P(K \geq 1|E_i)$ is independent of i by Assumption 2.

[13] Note that γ_k is independent of i and v by Assumptions 1 and 3.

Lemma 5: Suppose under a NE primary $i \in \{1, \ldots, n\}$ selects node v w.p. α_v if it has unused bandwidth. Then under that NE the price distribution of primary i at node v is the d.f. $\psi(.)$ in Section IV-A, with $\tilde{E}_1(\alpha_v), \ldots, \tilde{E}_n(\alpha_v)$ in place of $E_1, \ldots, E_n$ respectively all through.

Next, in place of the functions $w_i(t_j)$ introduced in Section II-C, we now let $w_i(t_j)$ be the probability that K_v or more out of primaries $\{1, \ldots, n\} \backslash i$ offer bandwidth at a given node $v \in I_j$ under the I.S. p.m.f. $\{t_j : j = 1, \ldots, d\}$, *conditioned on the event $\tilde{E}_i(t_j)$ that primary i has unused bandwidth and offers it at node v.*

Theorem 2 holds without change and characterizes a NE in the current context.

C. Proofs of Analytical Results

Proof of Lemma 3: First, we compute an expression for $f(y)$. Note that:

$$f(y) = \sum_k \gamma_k f_k(y), \tag{19}$$

where $f_k(y)$ is the value of $f(y)$ conditioned on the event $K = k$. By definition:

$$\begin{aligned} f_k(y) &= P(k \text{ or more out of primaries } 2, \ldots, n \\ &\quad \text{have pseudo-price } \leq x | E_1, K = k) \\ &= \sum_{S_1 \subset \{2,\ldots,n\}; |S_1| \geq k} \sum_{S_2 \subset \{2,\ldots,n\} \backslash S_1} \zeta(S_1, S_2) \end{aligned}$$

where $\zeta(S_1, S_2)$ is the probability of the event that primaries in set S_1 have pseudo-price $\leq x$, those in S_2 have unused bandwidth but set price $> x$, and those in $\{2, \ldots, n\} \backslash (S_1 \cup S_2)$ do not have unused bandwidth given E_1 and $K = k$. So:

$$\begin{aligned} & f_k(y) \\ = & \sum_{S_1 \subset \{2,\ldots,n\}; |S_1| \geq k} \sum_{S_2 \subset \{2,\ldots,n\} \backslash S_1} P(E_i : i \in S_1 \cup S_2, \\ & E_i^c : i \in \{2, \ldots, n\} \backslash (S_1 \cup S_2) | E_1, K = k) \\ & \times (y)^{|S_1|} (1-y)^{|S_2|} \end{aligned} \tag{20}$$

By (19) and (20), $f(\cdot)$ is a polynomial and hence continuous. We now show that $f(\cdot)$ is a strictly increasing function. Recall that each primary $j \in \{2, \ldots, n\}$ selects a price using the d.f. $\psi(\cdot)$ if it has unused bandwidth. However, from the point of view of primary 1, this is equivalent to primary j selecting a price using the d.f. $\psi(\cdot)$ irrespective of whether it has unused bandwidth, and offering the price if it has unused bandwidth. We adopt this second viewpoint in the following. Let the r.v. X_j be 1 if primary $j \in \{2, \ldots, n\}$ has unused bandwidth and 0 otherwise. Let $Y_j(y)$ be 1 if the price of primary j is $\leq x$ and 0 otherwise. Let:

$$N(y) = \sum_{j=2}^{n} X_j Y_j(y). \tag{21}$$

It can be checked that $f_k(y) = P(N(y) \geq k | K = k, E_1)$. Now, note that the r.v. $Y_j(y), j = 2, \ldots, n$ are independent of $X_j, j = 2, \ldots, n$, of the events E_1 and $K = k$, and of each other. Also, $Y_j(y)$ is Bernoulli with success probability y. So for $y' > y$:

$$P(Y_j(y') = 1) > P(Y_j(y) = 1).$$

Hence, for every $k \geq 1$ and fixed $T \subset \{2, \ldots, n-1\}$ such that $|T| \geq k$:

$$P\left(\sum_{j \in T} Y_j(y') \geq k\right) > P\left(\sum_{j \in T} Y_j(y) \geq k\right). \tag{22}$$

