Small Lifts of Expander Graphs are Expanding

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Abstract

A $k$-lift of an $n$-vertex base-graph $G$ is a graph $H$ on $n \times k$ vertices, where each vertex of $G$ is replaced by $k$ vertices and each edge $(u,v)$ in $G$ is replaced by a matching representing a bijection $\pi_{uv}$, so that the edges of $H$ are of the form $((u,i),(v,\pi_{uv}(i)))$. $H$ is a (uniformly) random lift of $G$ if for every edge $(u,v)$ the bijection $\pi_{uv}$ is chosen uniformly and independently at random. The main motivation for studying lifts has been understanding Ramanujan expander graphs via two key questions: Is a “typical” lift of an expander graph also an expander; and how can we (efficiently) construct Ramanujan expanders using lifts? Lately, there has been an increased interest in lifts and their close relation to the notorious Unique Games Conjecture [Kho02].

In this paper, we analyze the spectrum of random $k$-lifts of $d$-regular graphs. We show that, for random shift $k$-lifts, if all the nontrivial eigenvalues of the base graph $G$ are at most $\lambda$ in absolute value, then with high probability depending only on the number $n$ of nodes of $G$ (and not on $k$), the absolute value of every nontrivial eigenvalue of the lift is at most $O(\lambda)$. Moreover, if $G$ is moderately expanding, then this bound can be improved to $\lambda + O(\sqrt{d})$. While previous results on random lifts were asymptotically true with high probability in the degree of the lift $k$, our result is the first upperbound on spectra of lifts for bounded $k$. In particular, it implies that a typical small lift of a Ramanujan graph is almost Ramanujan, and we believe it will prove crucial in constructing large Ramanujan expanders of all degrees. We also establish a novel characterization of the spectrum of shift lifts by the spectrum of certain $k$ symmetric matrices, that generalize the signed adjacency matrix (e.g. see [BL06]). We believe this characterization is of independent interest.
1 Introduction

Expander graphs have spawned research in pure and applied mathematics during the last several years, with several applications to multiple fields including complexity theory, the design of robust computer networks, the design of error-correcting codes, de-randomization of randomized algorithms, compressed sensing and the study of metric embeddings. For a comprehensive survey of expander graphs see [Sar06, HILW06].

Informally, an expander is a graph where every small subset of the vertices has a relatively large edge boundary. Most applications are concerned with sparse \( d \)-regular graphs \( G \), where the largest eigenvalue of the adjacency matrix \( A_G \) is \( d \). In case of a bipartite graph, the largest and smallest eigenvalues of \( A_G \) are \( d \) and \( -d \), which are referred to as trivial eigenvalues. The expansion of the graph is related to the difference between \( d \) and \( \lambda \), the first largest (in absolute value) non-trivial eigenvalue of \( A_G \). Roughly, the smaller \( \lambda \) is, the better the graph expansion. The Alon-Boppana bound ([Nil91]) states that \( \lambda \geq 2\sqrt{d-1} - o(1) \), thus graphs with \( \lambda \leq 2\sqrt{d-1} \) are optimal expanders and are called Ramanujan.

A simple probabilistic argument can show existence of infinite families of expander graphs [Pin73]. However, constructing such infinite families explicitly has proven to be a challenging and important task. It is easy to construct Ramanujan graphs with a small number of vertices: \( d \)-regular complete graphs and complete bipartite graphs are Ramanujan. The challenge is to construct an infinite family of \( d \)-regular graphs that are all Ramanujan, which was first achieved by Lubotzky, Phillips and Sarnak [LPS88] and Margulis [Mar88]. They built Ramanujan graphs from Cayley graphs. All of their graphs are regular, have degrees \( p + 1 \) where \( p \) is a prime, and their proofs rely on deep number theoretic facts. In a recent breakthrough, Marcus, Spielman and Srivastava showed the existence of bipartite Ramanujan graphs of all degrees [MSS13]. A striking result of Friedman [Fri08] and a slightly weaker but more general result of Puder [Pud13], shows that almost every \( d \)-regular graph on \( n \) vertices is nearly Ramanujan i.e. it has \( \lambda = 2\sqrt{d-1} + o(1) \) (the \( o(1) \)-term tends to 0 as \( n \to \infty \)). It is still unknown whether the event that a random \( d \)-regular graph is exactly Ramanujan happens with constant probability. Despite the large body of work on the topic, all attempts to efficiently construct large Ramanujan expanders of any given degree have failed, and exhibiting such constructions remains an intriguing open problem.

A combinatorial approach to this problem, initiated by Friedman [Fri03], is to prove that one may obtain new (larger) Ramanujan graphs from smaller ones. In this approach one starts with a base graph \( G \) which one “lifts” to obtain a larger graph \( H \), which covers the original graph in the sense that there is a homomorphism from \( H \) to \( G \) such that all fibres in \( H \) of vertices of \( G \) are of equal size. If \( H \) is a cover of \( G \) and the fibres in \( H \) of vertices in \( G \) have size \( k \), then \( H \) is called a \( k \)-lift of \( G \), and we call \( k \) the degree of the lift. More concretely, a \( k \)-lift of an \( n \)-vertex base-graph \( G \) is a graph \( H \) on \( k \times n \) vertices, where each vertex \( u \) of \( G \) is replaced by \( k \) vertices \( u_1, \ldots, u_k \) and each edge \((u,v)\) in \( G \) is replaced by a matching between \( u_1, \ldots, u_k \) and \( v_1, \ldots, v_k \). In other words, for each edge \((u,v)\) of \( G \) there is a bijection \( \pi_{uv} \) so that the corresponding \( k \) edges of \( H \) are of the form \( ((u_1,v_1), (u_2,v_2), \ldots, (u_k,v_k)) \). \( H \) is a (uniformly) \( k \)-random lift of \( G \) if for every edge \((u,v)\) the bijection \( \pi_{uv} \) is chosen uniformly and independently at random from the set of permutations of \( k \) elements, \( S_k \).

Since we are focusing on Ramanujan graphs, it is enough to restrict our attention to lifts of \( d \)-regular graphs. It is easy to see that any lift \( H \) of a \( d \)-regular base-graph \( G \) is itself \( d \)-regular and inherits all the original eigenvalues of \( G \) (which, hereafter we refer to as “old” eigenvalues). One hopes that the lift would also inherit the expansion properties of its base-graph, and in particular that a typical lift of a Ramanujan graph will also be (almost) Ramanujan. One well-known connection between graph lifts and expansion is the fact that the universal cover of any \( d \)-regular connected graph is the infinite \( d \)-regular tree \( T_d \), whose spectral radius is \( \rho = 2\sqrt{d-1} \), the eigenvalue threshold in Ramanujan graphs. Namely, the infinite cover of \( d \)-regular graphs, is the best bounded-degree expander possible.

Let \( G \) be a \( d \)-regular graph with non-trivial eigenvalues at most \( \lambda \) in absolute value, and \( H \) be a (uniformly random) \( k \)-lift of \( G \). If we restrict our attention to the “new” eigenvalues of \( H \) (which are all the eigenvalues of \( H \) excluding the old ones), we would like to show that, with high probability, they are upperbounded in absolute value by roughly \( \lambda \). In this spirit, Friedman [Fri03] studied first the eigenvalues of random lifts of regular graphs and proved that every new eigenvalue \( \tilde{\lambda} \) is \( O(d^{3/4}) \) with high probability, and conjectured a bound of \( 2\sqrt{d-1} + o(1) \), which would be tight (see, e.g. [Gre95]). Linial and Puder [LP10] improved Friedman’ s bound to \( O(d^{2/3}) \). Lubetzky, Sudakov and Vu [LSV11] showed that the absolute value of every nontrivial eigenvalue of the lift is \( O(\lambda \log d) \) which improves on the previous results when
If one is interested in explicitly constructing Ramanujan graphs using lifts, then one would need to de-randomize the above probabilistic results in some clever way. However, such a de-randomisation might be infeasible if one is looking at lifts of large degree $k$, where $k \to \infty$ and thus it is essential to look at lifts with low degrees. Bilu and Linial [BL06] were the first to study lifts of graphs with bounded degree $k$ and suggested constructing Ramanujan graphs through a sequence of 2-lifts of a base graph: start with a good small $d$-regular expander graph on some finite number of nodes (e.g. $K_{d+1}$). Every time the 2-lift operation is performed, the size of the graph doubles. If there is a way to preserve expansion after lifting, then repeating this operation will give large good expanders of the same bounded degree $d$. The authors in [BL06] showed that if the starting graph $G$ is significantly expanding so that $|\lambda(G)| \leq O(\sqrt{d \log d})$, then with high probability in the number of vertices of $G$, a random 2-lift of $G$ has all its new eigenvalues upper bounded in absolute value by $O(\sqrt{d \log \frac{d}{2}})$. In the recent breakthrough work of Marcus, Spielman and Srivastava [MSS13], the authors showed that for every bipartite graph $G$, there exists a 2-lift of $G$, such that the new eigenvalues achieve the Ramanujan bound of $2\sqrt{d-1}$. The two results above indicate that understanding the expansion of typical bounded-degree lifts might be the right avenue towards constructing Ramanujan graphs of all degrees.

In this paper, we study 2-lifts and, more generally, $k$-lifts of graphs for bounded $k$. We significantly improve the results in [BL06] and present the first unconditional high probability result on expansion of random $k$-lifts for bounded $k$. Our main results are optimal up to constants:

**Theorem 1.** Let $G$ be a $d$-regular graph with non-trivial eigenvalues at most $\lambda$ in absolute value, and $H$ be a (uniformly random) 2-lift of $G$. Let $\lambda_{\text{new}}$ be the largest in absolute value new eigenvalue of $H$. Then

$$\lambda_{\text{new}} \leq O(\lambda)$$

with probability at least $1 - e^{-\Omega(n/d^2)}$. Moreover, if $G$ is moderately expanding such that $\lambda \leq \frac{d}{\log d}$, then

$$\lambda_{\text{new}} \leq \lambda + O(\sqrt{d})$$

with probability at least $1 - e^{-\Omega(n/d^2)}$

We also study $k$-lifts of a graph $G$ where the bijections $\pi_{uv}$ for each edge $(u, v)$ are chosen uniformly at random from the set of shift permutations on $k$ elements. We call such lifts “shift” lifts. We show a similar result:

**Theorem 2.** Let $G$ be a $d$-regular graph with non-trivial eigenvalues at most $\lambda$ in absolute value, and $H$ be a random shift $k$-lift of $G$. Let $\lambda_{\text{new}}$ be the largest in absolute value new eigenvalue of $H$. Then

$$\lambda_{\text{new}} \leq O(\lambda)$$

with probability at least $1 - k \cdot e^{-\Omega(n/d^2)}$. Moreover, if $G$ is moderately expanding such that $\lambda \leq \frac{d}{\log d}$, then

$$\lambda_{\text{new}} \leq \lambda + O(\sqrt{d})$$

with probability at least $1 - k \cdot e^{-\Omega(n/d^2)}$

These results, in particular, imply that if we start with $G$ being a small Ramanujan expander, then w.h.p. a random 2-lift will be almost Ramanujan, having all its new eigenvalues bounded by $O(\sqrt{d})$. In addition, we exhibit a bijection between the spectrum of shift $k$-lifts and the spectrum of certain $k$ matrices which generalize the signed adjacency matrix ([BL06]), which we believe might be of independent interest. We note that unlike the case of lifts of degree $k \to \infty$, the dependency on $\lambda$ is necessary for bounded $k$. This has previously been observed by the authors in [BL06] who gave the following example: Let $G$ be a disconnected
graph on \( n \) vertices that consists of \( n/(d + 1) \) copies of \( K_{d+1} \), and let \( H \) be a random 2-lift of \( G \). Then the largest non-trivial eigenvalue of \( G \) is \( \lambda = d \) and it can be shown that with high probability, \( \lambda_{\text{new}} = \lambda = d \). Therefore, our results are nearly tight.

1.1 Unique Games Conjecture and Small Lifts

As explained above, lifts of graphs have been a topic that has received a lot of attention in the context of spectral graph theory and expander constructions. However, our original motivation for studying graph lifts came from the Unique Games Conjecture [Kho02].

Khot’s Unique Games Conjecture (UGC) [Kho02] has, for a decade, been the focus of great attention. Resolution of the conjecture on either side will have implications for the hardness of approximating NP-hard problems. A Unique Game instance is specified by an undirected constraint graph \( G = (V, E) \), an integer \( k \) which is the alphabet size, a set of variables \( \{x_u\}_{u \in V} \), one for each vertex \( u \), and a set of permutations (constraints) \( \pi_{uv} : [k] \to [k] \), one for each \( (u, v) \) \( s.t. \) \( \{u, v\} \in E \), with \( \pi_{uv} = (\pi_{vu})^{-1} \). An assignment of values in \( [k] \) to the variables is said to satisfy the constraint on the edge \( \{u, v\} \) if \( \pi_{uv}(x_u) = x_v \). The optimization problem is to assign a value in \( [k] \) to each variable \( x_u \) so as to maximize the number of satisfied constraints.

Khot [Kho02] conjectured that it is NP-hard to distinguish between the cases when almost all, or very few, of the constraints of a Unique Game are satisfiable:

**Conjecture 1.** (UGC) For any constants \( \epsilon, \delta > 0 \), there is a \( k(\epsilon, \delta) \) such that for any \( k > k(\epsilon, \delta) \), it is NP-hard to distinguish between instances of Unique Games with alphabet size \( k \) where at least \( 1 - \epsilon \) fraction of constraints are satisfiable and those where at most \( \delta \) fraction of constraints are satisfiable.

