# Spin glasses with attitude: opinion formation in a partisan Erdős-Rényi world *

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*Abstract*—**We explore the role of partisan influence in the emergence of consensus opinions in connected populations. Such scenarios arise in many modern settings—political interactions and online forums provide two instances—and there is a long-standing interest in modeling the dynamics of opinion evolution in such systems. By overlaying an external party-based bias on a spin glass inspired structure we show how party discipline tends to persist in the presence of independents, explore the role of neighborhood size in influencing dynamics, and exhibit conditions under which defections from party ranks are enabled.**

## I. Introduction

Tools from statistical physics have been adapted for use in economics, models of neural computation, as well as to offer models of social interactions in network settings [3], [4], [5], [6].

In the latter setting, the opinion of an individual agent is influenced by the opinions of others in its network. In [2], we studied two different models of social interactions that captured biased interactions in a partisan network. Both models assumed a fully connected network of $n$ nodes (agents[1]), where each node has an opinion of $+1$ (for) or $-1$ (against) on a given issue. A node is influenced by its network neighbors and as a result, its opinion evolves over time. Neighbors have a symmetric influence on each other, which depends on their level of affinity. In [2] we considered a scenario where all nodes are members of one of two parties, with nodes from the same party more likely to exhibit a positive affinity bias.

This paper introduces two separate modifications to those models. It first studies the effect of a third group of independent nodes, *i.e.,* nodes who have unbiased affinities towards other nodes, independent of party affiliations. It then considers the impact of additional structure, in the form of an Erdős-Rényi graph (as opposed to a full graph), which determines which nodes influence other nodes.

The rest of the paper is structured as follows. Section II reviews the original models of [2]. Section III discusses the addition of a group of independent nodes to those models, and finally Section IV presents the effect of adding an Erdős-Rényi overlay structure on how nodes influence each other.

## II. Model Review

Two basic models were presented in [2], which capture the evolution of opinions arising from partisan interactions in a network of connected agents (nodes). Both models assume a network of $n$ fully connected *partisan* nodes, divided in two parties $G_1$ and $G_2$. Section III adds a third group comprising *independents, i.e.,* nodes unaffiliated with either party.

For $i = 1, \dots, n$, node $i$ holds a signed (for or against) *opinion* $x_i \in \{-1, 1\}$ about a given issue. The state of the system can then be represented by a state-vector $\mathbf{x} = (x_1, \dots, x_n) \in \{-1, +1\}^n$. As nodes are influenced by the opinions of their neighbors, they update their own opinions accordingly, resulting in an update to the state of the system. State updates are performed asynchronously: at any update epoch, a node $i$ is selected and a state update $x_i \mapsto x_i'$ performed according to the sign of a linear form of the node's current inputs,

$$x_i' = \operatorname{sgn} S_i = \operatorname{sgn}\left(\sum_{j=1}^{n} w_{ij} x_j\right), \tag{1}$$

while keeping all other nodal states unchanged.[2]

The weights $w_{ij}$ in Eq. (1) are symmetric interaction weights for any two nodes $i$ and $j$ and satisfy $-1 \leq w_{ij} \leq 1$. The interaction weight between two nodes is randomly selected in a manner that depends on their level of affinity, which is biased positively or negatively based on their respective party affiliations. If nodes $i$ and $j$ are both from the same party, then $w_{ij}$ is biased positively, otherwise, it is biased negatively.

The system is thus characterized by the symmetric stochastic matrix $[w_{ij}]$ of interaction weights, and it is the specifications of this matrix that sets apart the two models of [2]. We summarize these next.