Equation (22) follows from the fact that for every integer $m \geq k$, $\sum_{i=k}^{m} \binom{m}{i} y^i (1-y)^{m-i}$ is a strictly increasing function of y [3]. Next, let $X^{n-1} = (X_2, \ldots, X_n)$ and $x^{n-1} = (x_2, \ldots, x_n)$. Now:

$$\begin{aligned} & f_k(y') - f_k(y) \\ &= P(N(y') \geq k | K = k, E_1) - P(N(y) \geq k | K = k, E_1) \\ &= \sum_{x^{n-1}} \left[P(N(y') \geq k | K = k, E_1, X^{n-1} = x^{n-1}) \right. \\ &\quad \left. - P(N(y) \geq k | K = k, E_1, X^{n-1} = x^{n-1}) \right] \times \\ &\quad P(X^{n-1} = x^{n-1} | K = k, E_1). \end{aligned}$$

By (21) and (22), the term in square brackets is > 0 whenever k or more elements in x^{n-1} are 1. Thus, $f_k(y') > f_k(y)$ and $f_k(\cdot)$ is strictly increasing for each $k \in \{1, \ldots, n-1\}$. Hence by (19), $f(\cdot)$, is a strictly increasing function. The result follows. ∎

Proof of Lemma 4: By Lemma 3, $f(\cdot)$ is continuous and strictly increasing on $[0, 1]$. Also, $f(0) = 0$ and $f(1) = w_1$. So $f(\cdot)$ is invertible. Now, $f(\psi(x)) = g(x)$, $g(\tilde{p}) = 0$, $g(v) = w_1$ and $g(\cdot)$ is strictly increasing. So for fixed x, $\psi(x)$ is unique and is given by:

$$\psi(x) = f^{-1}(g(x)). \tag{23}$$

Since $f(\cdot)$ is a continuous and one-to-one map from $[0, 1]$ onto $[0, w_1]$, $f^{-1}(\cdot)$ is continuous by Theorem 4.17 in [4]. Also, $g(\cdot)$ is continuous. So by (23), $\psi(\cdot)$ is a continuous function of x since it is the composition of continuous functions $f^{-1}(\cdot)$ and $g(\cdot)$ (see Theorem 4.7 in [4]). Since $f(\cdot)$ is strictly increasing, so is $f^{-1}(\cdot)$. By (23), $\psi(\cdot)$ is the composition of two strictly increasing functions and hence is strictly increasing. Finally, since $g(\tilde{p}) = 0$, $g(v) = w_1$, $f(0) = 0$ and $f(1) = w_1$, we get $\psi(\tilde{p}) = 0$ and $\psi(v) = 1$ by (23). ∎

Proof of Theorem 3: Suppose every primary in $\{1, \ldots, n\} \backslash i$ uses the d.f. $\psi(\cdot)$ to select its price. By definition of the function $f(\cdot)$, the expected payoff of primary i if it sets a price $x \in [\tilde{p}, \nu]$ is given by:

$$\begin{aligned} E\{u_i(x, \psi_{-i})\} &= (x - c)(1 - f(\psi(x))) \\ &= (x - c)(1 - g(x)) \\ &= (\tilde{p} - c)r \text{ (by (6))}. \end{aligned}$$

At price $x < \tilde{p}$, primary i gets an expected payoff strictly less than $(\tilde{p} - c)r$. Thus, every price in $[\tilde{p}, \nu]$ is a best response for primary i. Under the strategy $\psi(\cdot)$, primary i randomizes only over prices in $[\tilde{p},\nu]$; hence, $\psi(\cdot)$ is a best response for primary i. Hence, the strategy profile in which each primary uses the d.f. $\psi(\cdot)$ constitutes a NE.

Now we show uniqueness of the NE. Suppose the strategy profile $(\psi_1(\cdot),\ldots,\psi_n(\cdot))$ constitutes a NE. By part 2 of Lemma 2, $\psi_1(x)=\ldots=\psi_n(x)=\psi'(x)$ (say) for $x\in[\tilde{p},\nu]$. By definition of the function $f(\cdot)$, the expected payoff of primary i if it sets a price $x\in[\tilde{p},\nu]$ and the other primaries use the strategy profile $\psi'(\cdot)$ is given by:

$$E\{u_i(x,\psi'_{-i})\}=(x-c)(1-f(\psi'(x))). \tag{24}$$

But:

$$E\{u_i(x,\psi'_{-i})\}=(\tilde{p}-c)r,\ \forall x\in[\tilde{p},\nu]. \tag{25}$$

Equation (25) holds because (i) each $x\in[\tilde{p},\nu]$ is a best response for primary i by part 3 of Lemma 2, (ii) primary i gets an expected payoff of $(\tilde{p}-c)r$ at price $x=\tilde{p}$ since $\tilde{p}$ is the lower endpoint of the support set of $\psi'(\cdot)$ and hence primary i's bandwidth is bought iff $K\geq 1$.