It is easy to see that such a Unique Games instance on a constraint graph \( G \) can be represented by a \( k \)-lift of \( G \), where for each edge \( (u, v) \) of \( G \), the matching between the fiber of \( u \) and the fiber of \( v \) is simply given by the constraint \( \pi_{uv} \). This representation is also referred to in the literature as the label-extended graph \( M_G \). Under this representation, an assignment of values to the variables of the instance corresponds to a special cut in \( M_G \), where one side of the cut contains exactly one vertex per fiber. The number of constraints that this assignment does not satisfy is equal to the number of edges that cross the cut. We note that the range of \( k \) that is relevant to Unique Games is always a constant, thus lifts of small degree \( k \) are of interest in this context. We also remark that the special case where the label-extended graph is a shift \( k \)-lift has been proven to be as hard as the general case [KKM04], indicating that shift lifts alone are a very interesting and rich class.

Understanding the spectral properties of the label-extended graph has led to significant advances in algorithms for Unique Games ([AKK+08, Kol11, ABS10]). One would expect that the reverse would also be true. Namely, the satisfiability of a Unique Games instance will also give valuable information about the spectra of the label-extended graph. Along those lines, we asked the following question: consider a random \( k \)-lift of a graph \( G \), which corresponds to a random instance of UG on \( G \) where all the constraints are chosen at random from some distribution. We know that a random instance of Unique Games is with high probability unsatisfiable, and this can be verified by a semidefinite programming (SDP) algorithm [KMM11]. As it turns out, the SDP for Unique Games and the second largest eigenvalue \( \lambda_2(M_G) \) of the label-extended graph are closely related, in the sense that \( \lambda_2(M_G) \) is a lowerbound to the dual (thus the primal) optimum. We could expect, then to be able to argue about the average value of \( \lambda_2(M_G) \), by using the (known) bound on the average case value of the SDP.

1.2 Proof Overview

Let \( G \) be an \( n \) vertex \( d \)-regular graph, with all non-trivial eigenvalues smaller than \( \lambda \) in absolute value. In order to prove Theorem 1 we need to first define the notion of a graph signing. A signing of the edges of \( G \) is a function \( s : E(G) \to \{\pm 1\} \). The signed adjacency matrix \( A_s \) for a signing \( s \) has rows and columns indexed by the vertices of \( G \). The \( (x, y) \) entry is \( s(x, y) \) if \( (x, y) \in E \) and 0 otherwise. It is easy to see that for every signing, there is a 2-lift \( H_s \) of \( G \) associated with it, where for each vertex \( x \) of \( G \) the fiber of \( x \) contains two vertices \( x_0 \) and \( x_1 \) and for \( (x, y) \in E(G) \), if \( s(x, y) = +1 \) the corresponding edges of \( H_s \) are \((x_0, y_0)\) and \((x_1, y_1)\) ("identity" permutation), whereas if \( s(x, y) = -1 \) the corresponding edges of \( H_s \) are \((x_0, y_1)\) and \((x_1, y_0)\) ("cross" permutation).
We use a fact that appeared in Bilu and Linial [BL06]. Namely, all the new eigenvalues of $H_s$ are bounded in absolute value by the spectral radius of $A_s$. The spectral radius of $A_s$ is defined as follows:

$$\|A_s\| = \max_{x \in \mathbb{R}^n} \frac{|x^T A_s x|}{\|x\|^2}$$

In order to show that a typical random 2-lift has bounded new eigenvalues, it is enough to provide an upper bound on the spectral radius of $A_s$ that holds with high probability (in $n$). We follow an approach taken by Bilu-Linial [BL06] and start by “rounding” each vector $x$ to a vector $y$, such that $y \in \{ \pm 1/2, \pm 1/4 \ldots \}$. It can be shown that $\frac{|x^T A_s y|}{\|y\|^2}$ approximates $\frac{|x^T A_s x|}{\|x\|^2}$ with a loss of at most a factor of 4. We next consider the diadic decomposition of $y$ to vectors $u_i \in \{0, \pm 1\}^n$, such that $y = \sum 2^{-i} u_i$ (for a formal definition of the diadic decomposition we refer the reader to section 2). Now it is easy to see that

$$|y^T A_s y| = \left| \sum_{i,j} (2^{-i} u_i)^T A_s (2^{-j} u_j) \right|$$


Let's consider an individual term $(2^{-i} u_i)^T A_s (2^{-j} u_j)$ in this sum. Over random choices of the signing, the product $(2^{-i} u_i)^T A_s (2^{-j} u_j)$ is a sum of independent, zero-mean random variables and a simple application of the Chernoff bound gives that

$$\Pr[|u_i^T A_s u_j| \geq \sqrt{d \log d} |S(u_i)||S(u_j)|] \leq d^{-(|S(u_i)|+|S(u_j)|)}$$

(1)

Here, for a vector $u$ we denote its support by $S(u)$.

Bilu-Linial [BL06] employed this simple bound in their proof, which sufficed for their purposes. However, in order to obtain our results, we are faced with two significant challenges. First, we need our argument to hold with high probability in $n$, and the probability term $d^{-|S(u_i)|+|S(u_j)|}$ is clearly not sufficient in the case where the supports of both vectors $u_i$ and $u_j$ are small. Second, we cannot afford to lose the factor of log $d$ in the above bound. To remedy these problems, we separate the sum $|y^T A_s y| = \left| \sum_{i,j} (2^{-i} u_i)^T A_s (2^{-j} u_j) \right|$ into different parts and apply different bounds at each of those parts.

First, we look at vectors $u_i$ and $u_j$ with small support, i.e. $|S(u_i)|, |S(u_j)| \leq \frac{n}{d^2}$. For such vectors we use a trivial bound and show that their total contribution to the (absolute value of the) sum is less than $\lambda \|y\|^2$.

Second, we look at the remaining part which consists of terms in which at least one of the $u_i, u_j$ has large ($> n/d^2$) support. In order to avoid the log $d$ factor loss, we need to further separate this remaining sum into parts. One part contains the set of all $(i, j)$ such that at least one of the three guarantees holds.

- $|i - j| > \frac{1}{2} \log d$
- $|S(u_i)| > E(S(u_j), V \setminus S(u_j))$
- $|S(u_j)| > E(S(u_i), V \setminus S(u_i))$

Here, for any two sets of nodes $A, B$ we denote by $E(A, B)$ the number of edges with one endpoint in $A$ and one endpoint in $B$. The last two cases represent the event where the support of one of the vectors $u_i$ or $u_j$ is larger than the total number of edges that leave the support of the other. We show, by using again a trivial bound, that the total contribution to the (absolute value of the) sum from terms that fall into one of the three cases above is no more than $\sqrt{d} \|y\|^2$. We note that both of the trivial bounds that we have used so far are non-probabilistic.

For the part of the sum that remains we need to employ a tighter bound on the deviation of the zero mean quantity $u_i^T A_s u_j$. As noted before, $u_i^T A_s u_j$ is a sum of independent variables whose total number is at most $E(S(u_i), S(u_j))$. To carry on with the proof, we apply a Chernoff bound. In order to get the right bound, we need to approximate $E(S(u_i), S(u_j))$ by $d |S(u_i)||S(u_j)|/n + \lambda \sqrt{|S(u_i)||S(u_j)|}$, using the Expander Mixing Lemma (EML).

To make the analysis easier we consider two cases according to which of the two terms in EML dominates the other i.e.
Case 1: \( \lambda \sqrt{|S(u_i)||S(u_j)|} \leq d|S(u_i)||S(u_j)|/n \Rightarrow E(S(u_i), S(u_j)) \leq 2d|S(u_i)||S(u_j)|/n \)

Case 2: \( \lambda \sqrt{|S(u_i)||S(u_j)|} \geq d|S(u_i)||S(u_j)|/n \Rightarrow E(S(u_i), S(u_j)) \leq 2\lambda \sqrt{|S(u_i)||S(u_j)|} \).

For Case 1 we prove that with probability at least \( 1 - e^{-\Omega(1/\lambda^2)} \) we have for each relevant term of the sum:

\[
|u_i^T A_s u_j| \leq 8 \sqrt{\lambda \sqrt{|S(u_i)||S(u_j)||S(u_j)| \log \left( \frac{2d|S(u_i)|}{|S(u_j)|} \right)}
\]

The quantity \( \lambda \sqrt{|S(u_i)||S(u_j)|} \) is chosen such that the term \( \lambda \sqrt{|S(u_i)||S(u_j)|} \) cancels out the term in the denominator which appears in the probability guarantee we get from the Chernoff bound and the term \( |S(u_j)| \log \left( \frac{2d|S(u_i)|}{|S(u_j)|} \right) \) allows us to apply the union bound.

Case 2 is slightly more complicated than Case 1, as we need to consider multiple terms \( |\sum_i u_i A_s u_j| \) for a fixed \( u_j \). If instead we considered each term separately, then for each \( u_j \) the term \( |S(u_i)| \) would get counted \( \log d \) times, which would result in a \( \log d \) factor loss we cannot afford. Instead we show that with probability at least \( 1 - e^{-\Omega(1/\lambda^2)} \) we have for each relevant \( u_j \):

\[
|\sum_i u_i^T A_s u_j| \leq 8 \sqrt{\frac{1}{n} d|S(u_j)|^2 (\sum_i |S(u_i)| 2^{2i}) \log \left( \frac{2n}{|S(u_j)|} \right)}
\]

Combining these two bounds we prove the following lemma which bounds the total contribution of the sum of terms that remain after removing vectors with small supports.

**Lemma 1.** Let \( u_1, u_2, \ldots \in \{0, \pm 1\}^n, v_1, v_2, \ldots \in \{0, \pm 1\}^n \) be two families of vector sets such that for all \((i, j), S(u_i) \cap S(u_j) = S(v_i) \cap S(v_j) = \emptyset \) and either for all \( i, |S(v_i)| > \frac{2\lambda}{d} \) or for all \( i, |S(u_i)| > \frac{2\lambda}{d} \). Let \( A_s \) be a random signing matrix. The following holds with high probability over random choices of signing.

\[
|\sum_{i \leq j} (2^{-i} u_i^T) A_s (2^{-j} * v_j)| \leq O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \sum_i |S(u_i)| 2^{-2i} + \left( \frac{\lambda}{2} + O(\sqrt{d}) \right) \sum_j |S(v_j)| 2^{-2j}
\]

In section 3 we combine the bound obtained by the above lemma and the bound on vectors with small support to prove Theorem 1.

For the proof of Theorem 2, we follow a similar path. However, we are no longer able to exploit the relation between the spectrum of lifts and the spectral radius of signed matrices. Instead, as presented in section 4, we find a novel complete characterization of the spectrum of shift \( k \)-lifts by the spectrum of certain \( k \) matrices which can be seen as a generalization of the signed matrix.

### 1.3 Paper Organization

The rest of the paper is organized as follows: In section 2 we give some preliminary definitions, notations and some facts we will use throughout the paper. Section 3 contains the proof of (a slightly weaker version of) Theorem 1. In section 4 we present a novel characterisation of the eigenvalues of shift \( k \)-lifts, by certain \( k \) symmetric matrices, generalising the notion of the signed adjacency matrix. We give the proof of (a slightly weaker version of) Theorem 2. Section 5 contains the proof of a main lemma that was used but not proved in section 3. The complete proofs of Theorems 1 and 2 are deferred to the Appendix.

## 2 Preliminaries

In this section we present in more detail a lot of the informal definitions and statements that were given in the introduction, introduce some notation and state some facts we will be using in the rest of the paper.
2.1 Notations

Let $G = (V, E)$ be a graph with vertex set $V$, $|V| = n$ and edge set $E$. Let $A$ be the adjacency matrix of the graph and let $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$ be its $n$ eigenvalues. Let $\lambda = \max_{i \in [2, n]} |\lambda_i|$. Note that since $A$ is a real, symmetric matrix its eigenvalues are also real. Moreover if $G$ is regular with degree $d$ it is easy to see that $\lambda_1 = d$ and that $\lambda \leq d$. Throughout the paper $G$ will be a $d$-regular graph and we will be concerned with eigenvalues of adjacency matrices. For any two subsets $S, T \subseteq V$ let $E(S, T)$ be the number of edges that go from $S$ to $T$. Let $\text{Spec}(G)$ denote the spectral gap of $G$ defined as $\text{Spec}(G) = \lambda_1 - \lambda_2 = d - \lambda_2$. For a matrix $M$ we denote by $||M||$ its spectral radius. For a vector $x$ the set $S(x)$ denotes its support, i.e. the set of coordinates where the vectors takes a non-zero value. In the paper we define $\log()$ to be the log function with base 2. Also for ease of presentation we assume that the quantity $1/2 \log(d)$ is an integer. The proof can be easily modified for the most general setting.

Also since our results for high probabilities require that $n >> d$, we can without loss of generality assume that $\lambda > \sqrt{d}$. This condition can be seen to be true on any graph for which diameter is greater than 4. Therefore if $n >> d$ in particular $n > d^5$ this condition holds. Therefore we assume in the rest of the paper that $\lambda > \sqrt{d}$.

