On the Diversity-Multiplexing Tradeoff of Unconstrained MIMO Fading Channels

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Abstract—In this work the optimal diversity-multiplexing tradeoff (DMT) is studied for the multiple-input multiple-output fading multiple-access channel with no power constraints (infinite constellations). For $K$ users ($K > 1$), $M$ transmit antennas for each user, and $N$ receive antennas, infinite constellations in general and lattices in particular are shown to attain the optimal DMT of finite constellations for the case $N \geq (K+1)M - 1$, i.e. user limited regime. On the other hand for the case $N < (K+1)M - 1$ it is shown that infinite constellations can not attain the optimal DMT. This is in contrast to the point-to-point case where infinite constellations are DMT optimal for any $M$ and $N$. In general, this work shows that when the network is heavily loaded, i.e. $K > \max (1, \frac{N-M+1}{M})$, taking into account the shaping region in the decoding process plays a crucial role in pursuing the optimal DMT.

I. INTRODUCTION

Employing multiple antennas in a point-to-point wireless channel increases the number of degrees of freedom available for transmission. This is illustrated for the ergodic case in [1],[2], where $M$ transmit and $N$ receive antennas increase the capacity by a factor of $\min (M, N)$. The number of degrees of freedom utilized by the transmission scheme is referred to as multiplexing gain. Another advantage of employing multiple antennas is the potential increase in the transmitted signal reliability. The fact that multiple antennas increase the number of independent links between antenna pairs, enables the error probability to decrease, i.e. add diversity. If for high signal to noise ratio (SNR) the error probability is proportional to $\text{SNR}^{-d}$, then we state that the diversity order is $d$.

For the point-to-point setting, Zheng and Tse [3] characterized the optimal diversity-multiplexing tradeoff (DMT) of the quasi-static Rayleigh flat-fading channel, i.e. for each multiplexing gain they found the best attainable diversity order. The optimal DMT is a piecewise linear function connecting the points $(M-l)(N-l), \ l = 0, \ldots, \min (M, N)$. The transmission scheme in [3] uses random codes. Subsequent works presented more structured schemes that attain the optimal DMT. El Gamal et al. [4] showed by using probabilistic methods that lattice space-time (LAST) codes attain the optimal DMT by using minimum-mean square error (MMSE) estimation followed by lattice decoding. Later, explicit coding schemes based on lattices and cyclic-division algebra [5], [6] were shown to attain the optimal DMT by using maximum-likelihood (ML) decoding, and also by using MMSE estimation followed by lattice decoding [7]. A subtle but very important fact is that these coding schemes take into consideration the power constraint in the decoder. A question that remained open was whether lattices can achieve the optimal DMT by using regular lattice decoding, i.e. decoder that takes into account the infinite lattice without considering the shaping region or the power constraint. In order to answer this question, the work in [8] presented an analysis of the performance of infinite constellations (IC’s) in the multi-input multiple-output (MIMO) fading channel. A new tradeoff was presented between the IC’s average number of dimensions per channel use, i.e. the IC dimensionality divided by the number of channel uses, and the best attainable DMT. By choosing the right average number of dimensions per channel use, it was shown [8] that IC’s in general and more specifically lattices using regular lattice decoding, attain the optimal DMT of finite constellations.

For the multiple-access channel, where a number of users transmit to a single receiver, the number of users in the network affects the multiplexing gain and the diversity order. For instance, for a network with $K$ users transmitting at the same rate, the number of available degrees of freedom for each user is $\min (M, N)$. Tse, Viswanath and Zheng [9] characterized the optimal DMT of a network with $K$ users, where each user has $M$ transmit antennas and the receiver has $N$ antennas. For the symmetric case, where the users transmit at the same multiplexing gain $r$, i.e. $r_1 = \cdots = r_K = r$, the optimal DMT takes the following elegant form [9]:

- For $r \in \left[0, \min \left(\frac{N}{K+1}, M\right)\right]$ the optimal symmetric DMT equals to the optimal DMT of a point-to-point channel with $M$ transmit and $N$ receive antennas $d_{M,N}^{(FC)} (r)$.
- For $r \in \left[\min \left(\frac{N}{K+1}, M\right), \min (M, N)\right]$ the optimal symmetric DMT equals to the optimal DMT of a point-to-point channel with all $K$ users pulled together $d_{K}^{(FC)} (\min (K, r))$.

Similar to the development in the point-to-point case, random codes were used in [9]. Later Nam and El Gamal [10] showed that a random ensemble of LAST codes attains the optimal DMT of the multiple-access channel using MMSE estimation followed by lattice decoding over the lattice induced by the
$K$ users. An explicit coding scheme based on lattices and cyclic division algebra that attains the optimal DMT using ML decoding was presented in [11].

In this paper we study the optimal DMT of lattices using regular lattice decoding, i.e. decoding without taking into consideration the power constraint, for the MIMO Rayleigh fading multiple-access channel. The result is rather surprising; unlike the point-to-point case where the tradeoff between dimensions and diversity enables to attain the optimal DMT, we show that for the multiple-access channel the optimal DMT is attained only when $N \geq (K + 1)M - 1$, i.e. user limited regime. On the other hand when the network is heavily loaded we show that IC's or lattices using regular lattice decoding, can not attain the optimal DMT.