By (24), (25) and (6), $f(\psi'(x))=g(x)\ \forall x\in[\tilde{p},\nu]$. By Lemma 4, $\psi'(x)=\psi(x)\ \forall x\in[\tilde{p},\nu]$ and the result follows. ∎

V. Limited Correlation

In Section IV, we considered the case of symmetric bandwidth availability events; we now relax the symmetry assumption. For tractability, we focus on a special case of the general model described in Section III, which is described below. The single location model and the model with spatial reuse are considered in Sections V-A and V-B respectively.

A. Single Location

Consider the following special case of the single location model described in Section III, which commonly arises in practice. The bandwidth availability events of different primaries as well as the number of secondaries that require bandwidth are correlated with each other through the load (subscriber demand) in the region, which is similar for different players at any given time. For example, the demand for bandwidth facing both primary and secondary users would be high during working hours in a commercial area or outside of working hours in a residential area, resulting in low probabilities of bandwidth availabilities with all the primaries and a high value of K, and vice versa. We model the above phenomenon as follows. We define a random variable $S\in[0,1]$, which is a measure of the overall load in the region. The events $E_i,\ i=1,\ldots,n$ and the random variable K are correlated *only through* the random variable S; more precisely, they are mutually independent given $S=s$ for every $s\in[0,1]$. Also, (i) $P(E_i|S=s)=q_i s,\ i=1,\ldots,n$, where $q_1\geq\ldots\geq q_n$, and (ii) $P(K=k|S=s)=\gamma_{k,s}$. For concreteness, we assume that S is a continuous r.v. with probability density function (p.d.f.) $g_S(\cdot)$; however, our results readily generalize to r.v.s S with arbitrary distributions. Similar to the technical assumptions in the second paragraph of Section II-A, we assume the following:

Assumption 4: (i) $\sum_{k=0}^{n-1}E\{\gamma_{k,S}\}>0$ and (ii) if $E(\gamma_{0,S})>0$, then $E(\gamma_{1,S})>0$.

Now, let $g_{S|E_i}(\cdot)$ denote the p.d.f. of S conditioned on the event E_i. Then:

$$\begin{aligned} g_{S|E_i}(s) &= \frac{P(E_i|S=s)g_S(s)}{P(E_i)} \\ &= \frac{(q_i s)g_S(s)}{\int_0^1 P(E_i|S=s)g_S(s)ds} \\ &= \frac{(q_i s)g_S(s)}{\int_0^1 (q_i s)g_S(s)ds} \\ &= \frac{s g_S(s)}{\overline{s}}, \end{aligned} \tag{26}$$

where $\overline{s}=\int_0^1 s g_S(s)ds$ is the mean of the p.d.f. $g_S(\cdot)$. Thus, $g_{S|E_i}(s)$ *is independent of i; the following NE analysis is enabled by this key observation.*

Remark 1: Recall our assumption that $P(E_i|S=s)=q_i s$. All the results in the rest of this section and in Section V-B readily generalize to the case where $P(E_i|S=s)=f_1(q_i)f_2(s)$, where $f_1(\cdot)$ and $f_2(\cdot)$ are arbitrary functions– it can be checked that $g_{S|E_i}(s)$ is independent of i in this case as well.

Next, we show that a unique NE exists, whose structure is similar to that in Section II-B; we now state the differences.

The pseudo-price, p'_i, of primary i is defined as in Section II-B. Let $\phi_i(x)=q_i\psi_i(x),\ i=1,\ldots,n$. Note that $\phi_i(\cdot)$ is *not* the d.f. of p'_i; however, it differs from $\psi_i(\cdot)$ only by a constant factor on $[c,\nu]$ and we use it in the rest of this section for convenience. The definition of w_i is modified as in Section IV-A. As in the NE in Section II-B, $\tilde{p}$ in (3) is the lower endpoint of the support sets of the NE strategies $\psi_1(\cdot),\ldots,\psi_n(\cdot)$. Let p'_{-i} be as defined in the last paragraph of Section II-B and let $F_{-i}(x)=P(p'_{-i}\leq x|E_i)$. Let $E\{u_i(x,\psi_{-i})\}$ be the expected payoff of primary i if it sets price x and the other primaries use the strategy profile ψ_{-i} *conditioned on* E_i. Equations (10), (11) and (6) go through without change.