2.2 Expansion Basics

There are many ways to characterize Expander Graphs, the most common among them being the combinatorial and the algebraic notions of expansion. The combinatorial notion measures the edge boundary of a subset of vertices in the graph. Formally, given a graph $G = (V, E)$ the expansion of the graph $H(G)$ is defined as

**Definition 1** (Combinatorial Expansion).

$$H(G) = \min_{S \subseteq V : |S| \leq |V|/2} \frac{E(S, V \setminus S)}{|S|}$$

Since $G$ is a $d$-regular graph $H(G) \leq d$. Another way to characterize expansion is via the spectral gap $\text{Spec}(G) = d - \lambda_2$, which is referred to as the algebraic expansion of the graph. The following fundamental fact known as the Cheeger’s Inequality gives a robust connection between the two notions of expansion above.

**Definition 2** (Cheeger’s Inequality).

$$\frac{d - \lambda_2}{2} \leq H(G) \leq \sqrt{d(d - \lambda_2)}$$

Expanders are also sometimes seen as graphs which are close to random graphs. This idea is quantified by the following well-known fact known as the Expander Mixing Lemma which bounds the deviation between the number of edges between two subsets and the expected number in a random graph.

**Theorem 3** (Expander-Mixing Lemma).

$$(\forall S, T \subseteq V) \quad |E(S, T) - \frac{d|S||T|}{n}| \leq \lambda \sqrt{|S||T|}$$

Bilu-Linial [BL06] in their work on lifts showed that the converse of the above statement is almost true as well.

**Theorem 4** (Converse of Expander Mixing Lemma). Given a graph such that for all $S, T \subseteq V$

$$|E(S, T) - \frac{d|S||T|}{n}| \leq \alpha \sqrt{|S||T|}$$

Then $\lambda = O(\alpha(1 + \log(d/\alpha)))$
2.3 Lifts - Definitions and Notations

In this section we formally define $k$-lifts of graphs and state some of their properties. A $k$-lift of graph corresponds to a set of permutations $\Pi = \{\pi_{u,v}\}$ which is indexed over the set of edges $E = \{(u, v)\}$, where each $\pi_{u,v} : [k] \to [k]$.

**Definition 3** ($k$-lift). Given a graph $G = (V, E)$ a $k$-lift of the graph corresponding to a set of permutations $\Pi$ is defined as a graph $H = (V \times [k], E')$ where

$$E' = \{((x, i), (y, j)) | (x, y) \in E, \pi_{x,y}(i) = j\}$$

For every vertex $x \in V$, we define the fiber of $x$ as $\text{fiber}(x) = \{x\} \times [k]$. Also let $A_H$ denote the adjacency matrix of $H$. $k$ would be referred to as the degree of the lift.

When the set of permutations $\Pi$ is chosen randomly (independently and uniformly for each edge) the corresponding lift is referred to as a random $k$-lift.

Some initial easy observations can be made about the structure of a $k$-lift. A $k$-lift is also regular with the same degree as the base graph. Also it is easy to see that $\mathcal{H}(H) \leq \mathcal{H}(G)$ by simply considering the set $S \times [k]$ for each subset $S \subseteq V$ of the original graph. It is easy to see that the eigenvalues of $A$ are also eigenvalues of $A_H$. Therefore we call the $n$ eigenvalues of $A$ the old eigenvalues and $n(k - 1)$ other eigenvalues of $A_H$ the new eigenvalues. We will denote by $\lambda_{\text{new}}$ the largest in absolute value new eigenvalue of $H$, which we also refer to as “first” new eigenvalue for simplicity.

We next define the notion of signing of a graph in a slightly different manner than the one in the Introduction, which is more convenient for our purposes.

**Definition 4** (Signing). Given an $n \times n$ adjacency matrix $A$, an $n \times n$ symmetric matrix $A_s$ is a signing of $A$ if for all $(i, j)$ such that $[A]_{ij} = 1$, $[A_s]_{ij} \in \{-1, 1\}$ and for all $(i, j)$ such that $[A]_{ij} = 0$, $[A_s]_{ij} = 0$.

An arbitrary signing of $A_s$ is obtained by choosing an arbitrary sign for each edge in $A$. It is easy to see that there is a simple bijection between 2-lifts and signings, i.e. for every edge there are two permutations to choose from which corresponds to the sign chosen in the signing.

A crucial property of signings observed by Bilu-Linial [BL06] which makes the study of new eigenvalues of a 2-lift convenient is that the new eigenvalues of the lift are exactly the eigenvalues of the signing $A_s$. To see this first note that the adjacency matrix of a two lift can be written as

$$A_H = 1/2 \ast \begin{bmatrix} A + A_s & A - A_s \\ A - A_s & A + A_s \end{bmatrix}$$

Now consider any eigenvector $v$ of $A$ with eigenvalue $\alpha$. It is easy to see that $[v, v]$ is an eigenvector of $A_H$ with the same eigenvalue $\alpha$. This is the set of old eigenvalues. Now consider any eigenvector $u$ of $A_s$ with eigenvalue $\beta$. It is easy to see that $[u, -u]$ is an eigenvector of $A_H$ with the same eigenvalue $\beta$. Since these are orthogonal eigenvectors we see that the spectrum of $A_s$ is precisely the set of new eigenvalues. Therefore $\lambda_{\text{new}} = ||A_s|| = \max_i \lambda_i(A_s))$. In this context Bilu and Linial[BL06] made the following conjecture about signings which is still open in its full generality.

**Conjecture 2** (Bilu-Linial Conjecture). For every adjacency matrix $A$ of a $d$-regular graph there exists a signing $A_s$ such that $||A_s|| \leq 2\sqrt{d - 1}$

2.4 Other Definitions/Lemmas

Given a vector $x \in \{0, \pm1/2, \pm1/4\ldots\}$ we define the **diadic decomposition** of $x$ as the set $\{2^{-i}u_i\}$ where each $u_i$ defined as

$$[u_i]_j = \begin{cases} 1, & \text{if } x_j = 2^{-i} \\ -1, & \text{if } x_j = -2^{-i} \\ 0, & \text{otherwise} \end{cases}$$

We use the following combinatorial identities. We have included their proofs in the appendix(Section 6) for completeness.
Lemma 2 (Discretization Lemma). For any \( x \in \mathbb{R}^n, \|x\|_\infty \leq 1/2 \) and \( M \) such that the diagonal entries of \( M \) are 0, there exists \( y \in \{ \pm 1/2, \pm 1/4, \ldots \}^n \) such that \( x^T M x \leq |y^T M y| \) and \( \|y\|^2 \leq 4 \times \|x\|^2 \). Moreover, each entry of \( x \) between \( \pm 2^{-i} \) and \( \pm 2^{-i-1} \) is rounded to either \( \pm 2^{-i} \) or \( \pm 2^{-i-1} \).

Similarly, for any \( x_1, x_2 \in \mathbb{R}^n, \|x_1\|_\infty, \|x_2\|_\infty \leq 1/2 \), there exists \( y_1, y_2 \in \{ \pm 1/2, \pm 1/4, \ldots \}^n \) such that \( x_1^T M x_2 \leq |y_1^T M y_2|, \|y_1\|^2 \leq 4 \times \|x_1\|^2, \|y_2\|^2 \leq 4 \times \|x_2\|^2 \) and each entry of \( x_1, x_2 \) between \( 2^{-i} \) and \( 2^{-i-1} \) is rounded to either \( 2^{-i} \) or \( 2^{-i-1} \).

Lemma 3. Assuming that \( r^t \leq z/2, r \geq 2, x > 0 \), we have the following inequality:

\[
\sum_{i=0}^{i=t} (r^i \log(z/r^i))^2 \leq c(r)(r^t \log(z/r^i))^2
\]

where \( c(r) \) is a constant depending only on \( r \).

3 Main Result

Our first main result of this paper is the following theorem:

**Theorem 1.** Let \( G \) be a \( d \)-regular graph with non-trivial eigenvalues at most \( \lambda \) in absolute value, and \( H \) be a (uniformly random) 2-lift of \( G \). Let \( \lambda_{\text{new}} \) be the largest in absolute value new eigenvalue of \( H \). Then

\[ \lambda_{\text{new}} \leq O(\lambda) \]

with probability at least \( 1 - e^{-\Omega(n/d^2)} \). Moreover, if \( G \) is moderately expanding such that \( \lambda \leq \frac{d}{\log d} \), then

\[ \lambda_{\text{new}} \leq \lambda + O(\sqrt{d}) \]

with probability at least \( 1 - e^{-\Omega(n/d^2)} \).

In this section, for the purpose of easing presentation, we will prove a slightly weaker theorem (weaker by a multiplicative factor of four). The complete proof of theorem 1 is deferred to the Appendix.

**Theorem 1*.** Let \( G \) be a \( d \)-regular graph with non-trivial eigenvalues at most \( \lambda \) in absolute value, and \( H \) be a (uniformly random) 2-lift of \( G \). Let \( \lambda_{\text{new}} \) be the largest in absolute value new eigenvalue of \( H \). Then

\[ \lambda_{\text{new}} \leq 4\lambda + O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \]

with probability at least \( 1 - e^{-\Omega(n/d^2)} \).

It is immediate to see that, similar to the statement of theorem 1, if \( G \) is moderately expanding such that \( \lambda \leq \frac{d}{\log d} \), then we get the bound

\[ \lambda_{\text{new}} \leq 4\lambda + O(\sqrt{d}) \]

We require the following main lemma. We defer the proof of this lemma to Section 5

**Lemma 1.** Let \( u_1, u_2, \ldots \in \{0, \pm 1\}^n, v_1, v_2, \ldots \in \{0, \pm 1\}^n \) be two families of vector sets such that for all \((i, j), S(u_i) \cap S(u_j) = S(v_i) \cap S(v_j) = \emptyset \) and either for all \( i, |S(v_i)| > \frac{n}{\sqrt{d}} \) or for all \( i, |S(u_i)| > \frac{n}{\sqrt{d}} \). Let \( A_s \) be a random signing matrix. The following holds with high probability over random choices of signing.

\[ |\sum_{i \leq j} (2^{-i} \cdot u_i^T) A_s (2^{-j} \cdot v_j)| \leq O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \sum_i |S(u_i)| 2^{-2i} + (\frac{\lambda}{5} + O(\sqrt{d})) \sum_j |S(v_j)| 2^{-2j} \quad (5) \]

Also if we know that for all \((i, j), |S(u_i)| \geq |S(v_j)|\), then the following holds with high probability

\[ |\sum_{i \leq j} (2^{-i} \cdot u_i^T) A_s (2^{-j} \cdot v_j)| \leq O(\max(\sqrt{\lambda \log d}, \sqrt{d}))(\sum_i |S(u_i)| 2^{-2i} + \sum_j |S(v_j)| 2^{-2j}) \quad (6) \]
We will now prove Theorem 1* assuming the lemma.

**Proof.** For any given vector \( x \in \mathbb{R}^n \) let \( R(x) = \frac{|x^T A x|}{\|x\|^2} \). We know that \( \lambda_{\text{new}} = \|A_s\| = \max_{x \in \mathbb{R}^n} R(x) \). To prove an upper bound on \( \lambda_{\text{new}} \) we will prove that the quantity \( R(x) \) is bounded for all \( x \). In particular we will show that for all \( x \), \( |x^T A_s x| \leq 4 \cdot (\lambda + c \sqrt{d}) \) with high probability. Also note that due to scaling we can look at only those vectors \( x \) for which \( |x_i| \leq 1/2 \).

Now given any vector \( x \) we will first obtain its discretized form \( y \in \{ \pm 1/2, \pm 1/4, \ldots \}^n \) as promised by Lemma 2. Note that \( |x^T A_s x| \leq |y^T A_s y| \) and \( |y|^2 \leq 4|x|^2 \). We will prove an upper bound on \( |y^T A_s y| \).

Consider the diadic decomposition of \( y = \{2^{-i} u_i\} \) where \( i \geq 1 \) and \( u_i \in \{-1,0,1\}^n \). Partition the vectors \( \{u_i\} \) into two sets \( A \) and \( B \) such that \( A = \{u_i|S(u_i)| \leq \frac{n}{d}\} \) and \( B = \{u_i|S(u_i)| > \frac{n}{d}\} \). Let \( y_A = \sum_{i: u_i \in A} 2^{-i} u_i \) and \( y_B = \sum_{i: u_i \in B} 2^{-i} u_i \). Note that \( y = y_A + y_B \) and \( |y|^2 = |y_A|^2 + |y_B|^2 = 2^{-2|S(u_i)|} \).

Now we have that \( |y^T A_s y| \leq |y_A^T A_s y_B| + 2|y_A^T A_s y_B| + |y_B^T A_s y_B| \). We will now consider each part of the above summation separately.