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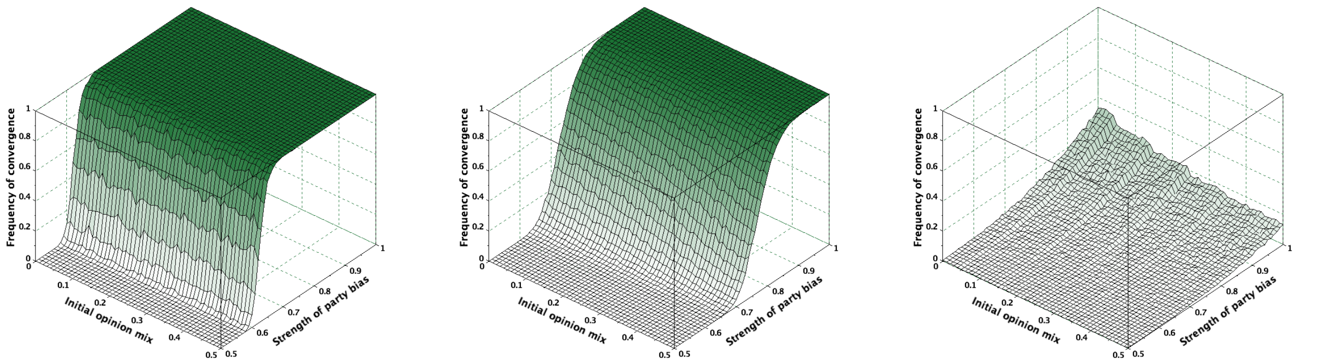
[1] In the paper, we use the terms agent and node interchangeably.

[2] The specifics of the asynchronous update schedule are not important. Any honest deterministic or stochastic update schedule which visits each node infinitely often (with probability one) may be selected.




(a) No independents (b) 60% independents (c) 90% independents

Fig. 1: Random Interactions Model: The frequency of convergence to (meta-)polarized equilibria in the presence of an independent population of nodes. Total network size is $n = 100$.

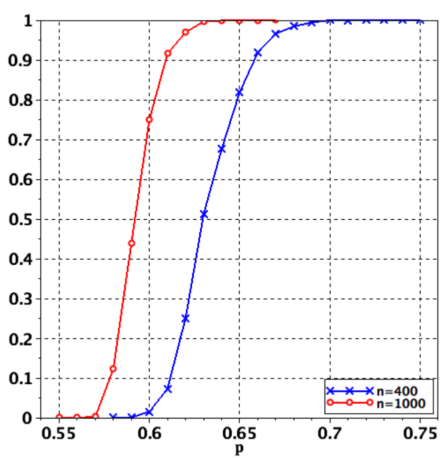



Fig. 2: Random Interactions Model: The frequency of convergence to a meta-polarized equilibrium, from a random starting point when 50% of the nodes are independents.

### A. Random Interactions Model

In a two-party system, the interaction weights in the Random Interactions model comprise a system of independent, signed Bernoulli variables with success parameter $p$ (for intra-party weights) or $1-p$ (for inter-party weights), where $1/2 < p \leq 1$. In other words, intra-party weights are $+1$ with probability $p$ (and $-1$ with probability $1-p$) for both parties, while inter-party weights are $+1$ with probability $1-p$ (and $-1$ with probability $p$). As such, two nodes are likely to positively influence each other if they belong to the same party; they are likely to negatively influence each other if they belong to different parties. We say that $p$ measures the *strength of party bias*.

With this specification, [2] shows that the system has two dominant equilibria, $x^+$ and $x^-$. At $x^+$, all the nodes in party $G_1$ have opinion $+1$, and all the nodes in party $G_2$ have opinion $-1$. The equilibrium $x^-$ is the symmetric opposite. The states $x^+$ and $x^-$ were termed *polarized* equilibria in [2] to illustrate that irrespective of the initial opinion mix inside each party, opinions typically converge to a state where all nodes in a party share the same opinion with nodes in the opposing party sharing the opposite opinion. This is illustrated in Fig. 1a, which shows the odds of such an outcome for different combinations of initial opinion mix and party bias (an initial opinion mix of 0.5 corresponds to completely random initial opinions in both parties, *i.e.*, half with opinion $+1$ and half with opinion $-1$).