In the first part of this paper an upper bound on the optimal symmetric DMT IC's can achieve is presented. The upper bound is attained by finding for each multiplexing gain $r$, the average number of dimensions per channel use for each user, that maximizes the diversity order. For the case $N < (K + 1)M - 1$ it is shown that the optimal DMT of IC's does not coincide with the optimal DMT of finite constellations. Moreover, for $N < (K - 1)M + 1$ it is shown that the optimal DMT of IC's in the symmetric case is inferior compared to the optimal DMT of finite constellations, for any value of $r$ except for the edges $r = 0, \frac{1}{M}$. On the other hand for the case $N \geq (K + 1)M - 1$, by choosing the correct average number of dimensions per channel use for each user, it is shown that the upper bound on the optimal DMT of IC's coincides with the optimal DMT of finite constellations $d_{M,N}^{\infty}(FC) (\max (r_1, \ldots, r_K))$.

In the second part of this paper, a transmission scheme that attains the optimal DMT for $N \geq (K + 1)M - 1$ is presented. Each user in this scheme transmits according to the DMT optimal scheme for the point-to-point channel, presented in [8]. By analyzing the receiver joint ML decoding performance, it is shown that this transmission scheme attains the optimal DMT of finite constellations. We wish to emphasize that the proposed transmission scheme is more involved than simply using orthogonalization between users, which in general is suboptimal for IC’s. The proposed transmission scheme requires $N + M - 1$ channel uses to attain the optimal DMT, which is smaller than $N + KM - 1$, the number of channel uses required in [9] (the dependence in the number of users lies in the fact that $N \geq (K + 1)M - 1$).

As a basic illustrative example of the results we consider the following two cases. In the first case assume a network with two users ($K = 2$), where each user has a single transmit antenna ($M = 1$), and a receiver with a single receive antenna ($N = 1$). In this case the optimal DMT of finite constellations in the symmetric case [9] equals $1 - r$ for $r \in [0, \frac{1}{2}]$, and $2 - 4r$ for $r \in \left[\frac{1}{2}, \frac{3}{4}\right]$. For IC’s it is shown in this setting that the optimal DMT for the symmetric case equals $1 - 2r$ for $r \in \left[0, \frac{1}{2}\right]$, which is strictly inferior except for $r = 0, \frac{1}{2}$. In the second case, by merely adding another receive antenna, i.e. $M = 1, N = K = 2$, the optimal DMT of IC’s coincides with finite constellations optimal DMT $d_{1,2}^{\infty}(FC) (\max (r_1, r_2))$.

The outline of the paper is as follows. In section II basic definitions for the fading multiple-access channel and IC’s are given. Section III presents an upper bound on the optimal DMT of IC’s, and shows the sub-optimality of IC’s for the case $N < (K + 1)M - 1$. Transmission scheme that attains the optimal DMT of finite constellations for the case $N \geq (K + 1)M - 1$ is presented in section IV. Finally, in section V we discuss the results in this paper and present for the multiple-access channel a geometrical interpretation to the DMT of IC’s. This paper contains the sketch of proofs. The full version with detailed proofs can be found in [12].

II. BASIC DEFINITIONS

A. Channel Model

We consider a $K$-user multiple access channel where each user has $M$ transmit antennas, and the receiver has $N$ antennas. We assume perfect knowledge of all channels at the receiver, and no channel knowledge at the transmitters. We also assume quasi static flat-fading channel for each user. The channel model is as follows:

$$y_i = \sum_{i=1}^{K} h_i(i) \cdot x_i(i) + \rho^{-\frac{1}{2}} n_i \quad t = 1, \ldots, T \quad (1)$$

where $x_i(i), t = 1, \ldots, T$ is user $i$ transmitted signal, $y_i \sim CN(0, \sqrt{\rho} I_N)$ is the additive noise where $CN$ denotes complex-normal, $I_N$ is the $N$-dimensional unit matrix, and $y_i \in \mathbb{C}^N$. $h_i(i)$ is the fading matrix of user $i$. It consists of $N$ rows and $M$ columns, where $h_i(i) \sim CN(0, 1)$. $0 \leq j \leq M$, are the entries of $h(i)$. The scalar $\rho^{-\frac{1}{2}}$ multiplies each element of $y_i$, where $\rho$ can be interpreted as the average SNR of each user in the receive antennas for power constrained constellations that satisfy $\frac{1}{T} \sum_{t=1}^{T} \|y_i(t)\|^2 \leq \frac{2}{N\rho}$.