### 3.1 Part 1 - \( |y_A^T A_s y_B| \)

Let's consider \( |y_A^T A_s y_B| \) first. Note that \( |y_A^T A_s y_A| \leq y_A^T A_B y_A \) where \( y_A \) is defined as the vector obtained by making each coordinate of \( y_A \) positive. Let \( J \) be the \( n \times n \) matrix with entries equal to 1. Therefore

\[
y_A^T A_B y_A = y_A^T (A - \frac{d}{n}J)y_A + y_A^T \left( \frac{d}{n}J \right)y_A \leq \lambda \|y_A\|^2 + y_A^T \left( \frac{d}{n}J \right)y_A
\]

Let's look at the term \( y_A^T \left( \frac{d}{n}J \right)y_A \). Let the diadic decomposition of \( y_A = \{2^{-i} u_i\} \). Note that since \( |S(u_i)| \leq \frac{n}{d} \)

\[
y_A^T \left( \frac{d}{n}J \right)y_A = \sum_{i} \sum_{j \geq i} \frac{d}{n} 2^{-i} |S(u_i)| 2^{-j} |S(u_j)| \leq 2 * \sum_{i} (1/d) 2^{-2i} |S(u_i)| \sum_{j \geq i} 2^{-j} \leq 4 * 1/d * \|y_A\|^2
\]

### 3.2 Part 2 - \( |y_B^T A_s y_B| \)

Let's look at the diadic decomposition of \( y_B = \{2^{-i} u_i\} \). Therefore \( |y_B^T A_s y_B| = 2 \sum_{i \leq j} \langle 2^{-i} u_i, 2^{-j} u_j \rangle \). Now since \( |S(u_i)| > \frac{n}{d} \), we can now apply Lemma 1 and we get that

\[
1/2 * |y_B^T A_s y_B| \leq \mathcal{O}(\max(\sqrt{\lambda \log d}, \sqrt{d})) \sum_{i} |S(u_i)| 2^{-2i} + (\frac{\lambda}{5} + \mathcal{O}(\sqrt{d})) \sum_{j} |S(u_j)| 2^{-2j} \leq (\frac{\lambda}{5} + \mathcal{O}(\max(\sqrt{\lambda \log d}, \sqrt{d}))) \|y_B\|^2
\]

### 3.3 Part 3 - \( |y_A^T A_s y_B| \)

Let's look at the diadic decomposition of \( y_A = \{2^{-i} u_i\} \) and \( y_B = \{2^{-j} v_j\} \). Therefore \( |y_A^T A_s y_B| = \sum_{i \leq j} \langle 2^{-i} u_i, 2^{-j} v_j \rangle + \sum_{i < j} \langle 2^{-i} u_i, 2^{-j} u_j \rangle \). Now since \( |S(v_i)| > \frac{n}{d} \) (by definition) and for all \( (i, j) \) \( |S(v_i)| \geq \)
if the edge \((x,y)\) with probability at least \(1\) given a signing of the edges \(s\) the ability to characterize its new eigenvalues as eigenvalues of the signed adjacency matrix. Just to recall, for \(2\)-lifts, since now we can treat \(2\)-lifts as a subcase.

First, we present a novel characterisation of the spectrum of shift \(k\) matrix. This characterisation unifies the study of shift \(k\) with probability at least \(1\):

\[
\|y^T A_s y\| \leq \sum_{i<j} (2^{-i} u_i) A_s (2^{-j} v_j) + \sum_{i<j} (2^{-i} u_i) A_s (2^{-j} u_j)
\]

\[
\leq O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \sum_i |S(u_i)| 2^{-2i} + \left( \frac{\lambda}{5} + O(\sqrt{d}) \right) \sum_j |S(v_j)| 2^{-2j}
\]

\[+ O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \sum_j (|S(v_j)| 2^{-2j} + |S(u_j)| 2^{-2j}) \]

\[
\leq O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \|y_A\|^2 + \left( \frac{\lambda}{5} + O(\max(\sqrt{\lambda \log d}, \sqrt{d})) \right) \|y_B\|^2
\]

### 3.4 Putting it all together

From the above arguments we get that \(\|y^T A_s y\|\) is bounded by \((\lambda + O(\max(\sqrt{\lambda \log d}, \sqrt{d})))\|y_A\|^2 + (\frac{\lambda}{5} + O(\max(\sqrt{\lambda \log d}, \sqrt{d})))\|y_B\|^2\). Therefore we have

\[
x^T A_s x \leq \|y^T A_s y\|
\]

\[
\leq (\lambda + O(\max(\sqrt{\lambda \log d}, \sqrt{d}))) \|y\|^2
\]

\[
\leq 4 * (\lambda + O(\max(\sqrt{\lambda \log d}, \sqrt{d}))) \|x\|^2
\]

Note that in the above proof the factor of 4 loss is only a by-product of the discretization of \(x\). However the part of the proof where we actually bound \(\|y^T_A (A - \frac{d}{n} J) y_A\|\) by \(\lambda \|y_A\|^2\) does not require \(y_A\) to be a discretized vector. Therefore it is possible to not discretize \(x\) straightaway, but push the discretization a little deeper into the proof instead. This is what we make use of in the appendix in order to avoid losing the factor of 4 and prove theorem 1.

### 4 Shifting Lift and Expansion Properties

Our second main result of the paper is the following theorem:

**Theorem 2.** Let \(G\) be a \(d\)-regular graph with non-trivial eigenvalues at most \(\lambda\) in absolute value, and \(H\) be a random shift \(k\)-lift of \(G\). Let \(\lambda_{new}\) be the be the largest in absolute value new eigenvalue of \(H\). Then

\[\lambda_{new} \leq O(\lambda)\]

with probability at least \(1 - k \cdot e^{-\Omega(n/d^2)}\). Moreover, if \(G\) is moderately expanding such that \(\lambda \leq \frac{d}{\log n}\), then

\[\lambda_{new} \leq \lambda + O(\sqrt{d})\]

with probability at least \(1 - k \cdot e^{-\Omega(n/d^2)}\).

In this section, for the sake of presentation, we will give the proof of a slightly weaker version of theorem 2. First, we present a novel characterisation of the spectrum of shift \(k\)-lifts, which extends the signed adjacency matrix. This characterisation unifies the study of shift \(k\)-lifts and allows us to generalize the previous proof for \(2\)-lifts, since now we can treat \(2\)-lifts as a subcase.

One of the major reasons that made the study of the new eigenvalues of a \(2\)-lift of a graph \(G\) easier, was the ability to characterize its new eigenvalues as eigenvalues of the signed adjacency matrix. Just to recall, given a signing of the edges \(s : E(G) \rightarrow \{-1, +1\}\), the signed adjacency matrix \(A_s\) has entries \(A_s(x, y) = 1\) if the edge \((x, y) \in E(G)\) has \(s(x, y) = +1\) (or equivalently, the chosen permutation for the edge \((x, y)\) is the
identity), $A_s(x, y) = -1$ if the edge $(x, y) \in E(G)$ has $s(x, y) = -1$ (or equivalently, the chosen permutation for the edge $(x, y)$ is the cross permutation), and $A_s(x, y) = 0$ if $(x, y) \notin E(G)$. It is easy to check that, for any eigenvector $v$ of the signed adjacency matrix the vector $[v, -v]$ is an eigenvector of the adjacency matrix of the corresponding 2-lift. This leads to the question of whether such characterization can be extended to $k$-lifts in general. We show that this characterization can indeed be extended to case where the permutations for each edge is a cyclic shift. In this context we first formally define a shift $k$-lift

**Definition 5 (Shift-k-Lift).** A shift $k$-lift of a graph is a $k$-lift such that the associated set of permutations $\Pi$ is such that for all $\pi_{u,v} \in \Pi$, $s \in [k]$ such that $\pi_{u,v}(i) = (i + s) \mod k$. That is every permutation is a cyclic shift. We denote by $\text{Shift}(u, v)$ the “magnitude” of the shift along the edge $(u, v)$. I.e. if $\pi_{u,v}(i) = (i + s) \mod k$, then $\text{Shift}(u, v) = s$. Note that $(i, j)$ here is an ordered pair and $\text{Shift}(v, u) = -\text{Shift}(u, v) \mod k$.

A natural avenue towards the characterizing such lifts is to look at the roots of unity and for each edge $(u, v)$, assign the value $\omega^{\text{Shift}(u, v)}$. Here $\omega$ is the $k^{th}$ root of unity.

Indeed, for any given shift $k$-lift instance, define the following family of Hermitian matrices $A_s(t)$ parameterized by $t$ where $t$ is the $k^{th}$ root of unity.

$$[A_s(t)]_{ij} = \begin{cases} 0, & \text{if } A_{ij} = 0 \\ t^{\text{Shift}(i,j)}, & \text{if } A_{ij} = 1 \end{cases}$$

We prove the following theorem about $A_s(t)$.

**Theorem 5.** Let $G(E, V)$ be a graph and $H$ any shift $k$-lift of $G$, with the corresponding shifts given by the set $\{\text{Shift}(i, j)\}_{(i,j) \in E}$. Let $\omega$ be a $k^{th}$ root of unity. Let $v$ be an eigenvector of the matrix $A_s(\omega)$ above, with eigenvalue $\alpha$. Then the vector $v^t = [v, \omega v, \omega^2 v, \ldots, \omega^{k-1} v]$ is an eigenvector of the adjacency matrix of $H$ with eigenvalue $\alpha$. Moreover, all eigenvectors created this way using different roots of unity are orthogonal.

**Proof.** Let $A_H$ be the adjacency matrix of $H$. Consider the above vector $v^t$. Since $v^t$ is a $1 \times kn$ dimensional vector, we will refer to its coordinates as a tuple $(x, i)$ where $x \in [n]$ and $i \in [k]$. Essentially, $(x, i)$ corresponds to the $i^{th}$ vertex in the fiber of the $x^{th}$ vertex in the original graph. Note that $v^t_{x,i} = \omega^i v_x$ Consider the term

$$[A_H * v^t]_{x,i} = \sum_{y:(x,y) \in E(G)} \omega^{i + \text{Shift}(x,y)} v_y$$

$$= \omega^i \sum_{y:(x,y) \in E(G)} \omega^{\text{Shift}(x,y)} v_y$$

$$= \alpha \omega^i v_x$$

Also note that for any two $x'(\omega), y'(\omega')$, $(x'(\omega), y'(\omega')) = (x, y)(1 + \beta + \beta^2 \ldots)$ where $\beta = \omega^* \cdot \omega'$. Note that if $\omega \neq \omega'$, $(1 + \beta + \beta^2 \ldots) = 0$, otherwise since $x, y$ are orthogonal eigenvectors corresponding to $A_s(\omega)$, therefore $(x, y) = 0$.

In order to prove theorem 2 we would proceed as follows. We note that $A_s(1)$ has spectrum equal to the set of old eigenvalues and from the theorem above we get that the new eigenvalues of the lift are eigenvalues of the matrices $A_s(\omega), A_s(\omega^2), \ldots, A_s(\omega^{k-1})$. Therefore if we can bound the spectral radius of each of these $A_s(\omega^i)$ we would bound the new eigenvalues of the lift. We first show that each of these matrices have low spectral radius with probability $\geq 1 - e^{-\Omega(n/d^2)}$ and then we take the union bound over all choices of $t$.

For ease of presentation, we will prove the following weaker version of theorem 2. The full proof is deferred to the Appendix.

**Theorem 2*. Let $G$ be a $d$-regular graph with non-trivial eigenvalues at most $\lambda$ in absolute value, and $H$ be a (uniformly random) shift $k$-lift of $G$. Let $\lambda_{\text{new}}$ be the largest in absolute value new eigenvalue of $H$. Then

$$\lambda_{\text{new}} \leq 16(\lambda + O(\max(\sqrt{\lambda \log d}, \sqrt{d})))$$

with probability at least $1 - k * e^{-\Omega(n/d^2)}$.\)
To prove the above theorem, we state a slightly general form of Theorem 1*

**Theorem 1**. Let $G$ be a $d$-regular graph with non-trivial eigenvalues at most $\lambda$ in absolute value with adjacency matrix $A$. Let $A'$ be a random real matrix each of whose entries $A'_{ij}$ is a random variable such that $\forall i, j, E[A'_{ij}] = 0$ and $\forall i, j$ if $A_{ij} = 0$ then $A'_{ij} = 0$ and if $A_{ij} = 1$ then $|A'_{ij}| \leq 1$ always. Then with probability at least $(1 - e^{-\Omega(n/d^2)})$

$$||A_s|| \leq 4(\lambda + O(\sqrt{\lambda \log(d)}))$$

The proof of the above Theorem is exactly the same as the proof of Theorem 1*. The only difference is that every entry in $A'$ may now have a smaller magnitude but that does not affect any of the arguments in the proof of Theorem 1*.

Using theorem 1**, we will now prove theorem 2*.

**Proof.** Note that for a shift lift $\lambda_{new} = \max_{\omega, \omega \neq 1} ||A_s(\omega)||$ where $\omega$ is the $k$th root of unity. Therefore, $P(\lambda_{new} \geq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d})))) \leq \sum_{\omega, \omega \neq 1} P(||A_s(\omega)|| \geq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d}))))$. Therefore if we can show that for a fixed $\omega P(||A_s(\omega)|| \geq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d})))) \leq e^{-\Omega(n/d^2)}$. By union bound, $P(\lambda_{new} \geq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d})))) \leq (k - 1)e^{-\Omega(n/d^2)}$ which implies the theorem. Therefore it is enough to show that for a fixed $\omega, P(||A_s(\omega)|| \geq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d})))) \leq e^{-\Omega(n/d^2)}$.