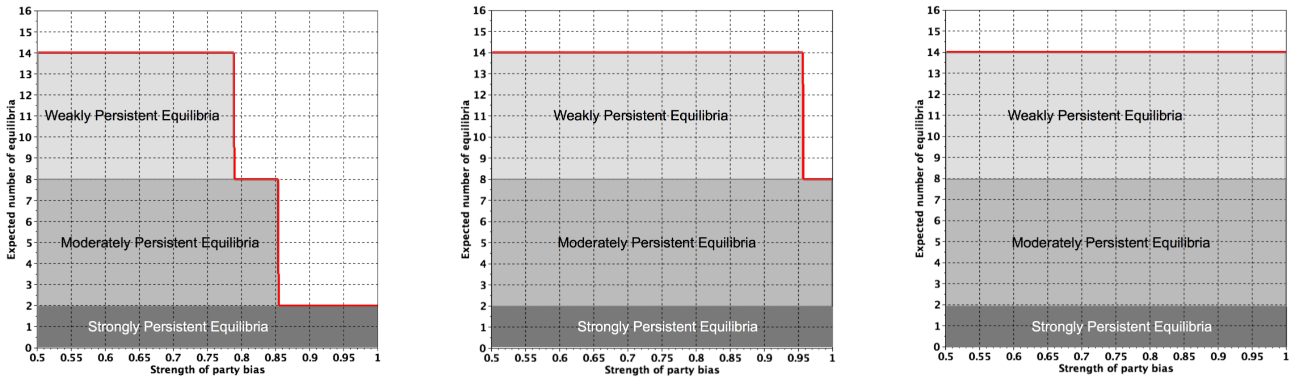
The Profile-based model, reviewed next, yields more nuanced outcomes.

### B. Profile-based Model

In the Profile-based model, the interaction weight between two nodes is based on the similarity of their *profile*. Node $i$'s profile is of the form $\pi_i = (\pi_{i1}, \ldots, \pi_{i\kappa}) \in \{-1,+1\}^\kappa$, where each entry in the profile reflects the node's stated opinion on one of $\kappa$ independent topics. The interaction weight $w_{ij}$ between nodes $i$ and $j$ is then obtained as the normalized dot product of the two profiles:

$$w_{ij} = \frac{1}{\kappa}\pi_i \cdot \pi_j = \frac{1}{\kappa}\sum_{k=1}^{\kappa}\pi_{ik}\pi_{jk}.$$

We assume that the sequence of profile bits (for each node and across nodes) $\{\pi_{ik}, 1 \leq i \leq n, 1 \leq k \leq \kappa\}$, constitutes a family of independent, signed Bernoulli variables. Reusing notation, $1/2 \leq p \leq 1$ is the "success"




(a) No independents

(b) 60% independents

(c) 90% independents

Fig. 3: Profile-based Model ($\kappa = 3$): Expected number of equilibria as a function of party bias $p$, in the presence of an independent population of nodes.

probability for profile opinions of nodes in party $G_1$, *i.e.,* $\mathbf{P}\{\pi_{ik} = +1\} = p$, $\mathbf{P}\{\pi_{ik} = -1\} = 1 - p$, $\forall i \in G_1$, and conversely $1 - p$ is the corresponding "success" probability for profile opinions of nodes in party $G_2$. As in the Random Interactions model, if nodes $i$ and $j$ belong to the same party, $w_{ij}$ is then more likely to be positive, while it is more likely to be negative if they are from opposite parties. Hence, as before, $p$ captures the strength of party bias.

In [2] we established that, unlike the Random Interactions model, a diversity of opinions within a party remains possible under the Profile-based model of nodal influence. This diversity manifests itself in two ways: diversity of opinions *within* a party at equilibria, as well as diversity in final opinions based on the initial opinion mix in each party.

This is illustrated in Fig. 3a, which identifies three batches of possible equilibria, labeled *weakly persistent*, *moderately persistent* and *strongly persistent*. Convergence to an equilibrium of a given type depends on the initial mix of opinions (and to some extent on the strength of party bias), and each equilibrium individually encompasses a mixture of opinions among nodes from both parties, *i.e.,* incorporates a certain level of *dissent* within each party (see [1] for details).