Next we wish to define an equivalent channel to (1). Let us define the extended transmission vector

$$x = \left( x_1^{(1)}, \ldots, x_K^{(K)}, \ldots, x_1^{(T)}, \ldots, x_T^{(K)} \right)^{\dagger} \quad (2)$$

i.e., first concatenate the users in each channel use, and then concatenate the vectors between channel uses. Now we define $H = (H^{(1)}, \ldots, H^{(K)})$ which is an $N \times KM$ matrix. By defining $H_{ex}$ as an $NT \times KM$ block diagonal matrix, where each block on the diagonal equals $I$, $y_{ex} = \rho^{-\frac{1}{2}} \cdot \left( n_1^{(1)}, \ldots, n_T^{(K)} \right)^{\dagger} \in \mathbb{C}^{NT}$ and $y_{ex} \in \mathbb{C}^{NT}$, we can rewrite the channel model in (1)

$$y_{ex} = H_{ex} \cdot x + y_{ex} \quad (3)$$

Let $L = \min (N, KM)$, and let $\sqrt{\lambda_i}, 1 \leq i \leq L$ be the real valued, non-negative singular values of $H$. Assume $\sqrt{\lambda_1} \geq \cdots \geq \sqrt{\lambda_L} > 0$. For large values of $\rho$, we state that $f(\rho) \geq \tilde{g}(\rho)$ when $\lim_{\rho \to \infty} \frac{\ln(\tilde{g}(\rho))}{\ln(\rho)} \geq \frac{\ln(g(\rho))}{\ln(\rho)}$, and also define $\leq, \doteq$ in a similar manner by substituting $\geq$ with $\leq$, respectively.
B. Infinite Constellations

Infinite constellation (IC) is a countable set $S = \{s_1, s_2, \ldots \} \in \mathbb{C}^n$. Let $\text{cube}_l(a) \subset \mathbb{C}^n$ be a (probably rotated) $l$-complex dimensional cube of length $a$ centered around zero. We define an IC $S_l$ to be $l$-complex dimensional if there exists a rotated $l$-complex dimensional cube $\text{cube}_l(a)$ such that $S_l \subset \lim_{a \to \infty} \text{cube}_l(a)$ and $l$ is minimal. $M(S_l, a) = |S_l \cap \text{cube}_l(a)|$ is the number of points of the IC $S_l$ inside $\text{cube}_l(a)$. In [13], the $n$-complex dimensional IC density was defined as

$$γ_G = \limsup_{a \to \infty} \frac{M(S_n, a)}{a^{2n}}$$

and the volume to noise ratio (VNR) for the additive white Gaussian noise (AWGN) channel was given as

$$μ_G = \frac{γ_G^{-\frac{1}{2}}}{2πeσ^2}$$

where $σ^2$ is the noise variance of each component.

We now turn to the IC definitions at the transmitters. We define the average number of dimensions per channel use as the IC dimension divided by the number of channel uses. Let us consider user $i$, where $1 \leq i \leq K$. We denote the average number of dimensions per channel use for $D_i$. Let us consider a $D_i \cdot T$-complex dimensional sequence of IC’s - $S_{D_i,T}^i(\rho)$, where $D_i \leq M, T$ is the number of channel uses, and $\sum_{i=1}^{K} D_i \leq L$. First we define $γ_i(\rho) = ρ_i^{-T}$ as the density of $S_{K,T}^i(\rho)$ in transmitter $i$. Similarly to the definitions in [8] the multiplexing gain of user’s $i$ IC is defined as

$$r_i = \lim_{ρ \to \infty} \frac{1}{T} \log_ρ(γ_i(ρ) + 1) = \lim_{ρ \to \infty} \frac{1}{T} \log_ρ(ρ_i^{-T} + 1), \quad 0 \leq r_i \leq D_i. \quad (4)$$

The VNR at the transmitter of user $i$ is

$$μ_i(\rho) = \frac{γ_i(\rho) - 1}{2πeσ^2} = ρ_i^{-\frac{1}{2}} \quad (5)$$

where $σ^2 = \frac{ρ_i^{-1}}{2πe}$ is each component’s additive noise variance. Now let us concatenate the users IC’s in accordance with (2). We denote $D = \sum_{i=1}^{K} D_i$. The concatenation yields an equivalent $D \cdot T$-complex dimensional IC, $S_{D,T}(\rho)$, that has multiplexing gain $\sum_{i=1}^{K} r_i$, density $γ_{tr} = ρ^{(\sum_{i=1}^{K} r_i)T}$ and VNR $μ_{tr} = ρ^{-\frac{1}{2}}$. In this case we get in (3) that the transmitted signal $x \in S_{D,T}(\rho) \subset \mathbb{C}^{KMT}$.

At the receiver we first define the set $H_{ex} \cdot \text{cube}_{D,T}(a)$ as the multiplication of each point in $\text{cube}_{D,T}(a)$ with the matrix $H_{ex}$. In a similar manner, the IC induced by the channel at the receiver is $S_{D,T}^r = H_{ex} \cdot S_{D,T}$. The set $H_{ex} \cdot \text{cube}_{D,T}(a)$ is almost surely $D \cdot T$-complex dimensional (where $D \leq L$). In this case

$$M(S_{D,T}, a) = |S_{D,T} \cap \text{cube}_{D,T}(a)| = |S_{D,T} \cap (H_{ex} \cdot \text{cube}_{D,T}(a))|.$$
III. UPPER BOUND ON THE BEST DIVERSITY-MULTIPLEXING TRADEOFF

In this section we show that for \( N < (K+1)M - 1 \) the DMT of the unconstrained multiple-access channel is suboptimal compared to the optimal DMT of finite constellations. On the other hand for \( N \geq (K+1)M - 1 \), we present an upper bound on the optimal DMT that coincides with the optimal DMT of finite constellations.