The spectral radius of $A_s(\omega) = \max_{x \in \mathbb{C}^n} |x^*A_s(\omega)x|$. We first split the vector $x$ and matrix $A_s(\omega)$ into its real and imaginary parts. Let $x = x_1 + ix_2$ and $A_s(\omega) = A^*_1(\omega) + iA^*_2(\omega)$ where $x_1, x_2$ are real vectors and $A^*_1(\omega)$ and $A^*_2(\omega)$ are real matrices. By theorem 1**, $|x^*A_s(\omega)x| \leq 4(\lambda + O(\max(\lambda \log(d), \sqrt{d}))(||x_1|| ||x_2||)$ w.h.p.

$$|x^*A_s(\omega)x| \leq \sum_{i,j,k \in \{1, 2\}} |x_i^*A^*_k(\omega)x_j|$$

$$\leq 8(\lambda + O(\max(\lambda \log(d), \sqrt{d}))(||x_1||^2 + ||x_2||^2) + 16(\lambda + O(\max(\lambda \log(d), \sqrt{d}))||x_1|| ||x_2||$$

$$\leq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d}))(||x_1||^2 + ||x_2||^2)$$

Therefore $\forall \omega, ||A_s(\omega)|| = \max_{x \in \mathbb{C}^n} |x^*A_s(\omega)x| \leq 16(\lambda + O(\max(\lambda \log(d), \sqrt{d})))$ w.p. greater than $1 - e^{-\Omega(\frac{n}{d^2})}$.

\[ \square \]

## 5 Proof of Lemma 1

To prove Lemma 1 we need to prove the following two lemmas. Note that these lemmas are the places where we use the argument of high probability. So once the conditions in these lemmas are satisfied the rest of the proof follows and we ensure that these conditions are met by a random lift with high probability.

**Lemma 4.** For a random 2-lift, let $A_s$ be the signed adjacency matrix the following property holds with probability $1 - e^{-\Omega(\frac{\log(d)}{d})}$

Let $u, v \in \{0, \pm 1\}^n$ s.t. $|S(u)| \leq |S(v)| \leq d|S(u)|$, $S(v) > \frac{\pi}{d}$ and $d/\lambda \sqrt{|S(u)||S(v)|} < n$. Then,

$$|u^T A_s v| \leq 8 \sqrt{\lambda \sqrt{|S(u)||S(v)||S(v)| \log(\frac{2d|S(u)|}{|S(v)|})}$$

(7)

**Lemma 5.** For a random 2-lift, let $A_s$ be the signed adjacency matrix the following property holds with probability $1 - e^{-\Omega(\frac{\log(d)}{d})}$

Let $v, u_0, u_1, \cdots \in \{0, \pm 1\}^n$ s.t. $|S(v)| \geq 2^{2i}|S(u_i)|$, and $d/\lambda \sqrt{|S(u_i)||S(v)|} \geq n$. Let $u = \sum_i u_i 2^i$. Then,

$$|v^T A_s u| \leq 8 \sqrt{\frac{1/n \ast d|S(v)|^2 (\sum_i |S(u_i)| 2^{2i}) \log(\frac{2n}{|S(v)|})}$$

(8)
Firstly, using lemma 4 and 5, we will prove lemma 1 and then prove these lemmas independently.

### 5.1 Proof of lemma 1

**Proof.** For the ease of presentation in this section we denote \(|S(u_i)|\) with \(y_i\) and \(|S(v_j)|\) with \(z_j\).

Since conditions 7 and 8 of lemma 4 and 5 hold true w.h.p. we can assume that both of the conditions hold for the matrix \(A_s\). To prove the lemma we need to bound the quantity \(X = \left| \sum_{i \leq j} (2^{-i} \cdot u_i) A_s (2^{-j} \cdot v_j) \right|\).

We will prove an upperbound on this quantity by partitioning the sum into multiple parts and proving upper bounds for all those parts. In the rest of the section we use \(C_i\) to denote sets of tuples \((i, j)\) of integers (that satisfy some conditions), and we use \(X_I\) to denote sums of the form \(\sum_{(i, j) \in C_I} u_i^T A_s v_j\).

We first partition the sum into two parts \(X_1\) and \(X_2\) where we show that the part \(X_2\) can be easily bound by using a trivial bound of \(|u_i^T A_s v_j| \leq \min(y_i, z_j)\) on each individual term of \(X_2\).

\[
C_1 = \{(i, j) | (i \leq j < i + \frac{1}{2} \log(d)) \land (\max(y_i, z_j) < d \min(y_i, z_j))\}
\]

\[
C_2 = \bar{C}_1 = \{(i, j) | (j \geq i + \frac{1}{2} \log(d)) \lor y_i \geq d z_j \lor z_j \geq d y_i)\}
\]

\[
X_1 = \left| \sum_{(i, j) \in C_1} 2^{-i-j} u_i^T A_s v_j \right|
\]

\[
X_2 = \left| \sum_{(i, j) \in C_2} 2^{-i-j} u_i^T A_s v_j \right|
\]

\[
X \leq X_1 + X_2
\]

We further analyze the sum \(X_1\) by breaking it into two parts \(X_3\) and \(X_4\) such that \(X_3\) contains the part of the sum where \(y_i \geq z_j\) and \(X_4\) contains the part of the sum where \(y_i < z_j\).

\[
C_3 = C_1 \cap \{(i, j) | (y_i \geq z_j)\}
\]

\[
C_4 = C_1 \cap \bar{C}_3 = C_1 \cap \{(i, j) | (y_i < z_j)\}
\]

\[
X_3 = \left| \sum_{(i, j) \in C_3} 2^{-i-j} u_i^T A_s v_j \right|
\]

\[
X_4 = \left| \sum_{(i, j) \in C_4} 2^{-i-j} u_i^T A_s v_j \right|
\]

\[
X_1 \leq X_3 + X_4
\]

We show the required bound on \(X_3\) then further analyze the sum \(X_4\). As guaranteed by Expander Mixing lemma, number of edges between \(S(u_i)\) and \(S(v_j)\) is bounded by \(\frac{d y_i}{n} \sqrt{\frac{y_i}{z_j}}\). As indicated in the proof overview we split our analysis based on which of the two terms in the right hand side of the EML is the dominating one. Therefore we split \(X_4\) into two cases \(X_5\) and \(X_6\) where \(X_5\) contains the terms where \(\frac{d y_i}{n} < \lambda \sqrt{\frac{y_i}{z_j}}\) and \(X_6\) contains the terms where \(\frac{d y_i}{n} \geq \lambda \sqrt{\frac{y_i}{z_j}}\). This separation helps us apply one of the two bounds in Lemma 4 and Lemma 5.

\[
C_5 = C_4 \cap \{(i, j) | (d \sqrt{\frac{y_i}{z_j}} < n)\}
\]

\[
C_6 = C_4 \cap \bar{C}_5 = C_4 \cap \{(i, j) | (d \sqrt{\frac{y_i}{z_j}} \geq n)\}
\]

\[
X_5 = \left| \sum_{(i, j) \in C_5} 2^{-i-j} u_i^T A_s v_j \right|
\]

\[
X_6 = \left| \sum_{(i, j) \in C_6} 2^{-i-j} u_i^T A_s v_j \right|
\]

\[
X_4 \leq X_5 + X_6
\]
To ease the calculations, we further split the sum $X_5$ into two parts depending on whether $y_i 2^{-2i}$ is significantly (in terms of $\lambda$) greater than $z_j 2^{-2j}$. In this regard we make the following separation of $X_5$ into $X_7$ and $X_8$. In the analysis of $X_7$ and $X_8$ we use the bound given by lemma 5 for individual entries.

$$C_7 = C_5 \cap \{(i,j) | (y_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} z_j 2^{-2j})\}$$

$$C_8 = C_5 \cap \overline{C}_7 = C_5 \cap \{(i,j) | (y_i 2^{-2i} \geq \frac{\lambda}{\sqrt{d}} z_j 2^{-2j})\}$$

$$X_7 = \sum_{(i,j) \in C_7} 2^{-i-j} u_i^T A_s v_j$$

$$X_8 = \sum_{(i,j) \in C_8} 2^{-i-j} u_i^T A_s v_j$$

$$X_5 \leq X_7 + X_8$$

Similar to the case of $X_5$ we separate $X_6$ into two parts $X_9$ and $X_{10}$ on the sizes of $2^{-2i} y_i$ and $2^{-2j} z_j$. In the analysis of $X_9$ and $X_{10}$ we use the bound given by lemma 4 for individual entries.

$$C_9 = C_6 \cap \{(i,j) | (y_i 2^{-2i} \leq z_j 2^{-2j})\}$$

$$C_{10} = C_6 \cap \overline{C}_9 = C_6 \cap \{(i,j) | (y_i 2^{-2i} > z_j 2^{-2j})\}$$

$$X_9 = \sum_{C_9} 2^{-i-j} u_i^T A_s v_j$$

$$X_{10} = \sum_{C_{10}} 2^{-i-j} u_i^T A_s v_j$$

$$X_6 \leq X_9 + X_{10}$$

These cases are summarized in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
<th>$X_7$</th>
<th>$X_8$</th>
<th>$X_9$</th>
<th>$X_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$i \leq j &lt; \frac{1}{2} \log(d) \land (\max(y_i, z_j) &lt; d \min(y_i, z_j))$</td>
<td>$y_1 \geq z_j$</td>
<td>$y_1 &lt; z_j$</td>
<td>$\frac{d}{\sqrt{\lambda}} \sqrt{y_i z_j} &lt; n$</td>
<td>$\frac{d}{\sqrt{\lambda}} \sqrt{y_i z_j} \geq n$</td>
<td>$y_i 2^{-2i} &lt; \frac{\lambda}{\sqrt{d}} z_j 2^{-2j}$</td>
<td>$y_i 2^{-2i} \geq \frac{\lambda}{\sqrt{d}} z_j 2^{-2j}$</td>
<td>$y_i 2^{-2i} &lt; z_j 2^{-2j}$</td>
<td>$y_i 2^{-2i} \geq z_j 2^{-2j}$</td>
<td>$y_i 2^{-2i} &lt; \frac{\lambda}{\sqrt{d}} z_j 2^{-2j}$</td>
<td>$y_i 2^{-2i} \geq \frac{\lambda}{\sqrt{d}} z_j 2^{-2j}$</td>
</tr>
</tbody>
</table>

We now prove bounds on the leaves $X_2, X_3, X_7, X_8, X_9, X_{10}$ of the above tree.
Upper bound on $X_2$

$$X_2 \leq \left| \sum_{j \geq i + \frac{1}{2} \log(d)} (2^{-i} u_i)^T A_s (2^{-j} v_j) \right| \quad (X'_2)$$

$$+ \left| \sum_{i \leq j < i + \frac{1}{2} \log(d), \max(y_i, z_j) \geq d \min(y_i, z_j)} (2^{-i} u_i)^T A_s (2^{-j} v_j) \right| \quad (X''_2)$$

Note that since the number of edges out of any set $S$ is bounded by $d |S|$, we have that $|u_i^T A v_j| \leq d \min(y_i, z_j)$ for any $u_i, v_j \in \{-1, 0, +1\}^n$. We avoid writing the complete conditions from the sum when otherwise understood.

$$X'_2 \leq \sum_{i} \sum_{j = i + \frac{1}{2} \log(d)}^{\infty} 2^{-i-2j} |u_i^T A v_j| \quad X''_2 \leq \sum_{i} 2^{-i-j} |u_i^T A v_j|$$

$$\leq \sum_{i} \sum_{j = i + \frac{1}{2} \log(d)}^{\infty} 2^{-i} \cdot 2^{-j} d y_i \quad \leq \sum_{i} 2^{-i-j} (y_i + z_j)$$

$$\leq O(\sqrt{d}) \sum_{i} 2^{-2i} y_i \quad \leq O(1) \sum_{i} y_i 2^{-2i} + O(\sqrt{d}) \sum_{j} z_j 2^{-2j}$$

Combining $X'_2$ and $X''_2$, we get

$$X_2 \leq O(\sqrt{d}) \left( \sum_{j} z_j 2^{-2j} + \sum_{i} y_i 2^{-2i} \right) \quad (9)$$

Upper bound on $X_3$

$X_3$ is the sum conditioned over the following set of $i, j$

$$C_3 = \{ (i, j) | (i \leq j < i + \frac{1}{2} \log(d)) \land (\max(y_i, z_j) \leq d \min(y_i, z_j)) \land (y_i \geq z_j) \}$$

If $d/\lambda \sqrt{y_i z_j} \geq n$, then by lemma 5 (substituting $v = u_i, u_0 = v_j, u_1 = u_2, \cdots = \phi$), and the fact that $\frac{n \log(n)}{\log(d)} \leq 2$, we get that $|u_i^T A v_j| \leq O(\sqrt{d}) y_i$.