## III. The Effect of Independents

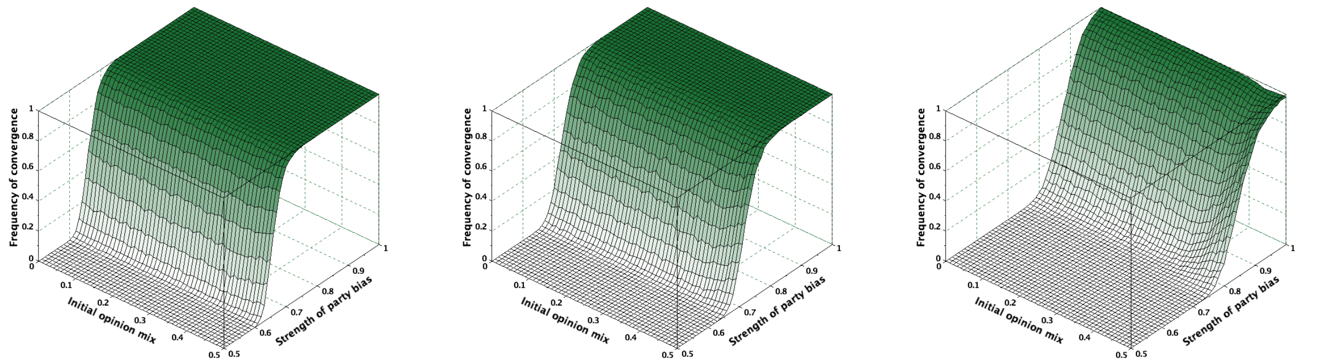
Assume that a subset $G_a, |G_a| = n_a \leq n$, of nodes are *independents, i.e.,* unaffiliated with a party. These independents have unbiased affinities towards other nodes, irrespective of their party affiliations. We explore next the impact these independents have on opinion formation in the context of the two models of Section II.

### *A. Random Interactions Model with Independents*

In the Random Interactions model, unbiased affinities of independent nodes mean that the interaction weights $\{w_{ij}, i \in G_a\}$ between an independent node $i$ and any other node $j$ in the network form a system of signed Bernoulli trials with success parameter $\mathbf{P}\{w_{ij} = 1\} = 1/2$.

The independents function like the nodes in a classical Ising spin glass and, as in that setting, an exponential number of fixed points can arise for opinions in the subset of independent nodes. We accordingly focus our attention on the impact of the independents on the opinions of party members. Specifically, we define $\mathbf{x}^{\mu+}$ as a *meta-polarized* set of states for which $x_i = +1, \forall i \in G_1$ and $x_i = -1, \forall i \in G_2$, with $\mathbf{x}^{\mu-}$ being the symmetric opposite. Note that under $\mathbf{x}^{\mu+}$ (or $\mathbf{x}^{\mu-}$) independent nodes can have any opinion. There is, therefore, a total of $2^{n_a}$ distinct states in a meta-polarized set, each corresponding to a different mix of opinions for independents, but all with unanimous opinions for members from the same party. It can be shown [1] that, with a suitable interpretation, the two meta-polarized sets of states are dominant equilibria similar to the polarized equilibria of the original model, *i.e.,* the presence of independents does not prevent party members from aligning along unanimous opposite opinions.

Fig. 1 illustrates the continued dominance of polarized outcomes even in the presence of independents (except for the extreme case where independents make-up 90% of the population as illustrated in Fig. 1c). Consistent with Section II-A, *initial opinion mix* reflects heterogeneity, but now only in the initial opinions of party members, *i.e.,* it does not extend to independents that are assigned random initial opinions. An initial opinion mix of 0 corresponds to all nodes within a party starting with the same opinion, while a value of 0.5 has

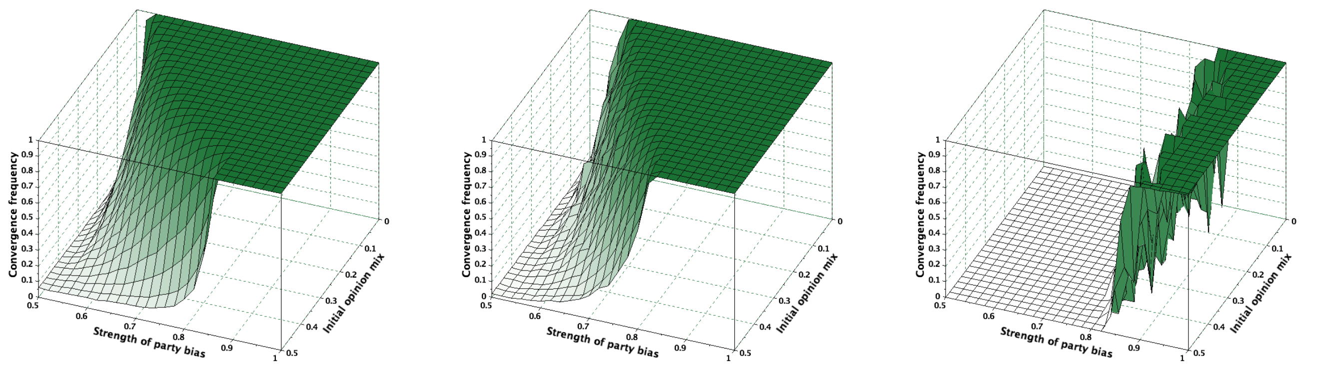



(a) $\rho = 0.6$

(b) $\rho = .03$

(c) $\rho = 0.1$

Fig. 4: Random Interactions Model: The frequency of convergence to polarized equilibria in an Erdős-Rényi graph with parameter $\rho$. Total network size is $n = 100$.