We begin by lower bounding the error probability of any IC for the multiple-access channel, by using lower bounds on the error probability of any IC in the point-to-point channel. We use those lower bounds to formulate an upper bound on the optimal DMT of IC’s for the multiple-access channel, in the form of an optimization problem. Then, we solve this optimization problem for the symmetric case, and compare the optimal DMT of IC’s to the optimal DMT of finite constellations, and find the cases where IC’s are suboptimal.

Finally, we give a convexity argument that shows for the symmetric case that whenever the optimal DMT is not a convex function IC’s are suboptimal.

Assume user \( i \) transmits \( D_i \cdot T \)-complex dimensional IC, with average number of dimensions per channel use \( D_i \) and \( T \) channel uses. The following lemma lower bounds the average decoding error probability of the \( K \)-users \( P_e (D_1, \ldots, D_K, T) (\rho, r_1, \ldots, r_K) \), where \( (D_1, \ldots, D_K) \) is the tuple of average number of dimensions per channel use, \( T \) is the number of channel uses and \( (r_1, \ldots, r_K) \) is the tuple of multiplexing gains.

**Lemma 1.**

\[
P_e (D_1, \ldots, D_K, T) (\rho, r_1, \ldots, r_K) \geq \max_{A \subseteq \{1, \ldots, K\}} \left( P_e (D_A, T) (\rho, R_A) \right)
\]

where \( P_e (D_A, T) (\rho, R_A) \) is the lower bound derived in [8] on the error probability of any IC with \( T \) channel uses, \( D_A = \sum_{a \in A} D_a \) average number of dimensions per channel use, and multiplexing gain \( R_A = \sum_{a \in A} r_a \), in a point-to-point channel with \( |A| \cdot M \) transmit and \( N \) receive antennas.

**Proof:** By considering the extended channel model (3), we get that the \( K \) distributed transmitters transmit an effective \( \left( \sum_{k=1}^K D_i \right) \cdot T \)-complex dimensional IC, over \( T \) channel uses, with multiplexing gain \( \sum_{i=1}^K r_i \). The error probability of this IC is lower bounded by the lower bound on the error probability of any IC with average number of dimensions per channel use \( \sum_{k=1}^K D_i \), \( T \) channel uses, and multiplexing gain \( \sum_{i=1}^K r_i \), in a point-to-point channel with \( K \cdot M \) transmit and \( N \) receive antennas. Such a lower bound on the error probability was derived in [8] for each channel realization (8) Theorem 1), and then for the average over all channel realizations when \( \rho \) is large (8) Theorem 2). Now consider the set \( A \subseteq \{1, \ldots, K\} \). In case a genie tells the receiver the transmitted messages of users \( \{1, \ldots, K\} \setminus A \), the optimal receiver attains an error probability that lower bounds the \( K \)-user optimal receiver error probability. Without loss of optimality, the optimal receiver can subtract them from the received signal, and get a new \( |A|\)-users unconstrained multiple-access channel with average number of dimensions per channel use \( \{D_a\}_{a \in A} \), \( T \) channel uses, and multiplexing gain \( \sum_{a \in A} r_a \). In a similar manner, the error probability of this \( |A|\)-users channel is lower bounded by the lower bound on the error probability of any IC with \( \sum_{a \in A} D_a \) average number of dimensions per channel use, \( T \) channel uses, and multiplexing gain \( \sum_{a \in A} r_a \), derived in [8]. Hence, the maximal lower bound on the error probability between all \( A \subseteq \{1, \ldots, K\} \) also sets a lower bound on the error probability. This concludes the proof.

Next we formulate an upper bound on the DMT of any sequence of IC’s in the \( K \)-user unconstrained multiple-access channel.

**Theorem 1.** The optimal DMT of any sequence of IC’s with multiplexing gains tuple \( (r_1, \ldots, r_K) \) is upper bounded by

\[
d_{\text{IC}}^{\text{opt}} (D_{1}, \ldots, D_{K}, M, N, (r_1), \ldots, (r_K)) = \max_{D \subseteq \{D_1, \ldots, D_K\}} \min_{R_1, \ldots, R_K} \left( d_{\text{IC}}^{D_1, \ldots, D_K, M, N} (R_1), \ldots, d_{\text{IC}}^{D_1, \ldots, D_K, M, N} (R_K) \right)
\]

where \( D = \{D_1, \ldots, D_K \mid 0 \leq D_1 \leq M, \sum_{i=1}^K D_i \leq L \} \).