And if $d/\lambda \sqrt{y_i z_j} < n$, then by lemma 4 (substituting $u = v_j, v = u_i$), and the fact that since $y_i \leq n$, we get that $|u_i^T A v_j| \leq O(\sqrt{\lambda \log(2d)}) y_i \leq O(\sqrt{\lambda \log(2d)}) y_i$.

$$X_3 \leq \sum_{(i, j) \in C_3} 2^{-i-j} |u_i^T A v_j|$$

$$\leq \sum_{i} \sum_{j = i}^{\infty} 2^{-i-j} O(\max(\sqrt{d}, \sqrt{\lambda \log(d)})) y_i$$

Therefore we get that

$$X_3 \leq O(\max(\sqrt{d}, \sqrt{\lambda \log(d)})) \sum_{i} y_i 2^{-2i} \quad (10)$$

Upper bound on $X_7$
$X_7$ is the sum conditioned over the following set of $i, j$

$$C_7 = \left\{ (i, j) | (i \leq j) \land (j < i + \frac{1}{2} \log(d)) \land (y_i < d y_i) \land \left( \frac{d}{\lambda \sqrt{y_i z_j}} < n \right) \land \left( y_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} z_j 2^{-2j} \right) \right\}$$

We will use lemma 4 (substituting $u = u_i$ and $v = v_j$) to bound $|u_i^T A_s v_j|$.

$$X_7 \leq O(1) \sum_{(i, j) \in C_7} 2^{-i-j} \sqrt{\frac{\lambda}{y_i z_j} z_j \log \left( \frac{2d y_i}{z_j} \right)}$$

$$\leq O(1) \sum_{C_7} \frac{(\lambda)^{3/4}}{d^{1/8}} z_j 2^{-i-j} \left( \frac{2^{-j/2} \log \left( \frac{\sqrt{2d \lambda}}{2^{j-i}} \right)}{2^{j-i}} \right) (y_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} z_j 2^{-2j})$$

$$\leq O(1) \frac{(\lambda)^{3/4}}{d^{1/8}} \sum_j z_j 2^{-2j} \sum_{i=j}^{i=j + \frac{1}{2} \log(d) + 1} 2^{j-i} \log \left( \frac{\sqrt{2d \lambda}}{2^{j-i}} \right)$$

$$\leq O(1) \frac{\lambda^{3/4}}{d^{1/8}} \sqrt{d \log \left( \frac{\sqrt{2d \lambda}}{\sqrt{d}} \right)} \sum_j z_j 2^{-2j}$$

(by lemma 3 and $\lambda \geq \sqrt{d}$)

$$= O(1) \lambda \sqrt{\frac{d}{\lambda}} \log \left( \frac{\sqrt{d}}{\sqrt{d}} \right) \sum_j z_j 2^{-2j}$$

It can be argued that for every $c_1 > 0$, there exists $c_2$ s.t. $\sqrt{\frac{2d}{\lambda}} \log \left( \frac{\sqrt{2d \lambda}}{\sqrt{d}} \right) \leq (c_1 + c_2 \sqrt{d}/\lambda)$ where $c_1, c_2$ are constants. Hence, we can chose $c_1$ s.t.

$$X_7 \leq \left( \frac{\lambda}{5} + \mathcal{O}(\sqrt{d}) \right) \sum_j z_j 2^{-2j} \quad (11)$$

**Upper bound on $X_8$**

$X_8$ is the sum conditioned over the following set of $i, j$

$$C_8 = \left\{ (i, j) | (i \leq j) \land (j < i + \frac{1}{2} \log(d)) \land (y_i < z_j < d y_i) \land \left( \frac{d}{\lambda \sqrt{y_i z_j}} < n \right) \land \left( y_i 2^{-2i} \geq \frac{\lambda}{\sqrt{d}} z_j 2^{-2j} \right) \right\}$$

Again using lemma 4 (substituting $u = u_i$ and $v = v_j$) to bound $|u_i^T A_s v_j|$.

$$X_8 \leq O(1) \sum_{(i, j) \in C_8} 2^{-i-j} \sqrt{\frac{\lambda}{y_i z_j} z_j \log \left( \frac{2d y_i}{z_j} \right)}$$

$$= O(1) \sum_{(i, j) \in C_8} 2^{-i-j} \sqrt{\frac{z_j}{y_i}} \log \left( \frac{2d y_i}{z_j} \right)$$

$$= O(1) \sum_{(i, j) \in C_8} 2^{-i-j} \frac{d^{3/8}}{\lambda^{1/4} y_i} \sqrt{2^{3j-3i} \log \left( \frac{2d \sqrt{d}}{2^{2j-2i}} \right)} (y_i 2^{-2j} \geq \frac{\lambda}{\sqrt{d}} z_j 2^{-2j})$$
Above holds since \( x^2 \log \left( \frac{c}{x} \right) \) is increasing if \( x \leq \frac{c}{2} \)

\[
\begin{align*}
&\leq O(1) \sum_i d^{3/8} \lambda_{1/2} y_i 2^{-2i} \sum_{j=i+1}^{y_i+1} \log(2^{j-i} - 2^{j-i-1}) \\
&\leq O(1) \sum_i d^{3/8} \lambda_{1/2} y_i 2^{-2i} \sqrt{\lambda} \log \left( \frac{\sqrt{2 \lambda \sqrt{d}}}{2^{j-i}} \right) \\
&\leq O(1) \sum_i d^2 y_i 2^{-2i} \sqrt{\lambda} \log \left( \frac{\sqrt{2 \lambda}}{\sqrt{d}} \right)
\end{align*}
\]  
(by lemma 3)

Using \( \lambda \geq \sqrt{d} \), we have \( \sqrt{\lambda} \log \left( \sqrt{\frac{2 \lambda}{\sqrt{d}}} \right) \leq c \)

\[
X_8 \leq O(\sqrt{d}) \sum_i y_i 2^{-2i}
\]  
(12)

Upper bound on \( X_9 \)

\( X_9 \) is the sum conditioned over the following set of \( i, j \)

\[
C_9 = \{(i, j) \mid (i \leq j) \land (j < i + \frac{1}{2} \log(d)) \land (y_i \leq z_j \leq d y_i) \land \left( \frac{d}{\lambda} \sqrt{y_i z_j} \geq n \right) \land (y_i 2^{-2i} < z_j 2^{-2j}) \}
\]

In this case, we will use lemma 5 to bound \( \left| \sum_{i=j-1/2}^{i=j} 2^{-i+j} u_i^T A_j u_j \right| \). In this case, we group \( v_j \) according to support sizes and then sum them together. For \( c = 0, 1, 2, \ldots, \log(n) \), let \( J_c \) be the set of indices \( j \) s.t. \( n/2^c \leq z_j < 2n/2^c \) and \( J_c = \min(J_c) \)

\[
X_9 \leq O(1) \sum_j 2^{-2j} \sqrt{\frac{d^2}{n} \sum_{i=j-1/2 \log(d)}^{i=j} y_i 2^{-2i} \log \left( \frac{2n}{z_j} \right)}
\]

\[
\leq O(\sqrt{d}) \sum_j 2^{-2j} \frac{z_j^2}{n} \log \left( \frac{2n}{z_j} \right) \sum_{i=j-1/2 \log(d)}^{i=j} y_i 2^{-2i}
\]

\[
\leq O(\sqrt{d}) \sum_{c} \sum_{j \in J_c} \sqrt{4n2^{-2j-2c} \log(2 \cdot 2^c)} \sum_{i=j-1/2 \log(d)+1}^{i=j} y_i 2^{-2i} (\frac{n}{2^c} \leq z_j < \frac{2n}{2^c})
\]

\[
\leq O(\sqrt{d}) \sum_{c} \sum_{j \in J_c} \frac{1}{2} \left( 4n2^{-j-j_c-c} + 2^{-j+j-c} \log(2 \cdot 2^c) \sum_{i=j-1/2 \log(d)+1}^{i=j} y_i 2^{-2i} \right) (A.M. \geq G.M.)
\]

\[
\leq O(\sqrt{d}) \sum_{c} \sum_{j \in J_c} \left( \frac{1}{2} \left( 4n2^{-j-j_c-c} + \sqrt{d} \sum_{c} \sum_{j \in J_c} 2^{-j+j-c} \log(2 \cdot 2^c) y_i 2^{-2i} \right) \right)
\]

\[
\leq O(\sqrt{d}) \left( 4 \sum_{c} \frac{n}{2^c} \sum_{j \in J_c} 2^{-j-j_c} + \sum_{c} \sum_{j \in J_c} y_i 2^{-2i} \sum_{c} \log(2 \cdot 2^c) \sum_{j \in J_c} 2^{-j+j_c} \right)
\]

18
Summing up various GP’s, we get

\[ X_9 \leq O(\sqrt{d}) \left( \sum_j z_j 2^{-2j} + \sum_i y_i 2^{-2i} \right) \]  

(13)

**Upper bound on \( X_{10} \)**

\( X_{10} \) is the sum conditioned over the following set of \( i, j \)

\[ C_{10} = \left\{ (i, j) | (i \leq j) \land (j \leq i + \frac{1}{2} \log(d)) \land (y_i \leq z_j \leq dy_i) \land \left( \frac{d}{\lambda} \sqrt{y_i z_j} \geq n \right) \land (y_i 2^{-2i} \geq z_j 2^{-2j}) \} \]

We divide \( X_{10} \) into two parts depending on the value of \( i \) and \( j \).

\[ X_{10} \leq \left| \sum_{(i,j) \in C_{11}} 2^{-i-j} u_i^T A_s v_j \right| \]  

(11)

\[ + \left| \sum_{(i,j) \in C_{12}} 2^{-i-j} u_i^T A_s v_j \right| \]  

(12)

\[ C_{11} = \left\{ (i, j) | (i, j) \in C_{10}, (j < i + \frac{1}{2} \log(n/y_i)) \right\} \]

\[ C_{12} = \left\{ (i, j) | (i, j) \in C_{10}, (j \geq i + \frac{1}{2} \log(n/y_i)) \right\} \]

First we analyze \( X_{11} \). We use lemma 5(substituting \( u_0 = u_i, v = v_j, u_1 = u_2 = \cdots = \emptyset \) for bounding \( |u_i^T A_s v_j| \).

\[ X_{11} \leq \sum_{(i,j) \in C_{11}} 2^{-i-j} |u_i^T A_s v_j| \]

\[ \leq O(1) \sum_{(i,j) \in C_{11}} 2^{-i-j} \sqrt{\frac{dy_i z_j^2}{n} \log \left( \frac{2n}{z_j} \right)} \]

\[ \leq O(\sqrt{d}) \sum_{(i,j) \in C_{11}} 2^{-2i} y_i \sqrt{2^{-2j+2i} \frac{1}{n y_i} z_j^2 \log \left( \frac{2n}{z_j} \right)} \]

\[ \leq O(\sqrt{d}) \sum_{i} y_i 2^{-2i} \sum_{j=i}^{j=i+\frac{1}{2} \log(n/y_i)} \sqrt{\frac{y_i 2^{2j-2i}}{n y_i^2 2^{2j-2i}}} \]  

\[ (z_j 2^{-2j} \leq y_i 2^{-2i}) \]

Above holds because \( x^2 \log \left( \frac{x}{2} \right) \) is increasing function if \( x \leq \frac{c}{2} \)

\[ \leq O(\sqrt{d}) \sum_{i} y_i 2^{-2i} \]  

(lemma 3)

Next, we analyze \( X_{12} \). We again use lemma 5(substituting \( u_0 = u_i, v = v_j, u_1 = u_2 = \cdots = \emptyset \) for bounding
\[ |u_i^T A_s v_j| \]

\[ X_{12} \leq \sum_{(i,j) \in C_{12}} 2^{-i-j} |u_i^T A_s v_j| \]

\[ \leq O(1) \sum_{i} \sum_{j=i+\frac{1}{2} \log(n/y_i)}^{\infty} 2^{-i-j} \sqrt{dy_i z_j} \sqrt{\frac{z_i n \log \left( \frac{2n}{z_j} \right)}{y_i}} \]

\[ \leq O(\sqrt{d}) \sum_{i} y_i 2^{-2i} \]

Combining \( X_{11} \) and \( X_{12} \) we get,

\[ X_{10} \leq O(\sqrt{d}) \sum_{i} y_i 2^{-2i} \]  \hspace{1cm} (14)

**Putting it all together**

Next we put together the multiple calculations in Equations 9,10,11,12,13,14

\[ X_6 \leq X_9 + X_{10} \leq O(\sqrt{d}) \sum_{i} y_i 2^{-2i} + O(\sqrt{d}) \sum_{j} z_j 2^{-2j} \]

\[ X_5 \leq X_7 + X_8 \leq (\frac{\lambda}{5} + O(\sqrt{d})) \sum_{j} z_j 2^{-2j} + O(\sqrt{d}) \sum_{i} y_i 2^{-2i} \]

\[ X_4 \leq X_5 + X_6 \leq (\frac{\lambda}{5} + O(\sqrt{d})) \sum_{j} z_j 2^{-2j} + O(\sqrt{d}) \sum_{i} y_i 2^{-2i} \]

\[ X_1 \leq X_3 + X_4 \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_{i} y_i 2^{-2i} + (\lambda/5 + O(\sqrt{d})) \sum_{j} z_j 2^{-2j} \]

\[ X \leq X_1 + X_2 \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_{i} y_i 2^{-2i} + (\lambda/5 + O(\sqrt{d})) \sum_{j} z_j 2^{-2j} \]

But, if we know that for all \( i, j, y_i \geq z_j \), then

\[ X_4 = 0 \]

\[ X_1 \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_{i} y_i 2^{-2i} \]

\[ X \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) (\sum_{i} y_i 2^{-2i} + \sum_{j} z_j 2^{-2j}) \]

\[ \square \]

**5.2 Proof of Lemma 4**

For the sake of presentation we make a slight change of notation here. Let \( exp(x) \) represent \( e^x \)

**Proof.** Without loss of generality we can assume that \( S(v) \subseteq N_G(S(u)) \). If not we simply look at the restriction of \( v \) on the set \( S(v) \cap N_G(S(u)) \).

Let \( Bad(u,v) \) be the event that \( |u^T A_s v| > 8\sqrt{\lambda \sqrt{|S(u)||S(v)||S(v)} \log \left( \frac{2d|S(u)|}{|S(v)|} \right)} \). We need to bound \( P(\cup_{u,v} Bad(u,v)) \)

Note that the sum \( u^T A_s v \) over random choices of \( A_s \) is a sum of independent variables with maximum value \( \pm 2 \) or \( \pm 1 \) and mean 0. The maximum number of non-zero entries in this sum could be \( E(S(u), S(T)), \)
i.e. the number of edges which go from \(S(u), S(v)\) when they are seen as subsets of vertices of the original graph.