Next, we modify the definition of the function $f_i(y)$ defined in Section II-B as follows:

Definition 2: For $0\leq y\leq 1$ and $s\in[0,1]$, let $f_{i,s}(y)$ be the probability of K_s or more successes out of $n-1$ independent Bernoulli events, $(i-1)$ of which have the same success probability ys and the remaining $(n-i)$ have success probabilities $q_{i+1}s,\ldots,q_n s$, where K_s is a r.v. with the p.m.f. $\{\gamma_{k,s}:k=0,1,2,\ldots\}$. Also, let:

$$f_i(y)=\int_0^1 f_{i,s}(y)\frac{sg(s)}{\overline{s}}ds. \tag{27}$$

Let $R_1,\ldots,R_{n+1}$ and the function $\phi(\cdot)$ be as defined in Section II-B with $f_i(\cdot)$ modified as in Definition 2 all through. Let the functions $\phi_i(\cdot),\ i=1,\ldots,n$ be as in (8). The properties of these functions stated after (8) continue to hold in the current context.

Theorem 1 continues to hold and characterizes the unique NE in the current context. Thus, a unique NE exists and has a structure similar to that in Section II-B.

B. Multiple Locations

We now consider the model with spatial reuse described in Section III. We consider the following generalization of the special case in Section V-A: the random variable S is as defined in Section V-A, and the events E_i, $i = 1, \ldots, n$ and the random variables $\{K_v : v \in V\}$ are correlated only through the random variable S. Also, for every $s \in [0, 1]$, $P(K_v = k|S = s) = \gamma_{k,s}$ for every $v \in V$. We will show that results similar to those in Section II-C hold; we state the differences. Equation (12) holds as before. The paragraph after (12) changes as follows. When $S = s$, primary i offers bandwidth at node v w.p. $(q_i s)\alpha_v^i = (q_i \alpha_v^i)s$. So the price selection problem at each node v is equivalent to that in Section V-A, with $q_i \alpha_v^i$ in place of q_i, $i = 1, \ldots, n$, throughout. Thus, Lemma 1 goes through with the d.f. $\psi_i(\cdot)$ in Section V-A in place of that in Section II-B.

Next, recall the functions $w_i(t_j)$ introduced in Section II-C. Instead, we now define $w_i(t_j)$ to be the probability that K_v or more out of primaries $\{1, \ldots, n\} \backslash i$ offer bandwidth at a given node $v \in I_j$ under the I.S. p.m.f. $\{t_j : j = 1, \ldots, d\}$, *conditioned on the event* E_i. Let $w_i^s(t_j)$ be the value of $w_i(t_j)$ conditioned on the event $S = s$. Then:

$$w_i(t_j) = \int_0^1 \frac{s g_S(s)}{\bar{s}} w_i^s(t_j) ds. \tag{28}$$

Theorem 2 holds without change and characterizes a NE in the current context.

VI. Conclusions and Future Work

We analyzed price competition among primaries in a CRN with multiple secondaries taking into account spatial reuse as well as uncertainty in the bandwidth availabilities and demands of primaries and secondaries, allowing these random quantities to be mutually correlated. We formulated a general model for the above scenario, in which the events that different primaries have unused bandwidth and the number of secondaries that require bandwidth at various locations have an arbitrary joint distribution. We characterized NE for two special cases of this general model– (i) symmetric bandwidth availability events of different primaries and (ii) asymmetric bandwidth availability events with limited correlation– considering the single location as well as multiple locations models in each case. For all the above cases, we generalized the NE analysis of the simple model studied in our prior work [22], which did not take into account correlation among the bandwidth availabilities and demands of different players. We showed that the structure of the NE in all the cases considered is similar to that in [22]; this shows that the structural results derived for the simple model in [22] are robust to several generalizations of the model. An open problem for future research is to investigate the existence, computation and uniqueness of NE for the general model formulated in Section III.

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