**Sketch of the proof:** From Lemma 1 we get a lower bound on the error probability of any sequence of effective IC’s \( S \sum_{i=1}^K D_i, T (\rho) \), transmitted by the \( K \) users. The lower bound on the error probability can be translated to an upper bound on the diversity order. In addition, the lower bound on the error probability depends on lower bounds on the error probabilities for the point-to-point channel. Hence, we can use the upper bound on the DMT in the point-to-point channel, presented in [8], to get the following upper bound on the DMT of a tuple of average number of dimensions per channel use \( (D_1, \ldots, D_K) \)

\[
\min_{D \subseteq \{D_1, \ldots, D_K\}} \left( d_{\text{IC}}^{D_1, \ldots, D_K, M, N} (R_1), \ldots, d_{\text{IC}}^{D_1, \ldots, D_K, M, N} (R_K) \right).
\]

Maximizing over \( (D_1, \ldots, D_K) \in D \) yields the upper bound on the optimal DMT.

Next we characterize an upper bound on the optimal DMT of IC’s in the symmetric case, i.e. \( r_1 = \cdots = r_K = r \), that later will be used to show the sub-optimality of the unconstrained multiple-access channel in the case \( N < (K+1)M - 1 \). In addition, we will show that this upper bound coincides with the optimal DMT of finite constellations in the case \( N \geq (K+1)M - 1 \). In order to present the upper bound we define the case \( N = (K-1)M + 1 + l < (K+1)M - 1, l = 0, \ldots, 2M - 3 \) the term

\[
d^*(r) = MN - \left\lfloor \frac{l}{2} \right\rfloor \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) - 2 \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \cdot \left( \frac{l}{2} - \left\lfloor \frac{l}{2} \right\rfloor \right) - (N + M - 1 - l) r.
\]

**Theorem 2.** The optimal DMT of any sequence of IC’s in the symmetric case is upper bounded by:
For $N \geq (K+1)M-1$
\[ d^*_{K,M,N}(r) = d^*_{M,N}(r). \]

For $N < (K-1)M+1$
\[ d^*_{K,M,N}(r) = M \cdot N - K \cdot M \cdot r. \]

For $N = (K-1)M+1$ and $l < (K+1)M-1$, where $l = 0, \ldots, 2M-3$
\[ d^*_{K,M,N}(r) = \begin{cases} 
 0 \leq r \leq \frac{M}{K} + 1, \\ 
 \frac{M}{K} + 1 \leq r \leq \frac{(K-1)M+1}{K}, \\ 
 \frac{(K-1)M+1}{K} \leq r \leq \frac{N}{K}. 
\end{cases} \]

Sketch of the proof: Based on Theorem 1 we can state that the DMT of any sequence of IC’s, in the symmetric case of $K$ users, is upper bounded by
\[ d^*_{K,M,N}(r) = \max_{(D_1, \ldots, D_K) \in D} \min_{A \subseteq \{1, \ldots, K\}} \left( d^*_{|A|,M,N}(|A| \cdot r) \right) \]

for any $0 \leq r \leq \frac{N}{K}$, i.e. we wish solve the optimization problem in (8) for each $0 \leq r \leq \frac{N}{K}$. In order to solve this optimization problem we first solve a simpler optimization problem for the case $D_1 = \cdots = D_K = D$, i.e. each user transmits $D$ average number of dimensions per channel use. In this case the upper bound in (8) takes a simpler form
\[ \max_D \min_{1 \leq i \leq K} \left( d^*_{i,M,N}(i \cdot r) \right) \]

where $0 \leq D \leq \frac{N}{K}$. This optimization problem is solved by analyzing the relations between $d^*_{i,M,N}(i \cdot r)$, $1 \leq i \leq K$, for different values of $D$, $M$ and $N$. After solving this optimization problem we prove that choosing $D_1 = \cdots = D_K = D$ also yields the optimal solution for (8).

Figure 1 presents $d^*_{K,M,N}(r)$ for the case $M = K = 2$ and $N = 4$ (which leads to $l = 1$).

Next we are ready to compare the upper bound on the optimal DMT of IC’s (in general, not only for the symmetric case) to the optimal DMT of finite constellations. This comparison enables us to show that for $N \geq (K+1)M-1$ the upper bound on the optimal DMT of IC’s coincides with the optimal DMT of finite constellations, where for $N < (K+1)M-1$ the upper bound is inferior compared to the optimal DMT of finite constellations. This leads to the conclusion that in case $N < (K+1)M-1$ ($K > 1$), the best DMT any sequence of IC’s can attain is suboptimal compared to the optimal DMT of finite constellations.

We begin by showing for the symmetric case when the upper bound on the optimal DMT of IC’s, $d^*_{K,M,N}(r)$, is suboptimal compared to the optimal DMT of finite constellations.

Lemma 2. For either $N \geq (K+1)M-1$ or $K = 2$, $M = s+1$, $N = 3 \cdot s$, where $s \geq 1$ and $s \in \mathbb{Z}$ we get
\[ d^*_{K,M,N}(r) = d^*_{M,N}(r) \]

For $N < (K-1)M+1$
\[ d^*_{K,M,N}(r) = d^*_{M,N}(r) \]

in the range $\left[ \frac{M}{K} + 1 \right] < r < \frac{(K-1)M+1}{K}$. 