Therefore for a fixed \(u, v\) by applying Chernoff bounds we get that

\[
P(Bad(u, v)) = \Pr \left( |u^T A_x v| > 8 \sqrt{\lambda \sqrt{|S(u)||S(v)|} \log \left( \frac{2d|S(u)|}{|S(v)|} \right)} \right)
\]

\[
\leq 2 * \exp \left( -2 * \frac{64\lambda \sqrt{|S(u)||S(v)|} \log \left( \frac{2d|S(u)|}{|S(v)|} \right)}{4E(S,T)} \right) \quad (15)
\]

Now given the condition of the lemma and the expander mixing lemma we have that

\[
E(S(u), S(v)) \leq 2 * \lambda \sqrt{|S(u)||S(v)|}
\]

Putting this in the previous expression we get that the probability is bounded by

\[
2 * \exp \left( -16|S(v)| \log \left( \frac{2d|S(u)|}{|S(v)|} \right) \right)
\]

Note that we want to put an upper bound on \(P(\cup Bad(u, v))\) for all choices of \(u, v\). For this purpose we would first fix the size of support of \(u, v\) and union bound over all possible choices of \(u, v\) of that fixed support and then union bound over all choices of the support. For fixed support sizes \(|S(u)|, |S(v)|\), note that the total number of choices for the support sets for \(u\) are \(\binom{n}{|S(u)|}\) and for a fixed \(S(u)\) number of choices of \(S(v)\) for a fixed \(S(u)\) are bounded by \(\binom{d|S(u)|}{|S(v)|}\). Also since each entry in \(u, v\) is 0 or \(\pm 1\) the total number of choices for \(u\) and \(v\) are bounded by

\[
\left( \frac{n}{|S(u)|} \right) * 2^{|S(u)|} * \left( \frac{d|S(u)|}{|S(v)|} \right) * 2^{|S(v)|} \leq \exp \left( |S(u)| \log \left( \frac{n}{|S(u)|} \right) + (\ln(2) + 1)|S(u)| \right)
\]

\[
* \exp \left( |S(v)| \log \left( \frac{d|S(u)|}{|S(v)|} \right) + (\ln(2) + 1)|S(v)| \right) \quad (16)
\]

We will first show upper bounds on each of these terms. Note that since \(|S(v)| \geq \frac{n}{d}\) and hence \(|S(u)| \geq \frac{n}{d}\) we get that (assuming \(d \geq 2\))

\[
\exp \left( |S(u)| \log \left( \frac{n}{|S(u)|} \right) + (\ln(2) + 1)|S(u)| \right) \leq \exp \left( 3|S(u)| \log(d) \right)
\]

\[
= \exp \left( 3 \frac{|S(u)|}{|S(v)|} \log(d) \right) \cdot |S(v)| \log \left( \frac{2d|S(u)|}{|S(v)|} \right)
\]

\[
\leq \exp \left( 3 \cdot |S(v)| \log \left( \frac{2d|S(u)|}{|S(v)|} \right) \right)
\]

The last line follows by noting that \(x \log(d)/\log(2dx)\) is bounded by 1 for \(x \in [1/d, 1]\) and that \(\frac{|S(u)|}{|S(v)|} \in [1/d, 1]\)

Also note that

\[
\exp \left( |S(v)| \log \left( \frac{2d|S(u)|}{|S(v)|} \right) + (\ln(2) + 1)|S(v)| \right) \leq \exp \left( 3 \cdot |S(v)| \log \left( \frac{2d|S(u)|}{|S(v)|} \right) \right)
\]

Therefore by union bound we get that the probability of a bad event for fixed support sizes \(|S(u)|, |S(v)|\) is bounded by
\[2 \cdot \exp \left( -6|S(v)| \log \left( \frac{2d|S(u)|}{|S(v)|} \right) \right) \leq \exp \left( -6 \frac{n}{d^2} \log(2) \right)\]

Now the number of choices of the supports are \(n^2\) at best and we get that

\[\Pr \left( |v^T A_s u| \leq 8 \sqrt{\lambda \frac{d|S(u)||S(v)| \log(\frac{2d|S(u)|}{|S(v)|})}{n}} \right) \geq 1 - 2n^2 \exp \left( -6 \frac{n}{d^2} \log(2) \right) \geq 1 - \exp \left( -\Omega \left( \frac{n}{d^2} \right) \right)\]

Hence proved.

5.3 Proof of Lemma 5

Proof. As in the proof of the previous lemma. We will once again use the Chernoff bound to bound the probability of bad events. We will fix the size of the supports of \(v, u_1, u_2\ldots\) and prove that the probability is small and then union bound over the choices of the support.

Let first fix \(v, u_1, u_2, \ldots\). The sum \(|v^T A_s u|\) is once again a sum of independent random variables with mean 0. This is so because note that the intersection between any two sets in \(\{u_1\}\) is \(\phi\). It is easy to see that the number of squres of the maximum values of these variables is \(\sum_i 4 \cdot E(S(u_i), S(v)) \cdot 2^{d_i}\). Now we know that given the conditions of the lemma and the Expander Mixing Lemma \(E(S(u_i), S(v)) \leq 2 \cdot \frac{d|S(u)| \cdot |S(v)|}{n}\)

Therefore using the above we get (via Chernoff Bound) that for a fixed \(v, u_1, u_2, \ldots\)

\[\Pr \left( |v^T A_s u| > 8 \sqrt{\frac{6}{n}} \log(\frac{2n}{|S(v)|}) \right) \leq 2 \cdot \exp \left( -16S(v) \log(\frac{2n}{|S(v)|}) \right)\]

Now fixing the values of the support sizes \(|S(v)|, |S(u_1)|, |S(u_2)|, \ldots\) the number of possible choices for \(v\) are \(\binom{n}{|S(v)|} \cdot 2^{3|S(v)|} \leq \exp \left( 3|S(v)| \log(\frac{2n}{|S(v)|}) \right)\). Similarly the number of possible choices for each \(u_i\) are \(\exp \left( 3|S(u_i)| \log\left( \frac{2n}{|S(u_i)|} \right) \right)\). Therefore the total number of choices for all \(u_i\) are \(\exp \left( \sum |S(u_i)| \log\left( \frac{2n}{|S(u_i)|} \right) \right)\).

Note that since each \(|S(u_i)|, |S(v)| \leq n\) we can replace each \(|S(u_i)|\) by its upper bound \(\frac{|S(v)|}{2^{2i}}\). Therefore

\[\exp \left( \sum |S(u_i)| \log\left( \frac{2n}{|S(u_i)|} \right) \right) \leq \exp \left( 3 \sum \frac{|S(v)|}{2^{2i}} \log\left( \frac{2n}{|S(v)|} \right) \right) \]

\[\leq \exp \left( 10 \cdot |S(v)| \log\left( \frac{2n}{|S(v)|} \right) \right)\]

The last inequality follows by applying Lemma 3

Therefore the total number of choices of \(v, u_1 \ldots\) fixing \(|S(v)|, |S(u_1)| \ldots\) are bounded by \(\exp \left( 13 \cdot |S(v)| \log\left( \frac{2n}{|S(v)|} \right) \right)\). Therefore by union bound fixing the support sizes the probability of the bad event is bounded by

\[\exp \left( -3|S(v)| \log\left( \frac{2n}{|S(v)|} \right) \right) \leq \exp \left( -3 \frac{n}{d^2} \log(d) \right)\]

Now the number of choices for sizes of these supports are at best \(n \cdot n^\log(n)\). To see this since the size of each \(|S(u_i)|\) decreases exponentially there can be at best \(\log(n)\) such sets. Therefore putting together the union bound we get that the total probability of the bad event is bounded by

\[\exp \left( (\log(n)) \log(n) + 1 \right) \cdot \exp \left( -3 \frac{n}{d^2} \log(d) \right) \leq \exp \left( -\Omega \left( \frac{n}{d^2} \right) \right)\]

Hence proved. \(\square\)
References


6 Appendix: Full Proofs of Theorems 1 and 2

We will first prove theorem 2 and then theorem 1 will follow as a special case. To prove Theorem 2, we need a modified version of lemma 1.

**Lemma 6.** Let \( u_1, u_2, \ldots \in \{0, \pm 1, \pm \frac{1}{2}\}^n \), \( v_1, v_2, \ldots \in \{0, \pm 1, \pm \frac{1}{2}\}^n \) be two families of vector sets such that for all \((i, j), S(u_i) \cap S(v_j) = S(v_j) \cap S(u_i) = \emptyset \) and either for all \( i, |S(u_i)| > \frac{2n}{3} \) or for all \( i, |S(u_i)| > \frac{n}{3} \). Let \( A' \) be a random real matrix each of whose entry \( A'_{ij} \) is a random variable such that \( \forall i, j, E[A'_{ij}] = 0 \) and \( \forall i, j \) if \( A_{ij} = 0 \) then \( A'_{ij} = 0 \) and if \( A_{ij} = 1 \) then \( |A'_{ij}| \leq 1 \) always. Then with probability atleast \((1 - e^{-\Omega(n/d^2)})\)

**Proof.** For a shift lift \( \lambda_{new} = max_{\omega, \omega \neq 1} ||A_\omega|| \). So, if we prove that for a fix \( \omega, \lambda_f(||A_\omega||) \geq \lambda + O(max(\sqrt{\lambda \log d}, \sqrt{d}))) \leq e^{-\Omega(\frac{d}{n})}, \) then by union bound \( \lambda_{new} = \lambda + O(max(\sqrt{\lambda \log d}, \sqrt{d}))) \leq e^{-\Omega(\frac{d}{n})}. \)

Spectral radius of \( A_\omega \) is max \( \frac{||x^* A_\omega x||}{||x||} \) where \( x \in \mathbb{C}^n \). Let \( x = q + iw \in \mathbb{C}^n \) where \( q, w \in \mathbb{R}^n \).

Consider decomposition of \( y, w \) similar to diadic decomposition.

\[
[y_{ij}] = \begin{cases} 
q_{ij}, & \text{if } 2^{-i-1} < |q_{ij}| \leq 2^{-i} \\
0, & \text{otherwise}
\end{cases}
\]

\[
[z_{ij}] = \begin{cases} 
w_{ij}, & \text{if } 2^{-i-1} < |w_{ij}| \leq 2^{-i} \\
0, & \text{otherwise}
\end{cases}
\]

Partition the set of \( y_i \)'s into two sets \( A \) and \( B \) s.t. \( y_i \in A \text{ if } |S(y_i)| < n/d^2 \) and \( y_i \in B \text{ if } |S(y_i)| \geq n/d^2 \).

Define \( y_A = \sum_{y_i \in A} y_i \) and \( y_B = \sum_{y_i \in B} y_i \). Similarly, define \( z_A \) and \( z_B \). Call vectors \( y_A, z_A \) to be type \( A \) and \( y_B \) \& \( z_B \) to be type \( B \) vectors. Splitting the terms in \( ||x^* A_\omega x|| \), we get

\[
||x^* A_\omega x|| \leq ||(y_A + iz_A)^* A_\omega (y_A + iz_A)|| + ||z_B^* A_\omega z_B|| + ||y_B^* A_\omega y_B|| + ||z_B^* A_\omega y_A|| + ||y_B^* A_\omega z_B|| + ||z_B^* A_\omega z_B||
\]

To prove upper bound on \( ||x^* A_\omega x|| \), we will prove the following upper bounds

- \( ||(y_A + iz_A)^* A_\omega (y_A + iz_A)|| \leq (\lambda + O(\frac{1}{d})) ||y_A + iz_A||^2 \) (w.p.1)

- \( ||a^T A_\omega b|| \leq (\frac{3}{32} + O(\lambda)) (||a||^2 + ||b||^2) \) where \( a \) and \( b \) are type \( A \) vector (w.h.p.)

- \( ||b^T A_\omega a|| \leq (\frac{3}{32} ||b||^2 + O(\lambda)) ||a||^2 + ||b||^2 \) when \( a \) is type \( A \) and \( b \) is type \( B \) (w.h.p.)

Note that all terms of equation 19 fall into one of these categories. Using the above three upper bounds, we get \( ||x^* A_\omega x|| \leq (\lambda + O(\frac{1}{d})) ||y_A + iz_A||^2 + (\frac{3}{32}) (||y_B||^2 + ||z_B||^2) + ||y_B||^2 + ||z_B||^2 + O(\lambda) \) which in turn is less than \( (\lambda + O(\lambda)) ||x||^2 \) (w.h.p.). This proves that spectral radius of \( A_\omega \) is bounded by \( \lambda + O(\lambda) \). Now, we will prove the three bounds stated above
\[ ||(y_A + iz_A)^* A_s(\omega)(y_A + iz_A)|| \]

Easy to argue that \[ ||(y_A + iz_A)^* A_s(\omega)(y_A + iz_A)|| \leq y^T Ay' \] where \( y' \) is such that \( j \)-th element of \( y' \) is magnitude of \( j \)-th element in \( (y_A + iz_A) \). Let \( J \) be a \( n \times n \) matrix with all 1’s. Then \( y^T Ay = y^T (A - \frac{d}{n} J) y + y^T \frac{d}{n} J y' \). Spectral radius of \( A - \frac{d}{n} J \) is \( \lambda \). Hence, \( y^T (A - \frac{d}{n} J) y' \leq \lambda ||y'||^2 = \lambda ||(y_A + iz_A)||^2 \).

\[ y^T \frac{d}{n} J y' \leq (y_A' + z_A')^T \frac{d}{n} J (y_A' + z_A') \] where \( j \)-th entry of \( y_A' \) and \( z_A' \) are absolute value of \( j \)-th entry in \( y_A \) and \( z_A \) respectively. Note that number of entries between \( 2^{-i-1} \) and \( 2^{-i} \) in \( y_A' \) and \( z_A' \) are less than \( \frac{n}{2^i} \). Next, we prove \([u^T \frac{d}{n} J v] \leq \frac{1}{n} \sum ||u||^2 + ||v||^2 \) where \( u, v \in \{y_A', z_A'\} \).