(a) Fully connected network

(b) $\rho = 0.3$

(c) $\rho = 0.05$

Fig. 5: Profile-based Model: The frequency of convergence to the strongly persistent equilibria for $\kappa = 3$ in an Erdős-Rényi graph with parameter $\rho = 0.3$. Total network size is $n = 400$.

nodes in both party equally likely to have either opinion. The figure shows the frequency of convergence to either of the two meta-polarized sets for a population of size $n = 100$. Furthermore and as shown in Fig. 2, as in [2], the dominance of meta-polarized outcomes grows stronger as the network size $n$ becomes larger.

We defer formal theorems and proofs of the results for elsewhere [1].

### B. *Profile-based Model with Independents*

In the Profile-based model, unbiased affinities for independent nodes are realized by randomly (with probability 1/2) choosing the value of each of their profile opinions. It can then be shown that the presence of such a group of independents nodes has a similar effect on the number of possible distinct equilibria as decreasing the strength of party bias.

Fig. 3 illustrates this for the case of $\kappa = 3$. Fig. 3a restates the original results in the absence of independents, and shows a steady decrease in the number of possible equilibria as the strength of party bias increases. As the relative size of the group of independents grows (Fig. 3b and Fig. 3c), sensitivity to party bias weakens. Proofs again are provided in [1].

## IV. Erdős-Rényi Graph Overlay

Sections II and III assumed a fully connected graph of $n$ nodes. We now extend the results to an Erdős-Rényi graph $G(n, \rho)$. The effect of an Erdős-Rényi structure is that a node is not influenced by the entire population anymore. Rather, it only takes into account the opinions of its immediate graph neighbors. Therefore, the update step from Eq. (1) becomes

$$x_i' = \operatorname{sgn} S_i = \operatorname{sgn}\left(\sum_{j\in N_i} w_{ij}x_j\right), \tag{2}$$

where $N_i$ is the set of nodes directly connected to node $i$. The extension of the results to this new structure leverages the fact as $n$ grows large, the size of $N_i$ gets concentrated around $\rho n$. We defer the proofs [1].

### A. Random Interactions Model and Erdős-Rényi

As Fig. 2 showed, convergence to a meta-polarized state in the fully connected network happens less frequently when the network size is smaller. Similarly, an Erdős-Rényi structure decreases the number of neighbors that a node interacts with, on average, by a factor $\rho$. A value of $\rho < 1$ has the same effect as that of a smaller network size in the original model, and therefore decreases the dominance of the polarized states. This is illustrated in Fig. 4 for different values of the Erdős-Rényi parameter $\rho$.

### B. Profile-based Model and Erdős-Rényi

As alluded to earlier, the main impact of an Erdős-Rényi structure is to reduce the number of nodes influencing each other. This is somewhat akin to reducing the size $n$ of the population of interacting nodes. The investigation of [2] showed that the profile-based model exhibited only limited sensitivity to the size of the population of interacting nodes, *i.e.,* it converged quickly to the large population behavior. As a result, one should expect a similar lack of sensitivity to the introduction of an Erdős-Rényi structure.

This is illustrated in Fig. 5, which focuses on convergence to one type of equilibrium (strongly persistent) for different values of the Erdős-Rényi parameter $\rho$. Fig. 5a is for the original fully connected graph of [2], while Fig. 5b and Fig. 5c are for Erdős-Rényi graphs with parameters $\rho = 0.3$ and $\rho = 0.05$, respectively. Fig. 5b is nearly identical to Fig. 5a, which confirms the limited impact of the Erdős-Rényi structure, at least for values of $\rho$ that are not too small. Fig. 5c illustrates that once $\rho$ is too small, the outcome becomes mostly unpredictable due to the very small sizes of the interacting sub-populations of nodes.

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