Sketch of the proof: In a nutshell the proof is based on the properties of $d^*_{M,N}(r)$ derived in [8], and also on the results in Theorem 2. It is important to note that for $K = 2$, $M = s+1$ and $N = 3 \cdot s$ we get $d^*_{K,M,N}(r) = d^*_{M,N}(r)$ since in this case $\left[ \frac{M}{K} + 1 \right] = \frac{(K-1)M+1}{K}$.

The sub-optimality of $d^*_{K,M,N}(r)$ for the case $N < (K-1)M+1$ is illustrated in Figure 2, where the sub-optimality for the case $N = (K-1)M+1+l$, $l = 0, \ldots, 2M-3$ is illustrated in Figure 1.
constellations, and the cases where the optimal DMT in this setting is suboptimal compared to the optimal DMT of finite constellations.

**Theorem 3.** For $N \geq (K + 1) M - 1$ the optimal DMT of IC’s in the unconstrained multiple-access channel is upper bounded by $d_{K,M,N}^{*,(FC)}$ (max $(r_1, \ldots, r_K)$) the optimal DMT of finite constellations. For $N < (K + 1) M - 1$ the best DMT that can be attained in the unconstrained multiple-access channel is inferior compared to the optimal DMT of finite constellations.

**Sketch of the proof:** Recall that in Theorem 1 we have shown that the optimal DMT of IC’s is upper bounded by

$$d_{K,M,N}^{*,(IC)}(r_1, \ldots, r_K) = \max_{(D_1, \ldots, D_K) \in \mathcal{A}} \min_{A} \left( d_{A|M,N}^{*,DA} \left( R_A \right) \right).$$

For $N \geq (K + 1) M - 1$ we show that this term is upper and lower bounded by $d_{K,M,N}^{*,(IC)}(\max (r_1, \ldots, r_K))$, which is the optimal DMT of finite constellations in this case.

For $N < (K + 1) M - 1$ we show that the optimal DMT is not attained, by finding a set of multiplexing gain tuples $(r_1, \ldots, r_K) \in B$ for which $d_{K,M,N}^{*,(IC)}(r_1, \ldots, r_K) < d_{K,M,N}^{*,(IC)}(r_1, \ldots, r_K)$. Based on Lemma 2 we get for $r_1 = \cdots = r_K = r$ that there exists a set of multiplexing gains for which $d_{K,M,N}^{*,(IC)}(r) < d_{K,M,N}^{*,(IC)}(r)$, except for the case $K = 2$, $M = s + 1$ and $N = 3 + s$, where $s \geq 1$ is an integer. For this case showing that $d_{2,s+1,3,s}^{*,(IC)}(r_1, r_2) < d_{2,s+1,3,s}^{*,(IC)}(r_1, r_2)$ is more involved and requires considering the case $r_1 \neq r_2$ (see [12] for the full proof).

It is interesting to note that the upper bound on the optimal DMT of IC’s in the symmetric case is a convex function, where the optimal DMT of finite constellations is not necessarily so [9] (see Figure 1 for example). In fact the optimal DMT is not a convex function whenever $N < (K - 1) M + 1$, or when $N = (K - 1) M + 1 + l < (K + 1) M - 1$ and $\frac{l}{2} + 1 \neq \frac{(K-1)M+1}{K}$ where $l = 0, \ldots, 2M - 3$. Therefore, we can state that in the symmetric case, whenever the optimal DMT of finite constellations is not a convex function, IC’s are suboptimal.

**IV. ATTAINING THE OPTIMAL DMT FOR $N \geq (K + 1) M - 1$**

In this section we show that the upper bound on the DMT of IC’s in the unconstrained multiple-access channel, derived in section III, is achievable for the case $N \geq (K + 1) M - 1$ by a sequence of IC’s in general and lattices using regular lattice decoding at the receiver in particular. Essentially, we show for $N \geq (K + 1) M - 1$ that IC’s attain DMT that equals to $d_{K,M,N}^{*,(FC)}(r_1, \ldots, r_K) = d_{K,M,N}^{*,(FC)}(\max (r_1, \ldots, r_K))$.

We begin by introducing in Subsection IV-A the transmission scheme for each user, followed by presentation of the effective channel induced by the transmission scheme in Subsection IV-B. We derive in Subsection IV-C for each channel realization an upper bound on the error probability of the ML decoder of an ensemble of $K$ IC’s, and then average this upper bound over the channel realizations to show that the optimal DMT is attained for $N \geq (K + 1) M - 1$.

**A. The Transmission Scheme**

Essentially, in the proposed transmission scheme each user transmits as if the channel was a point-to-point channel with $M$ transmit and $N$ receive antennas. Hence, each user transmission matrix is identical to the transmission matrix presented in [8].

We denote the transmission matrix of user $i$ by $G_{1}^{(i)}$, where $l = 0, \ldots, M - 1$ and $i = 1, \ldots, K$. $G_{1}^{(i)}$ has $M$ rows that represent the transmission antennas, and $T_{l} = N + M - 1 - 2 \cdot l$ columns that represent the number of channel uses. As in [8] $G_{1}^{(i)}$ transmits $D_{l} = \frac{N}{N+M-1-2l}$ average number of dimensions per channel use in the following manner.