Using Discretization lemma 2, there exists \( u', v' \) s.t. \([u^T \frac{d}{n} J v] \leq [u^T \frac{d}{n} J v'] \) where \( u', v' \in \{0, \pm \frac{1}{2}, \pm \frac{1}{4}, \ldots \} \), \([u']^2 \leq 4||u||^2, [v']^2 \leq 4||v||^2 \). Consider the diadic decomposition of \( u' = \{2^{-i} u_i \} \) and \( v' = \{2^{-j} v_j \} \). Since, all entries between \( \pm 2^{-i-1} \) and \( \pm 2^{-i} \) in \( u' \) and \( v' \), we get \([S(u_i)], [S(v_j)] < \frac{2n}{2^i}\).

\[ |u^T \frac{d}{n} J v'| = \sum_{i,j} 2^{-i-2-j} \left| \frac{d}{n} J_{ij} \right| \leq \sum_{i,j} 2^{-i-j} \frac{|S(u_i)||S(v_j)|}{n} \leq 2 \sum_{i,j} 2^{-2i} \left| S(u_i) \right| \left| S(v_j) \right| \]

For \( u, v \in \{y_A', z_A'\}, u^T \frac{d}{n} J v \leq u^T \frac{d}{n} J v' \leq O(\frac{1}{d})(||u'||^2 + ||v'||^2) \leq O(\frac{1}{d})(||u||^2 + ||v||^2) \)

\[ y^T \frac{d}{n} J y' \leq (y_A' + z_A')^T \frac{d}{n} J (y_A' + z_A') \leq y_A^T \frac{d}{n} J y_A' + y_A^T \frac{d}{n} J z_A' + z_A^T \frac{d}{n} J y_A' + z_A^T \frac{d}{n} J z_A' \leq y_A^T \frac{d}{n} J y_A' = O(\frac{1}{d})(||y_A'||^2 + ||y_A'||^2 + ||z_A'||^2 + ||z_A'||^2 + ||z_A'||^2 + ||z_A'||^2) \]

Combining \( y^T (A - \frac{d}{n} J) y' \) and \( y^T \frac{d}{n} J y' \), we get \([y_A + iz_A]^* A_s(\omega)(y_A + iz_A)|| \leq y' A y' \leq (\lambda + O(\frac{1}{d}))(||y_A + iz_A||^2) \).

\[ \|a^T A_s(\omega)b\| \] where \( a, b \in \mathbb{R}^n \). Let \( A_s(\omega) = A_1^s(\omega) + i A_2^s(\omega) \) where \( A_1^s(\omega) \) and \( A_2^s(\omega) \) are real matrices.

\[ ||a^T A_s(\omega)b|| \leq ||a^T A_1^s(\omega)b|| + ||a^T A_2^s(\omega)b|| \]

\[ |a^T A_1^s(\omega)b| \] By discretization lemma 2, there exists \( y, z \) s.t. \(|a^T A_1^s(\omega)b| \leq |y^T A_1^s(\omega)z| \) where \( y, z \in \{0, \pm \frac{1}{2}, \pm \frac{1}{4}, \ldots \}^n \) and \(|y||z| \leq 4 ||a||^2 \) and \(|z|^2 \leq 4 ||b||^2 \). Moreover, every entry of \( a \) and \( b \) between \( \pm 2^{-i-1} \) and \( \pm 2^{-i} \) is rounded to either \( \pm 2^{-i-1} \) or \( \pm 2^{-i} \) in \( y \) and \( z \) respectively. Consider the following decomposition of \( y \) and \( z \) into \( \{2^{-i} u_i \} \) and \( \{2^{-j} v_j \} \) respectively.

\[ \begin{align*}
[u_{ij}] &= \begin{cases}
\frac{u_j}{2^i}, & \text{if } 2^{-i-1} \leq |a_j| < 2^{-i} \\
0, & \text{otherwise}
\end{cases} \\
[v_{ij}] &= \begin{cases}
\frac{v_j}{2^i}, & \text{if } 2^{-i-1} \leq |b_j| < 2^{-i} \\
0, & \text{otherwise}
\end{cases}
\end{align*} \]
Some easy observations include $u_i, v_j \in \{0, \pm \frac{1}{2}, \pm 1\}^n$, $|y^T A_s^i(\omega)z| = |\sum_{i,j} 2^{-i-j} u_i^T A_s^i(\omega)v_j|$. Also, $||y||^2 = \sum_i 2^{-2i} |u_i|^2 \geq \frac{1}{4} \sum_i 2^{-2i} |S(u_i)|$

$$\left| \sum_{i \leq j} 2^{-i-j} u_i^T A_s^i(\omega)v_j \right| \leq \sum_{i \leq j} 2^{-i-j} u_i^T A_s^i(\omega)v_j + \sum_{i < j} 2^{-i-j} v_i^T A_s^i(\omega)u_j$$

- If both $u$ and $v$ are of type B, then $|S(u_i)|, |S(v_j)| \geq \frac{\lambda}{2^2}$ for all $i, j$. By lemma 6,

$$\left| \sum_{i \leq j} 2^{-i-j} u_i^T A_s^i(\omega)v_j \right| \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_i |S(u_i)|2^{-2i} + \left( \frac{\lambda}{384} + O(\sqrt{d}) \right) \sum_j |S(v_j)|2^{-2j}$$
$$\left| \sum_{i < j} 2^{-i-j} v_i^T A_s^i(\omega)u_j \right| \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_i |S(v_i)|2^{-2i} + \left( \frac{\lambda}{384} + O(\sqrt{d}) \right) \sum_j |S(u_j)|2^{-2j}$$

Combining the above two, we get

$$|y^T A_s^i(\omega)z| = \left| \sum_{i,j} 2^{-i-j} u_i^T A_s^i(\omega)v_j \right|$$

$$\leq \left( \frac{\lambda}{384} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \right) \sum_i |S(u_i)|2^{-2i} + \sum_j |S(v_j)|2^{-2j}$$
$$\leq \left( \frac{\lambda}{96} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \right) ||y||^2 + ||z||^2$$

$$|a^T A_s^i(\omega)b| \leq |y^T A_s^i(\omega)z| \leq \left( \frac{\lambda}{96} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \right) ||y||^2 + ||z||^2$$

$$\leq \left( \frac{\lambda}{24} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \right) ||a||^2 + ||b||^2$$

Similarly, we get $|a^T A_s^i(\omega)b| \leq \left( \frac{\lambda}{24} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \right) ||a||^2 + ||b||^2$.

Combining $|a^T A_s^i(\omega)b|$ and $|a^T A_s^i(\omega)b|$, we get $|a^T A_s^i(\omega)b| \leq \left( \frac{\lambda}{12} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \right) ||a||^2 + ||b||^2$ if both $a$ and $b$ are of type B

- If $a$ is of type A and $b$ is of type B, $|S(u_i)| < \frac{\lambda}{2^2}$ and $|S(v_j)| \geq \frac{\lambda}{2^2}$ for all $i, j$. By lemma 6,

$$\left| \sum_{i \leq j} 2^{-i-j} u_i^T A_s^i(\omega)v_j \right| \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_i |S(u_i)|2^{-2i} + \left( \frac{\lambda}{384} + O(\sqrt{d}) \right) \sum_j |S(v_j)|2^{-2j}$$
$$\left| \sum_{i < j} 2^{-i-j} v_i^T A_s^i(\omega)u_j \right| \leq O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_i |S(v_i)|2^{-2i} + \sum_j |S(u_j)|2^{-2j}$$

Combining the above two, we get

$$|y^T A_s^i(\omega)z| \leq \left| \sum_{i,j} 2^{-i-j} u_i^T A_s^i(\omega)v_j \right|$$

$$\leq \frac{\lambda}{384} \sum_j |S(v_j)|2^{-2j} + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) \sum_j |S(v_j)|2^{-2j} + \sum_i |S(u_i)|2^{-2i}$$
$$\leq \frac{\lambda}{96} ||z||^2 + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) ||z||^2 + ||y||^2$$

$$|a^T A_s^i(\omega)b| \leq |y^T A_s^i(\omega)z| \leq \frac{\lambda}{96} ||z||^2 + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) ||z||^2 + ||y||^2$$
$$\leq \frac{\lambda}{24} ||b||^2 + O(\max(\sqrt{\lambda \log(d)}, \sqrt{d})) ||b||^2 + ||a||^2$$
Similarly, we get $|a^TA_2(\omega)b| \leq \frac{\lambda}{22}||b||^2 + \mathcal{O}(\max(\sqrt{\lambda\log(d)}, \sqrt{d}))(||b||^2 + ||a||^2)$ and $|a^TA_\ast(\omega)b| \leq \frac{\lambda}{22}||b||^2 + \mathcal{O}(\max(\sqrt{\lambda\log(d)}, \sqrt{d}))(||b||^2 + ||a||^2)$ if $u$ is of type A and $v$ is of type B.

Putting the above bounds in Equation 19 concludes the proof of Theorem 2.

Proof. (Theorem 1) 2-lift is a special case of shift $k$-lift and hence, it follows from Theorem 2.

We now prove the combinatorial identities Lemma 2 and Lemma 3

Proof of Lemma 2. To obtain such a vector $y$ we simply take a vector $x$ and round its coordinates independently with the following probabilistic rule. Let $x_i = \pm(1 + \delta_i)2^{-i}$ be the $i^{th}$ coordinate of $x$. We round $x_i$ to $\text{sign}(x_i) \cdot 2^{-i+1}$ with probability $\delta_i$ and $\text{sign}(x_i) \cdot 2^{-i}$ with probability $1 - \delta_i$. Let the rounded vector be $x'$. Note that $E[x'_i] = x_i$. Now since each coordinate is rounded independently and the diagonal entries of $M$ are 0, we get that $E[x'^T M x'] = x'^T M x$. This implies there exists a $y \in \{\pm 1/2, \pm 1/4, \ldots\}^n$ that can be generated by this rounding such that $|x'^T M x| \leq |y^T M y|$. Also it is easy to see that $||y||^2 \leq 4 \cdot ||x||^2$ and by definition in $y$ every coordinate value between $\pm 2^{-i}$ and $\pm 2^{-i-1}$ is rounded to either $\pm 2^{-i}$ or $\pm 2^{-i-1}$. The proof of the second part of the lemma is the same as the first part. Here we obtain $x'_1$ and $x'_2$ by the same procedure and follow the exact same argument to get $y_1$ and $y_2$.

Proof of Lemma 3. For all $i$ define $a_i = (r^i \log(z/r^i))^x$. Let consider the ratio of consecutive terms $a_{i+1}/a_i$ for $i \in [0, t - 1]$.

$$\frac{a_{i+1}}{a_i} = \left(\frac{r^{i+1} \log(z/r^{i+1})}{r^i \log(z/r^i)}\right)^x$$

$$= \left(r \left(1 - \frac{\log(r)}{\log(z) - i \log(r)}\right)^x\right)$$

$$\geq \left(r \left(1 - \frac{\log(r)}{1 + (t - i) \log(r)}\right)^x\right)$$

$(r^t \leq z/2)$

If $i \leq t - 2$ we get that $a_{i+1}/a_i \geq r^x \left(\frac{1+\log(r)}{1+2 \log(r)}\right)^x = \alpha(r)$. It is easy to see that $\alpha(r) > 1$ for $r \geq 2$. Also for $i = t - 1$ we get that $a_{i+1}/a_i \geq (r/(1 + \log(r)))^x \geq 1$.

Now consider the sum $S_{-1}$ defined as

$$S_{-1} = a_0 + a_1 + \ldots + a_{t-1}$$

$$\Rightarrow \alpha(r) S_{-1} = \alpha(r) (a_0 + a_1 + \ldots + a_{t-1})$$

$$\Rightarrow (\alpha(r) - 1) S_{-1} = -a_0 + (\alpha(r)a_0 - a_1) + (\alpha(r)a_1 - a_2) + \ldots + a_{t-1} \alpha(r)$$

$$\Rightarrow (\alpha(r) - 1) S_{-1} \leq a_{t-1} \alpha(r)$$

$$\Rightarrow S_{-1} \leq a_{t-1} \left(\frac{\alpha(r)}{\alpha(r) - 1}\right)$$

$(a_{i+1} \geq \alpha(r)a_i)$

Therefore

$$\sum_{i \in [t]} a_i \leq S_{-1} + a_t \leq \left(1 + \left(\frac{\alpha(r)}{\alpha(r) - 1}\right)\right) a_t$$

Setting $c(r) = \left(1 + \left(\frac{\alpha(r)}{\alpha(r) - 1}\right)\right)$ we get the required result.

27