Consider a channel with $M$ transmit and $N$ receive antennas.

1) For $D_{M-1} = \frac{M(N-M+1)}{N+M-1} = M$: the matrix $G_{1}^{(i)}$ has $N + M - 1$ columns (channel uses). In the first column transmit symbols $x_1, \ldots, x_M$ on the $M$ antennas, and in the $N + M + 1$ transmit symbols $x_M(N-M+1)+1, \ldots, x_M(N-M+1)$ on the $M$ antennas.

2) For $D_{l}$, $l = 0, \ldots, L - 2$: the matrix $G_{1}^{(i)}$ has $M + N - 1 - 2 \cdot l$ columns. We add to $G_{1}^{(i)}$, the transmission scheme for $D_{l+1}$, two columns in order to get $G_{1}^{(i)}$. In the first added column transmit $l+1$ symbols on antennas $1, \ldots, l + 1$. In the second added column transmit different $l+1$ symbols on antennas $M - l, \ldots, M$.

According to the definition of the transmission scheme we can see that the different users transmit the same average number of dimensions per channel use. Let us denote the transmission scheme of the first $k$ users by

$$G_{1}^{(1, \ldots, k)} = \left( G_{1}^{(1)}, \ldots, G_{1}^{(k)} \right)^{t} k = 1, \ldots, K. \quad (10)$$

$G_{1}^{(1, \ldots, k)}$ is a $k \cdot M \times T_{1}$ matrix. Note that $G_{1}^{(1, \ldots, k)}$ transmits $k \cdot D_{1} \cdot T_{1}$ average number of dimensions per channel use. Later in this section we show that $G_{1}^{(1, \ldots, K)}$ attains the optimal DMT in the range $l \leq r_{\text{max}} \leq l + 1$.

**Example:** $M = 2$, $N = 5$ and $K = 2$. In this case the transmission scheme for $D_{0} = \frac{10}{5}$, $D_{1} = \frac{8}{5}$ ($G_{0}^{(1,2)}, G_{1}^{(1,2)}$ respectively) as follows:

$$G_{1}^{(1,2)} = \begin{pmatrix} G_{1}^{(1)} \\ G_{1}^{(2)} \end{pmatrix} = \begin{pmatrix} x_{1} & x_{3} & x_{5} & x_{7} & x_{17} & 0 \\ x_{2} & x_{4} & x_{6} & x_{8} & 0 & x_{18} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{9} & x_{11} & x_{13} & x_{15} & x_{19} & 0 \\ x_{10} & x_{12} & x_{14} & x_{16} & 0 & x_{20} \end{pmatrix}. \quad (11)$$

$$D_{1} = \frac{8}{5}, G_{1}^{(1,2)}$$

$$D_{0} = \frac{10}{5}, G_{0}^{(1,2)}$$
B. The Effective Channel

Next we define the effective channel matrix induced by the transmission scheme of the first \(k\) users \(G_l^{(1),...,(k)}\), where \(k = 1, \ldots, K\). Let us denote the first \(k\) users transmission at time instance \(t\) by

\[
\tilde{z}_t = \left( \tilde{z}_t^{(1)}, \ldots, \tilde{z}_t^{(k)} \right) \quad t = 1, \ldots, T_i.
\]

In accordance with the channel model from (1) we get

\[
y_m = H^{(1),\ldots,k} \cdot \tilde{z}_t = 1, \ldots, T_i,
\]

where \(H^{(1),\ldots,k} = (H^{(1)}, \ldots, H^{(k)})\), is an \(N \times k \cdot M\) matrix. The multiplication \(H^{(1),\ldots,k} \cdot G_l^{(1),\ldots,k}\) yields a matrix with \(N\) rows and \(T_i\) columns, where each column equals to \(H^{(1),\ldots,k} \cdot \tilde{z}_t\), \(t = 1, \ldots, T_i\). Each user is transmitting \(D_l \cdot T_i\)-complex dimension IC with \(D_l \cdot T_i\)-complex symbols, i.e., \(G_l^{(i)}\) has exactly \(D_l \cdot T_i\) non-zero values representing the \(D_l \cdot T_i\)-complex dimension IC within \(\mathbb{C}^{M \cdot T_i}\). Together, the first \(k\) users transmit an effective \(k \cdot D_l \cdot T_i\)-dimensional complex IC within \(\mathbb{C}^{M \cdot T_i}\). For each column of \(G_l^{(1),\ldots,k}\), denoted by \(g_m^{(k)}\), \(m = 1, \ldots, T_i\), we define the effective channel that \(g_m^{(k)}\) sees as \(\tilde{H}_m\). It consists of the columns of \(H^{(1),\ldots,k}\) that correspond to the non-zero entries of \(g_m^{(k)}\), i.e., \(H^{(1),\ldots,k} \cdot \tilde{z}_t = \tilde{H}_m \cdot g_m^{(k)}\), where \(\tilde{H}_m\) equals to the non-zero entries of \(g_m^{(k)}\). As an example assume without loss of generality that only the first \(l_m\) entries of \(g_m^{(k)}\) are non-zero. In this case \(\tilde{H}_m\) is an \(N \times l_m\) matrix that equals to the first \(l_m\) columns of \(H^{(1),\ldots,k}\). In accordance with (3), \(H^{(1),\ldots,k}\) is an \(NT_i \times k \cdot D_l \cdot T_i\) block diagonal matrix consisting of \(T_i\) blocks. Since each block in \(H^{(1),\ldots,k}\) corresponds to the multiplication of \(H^{(1),\ldots,k}\) with a different column in \(G_l^{(1),\ldots,k}\), the blocks of \(H^{(1),\ldots,k}\) equal \(\tilde{H}_m\), \(m = 1, \ldots, T_i\). Note that in the effective matrix \(\tilde{H}_m \geq k \cdot D_l \cdot T_i\).

Next we elaborate on the structure of the blocks of \(H^{(1),\ldots,k}\). For this reason we denote the \(m\)th column of \(H^{(1),\ldots,k}\) by \(\tilde{h}_m\), \(m = 1, \ldots, k \cdot M\). The transmission scheme has \(N + M - 1\) columns. The entries of the first \(N - M + 1\) columns of \(G_l^{(1),\ldots,k}\), \(g_1^{(k)}, \ldots, g_{N-M+1}^{(k)}\), are all different from zero. Hence, the first \(N - M + 1\) blocks of \(H^{(1),\ldots,k}\) are

\[
\tilde{H}_m = H^{(1),\ldots,k} \quad m = 1, \ldots, N - M + 1. \tag{12}
\]

After the first \(N - M + 1\) columns we have \(M - 1\) rows of columns. For each pair of columns

\[
\tilde{H}_{N-M+2v} = \tilde{H}_{N-M+2v-1} = \tilde{H}_{N-M+2v-1} \quad \forall \in \{0, \ldots, M - 2 \cdot l\} = \{l, \ldots, M-v, M+(v-1)\} + \{1, \ldots, M-v\}
\]

and

\[
\tilde{H}_{N-M+2v+1} = \tilde{H}_{N-M+2v+1} = \tilde{H}_{N-M+2v+1} \quad \forall \in \{0, \ldots, M - 1 \} = \{l, \ldots, M-v, M+(v-1)\}
\]

where \(v = 1, \ldots, M - 1\).

Example: consider \(M = 2, N = 5\) and \(K = 2\) as presented in (11). In this case \(l = 0.1\) and we have \(D_0 = \frac{10}{6}\) and \(D_1 = \frac{8}{4} = 2\) respectively. In addition \(H^{(1),2} = (H^{(1)}, H^{(2)}) = (\hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4)\). We begin with \(k = 1\). In this case we get a point-to-point channel with 2 transmit and 5 receive antennas \(H^{(1)} = (\hat{h}_1, \hat{h}_2)\), which leads to the following effective channels

1) \(D_1 = 2; H^{(1),1}_{\text{eff}}(k=1)\) is generated from the multiplication of the \(5 \times 2\) matrix \(H^{(1)}\) with the four columns of the transmission matrix \(G_l^{(1)}\). In this case \(H^{(1),1}_{\text{eff}}\) is a \(20 \times 8\) block diagonal matrix, consisting of four blocks, where each block equals to \(H^{(1)}\).

2) \(D_0 = \frac{10}{6}; H^{(0),0}_{\text{eff}}(k=1)\) is a \(30 \times 10\) block diagonal matrix consisting of six blocks. The first four blocks are equal to \(H^{(1)}\). The additional two blocks (induced by columns 5-6 of \(G_0^{(1)}\)) are vectors. We get that \(\tilde{H}_5 = \hat{h}_1\) and \(\tilde{H}_6 = \hat{h}_2\).

For the case \(k=2\) the effective channel induced by \(G_l^{(1),2}\) is as follows.

1) \(D_1 = 2; In this case the effective channel \(H^{(2)}_{\text{eff}}(k=2)\) is a \(20 \times 16\) matrix consisting of four blocks, where each block equals \(H^{(1),2} = (H^{(1)}, H^{(2)})\).

2) \(D_0 = \frac{10}{6}; In this case the effective channel \(H^{(0),0}_{\text{eff}}(k=2)\) is a \(30 \times 20\) matrix consisting of six blocks. The first four blocks equal to \(H^{(1),2}\), where the other two blocks are \(\tilde{H}_5 = (\hat{h}_1, \hat{h}_2)\) and \(\tilde{H}_6 = (\hat{h}_2, \hat{h}_3)\).

We present \(H^{(1),2}\) of our example in equation (15).

To conclude, each row of the transmission matrix is related to the column of \(H^{(1),\ldots,k}\) that multiplies it, i.e., row \(j\) in \(G_l^{(1),\ldots,k}\) corresponds to column \(\tilde{h}_j\). In case there is a non zero entry of row \(j\) in column \(m\) of \(G_l^{(1),\ldots,k}\), it means that \(\tilde{h}_j\) occurs in \(\tilde{H}_m\).

C. Achieving the Optimal DMT

In this subsection we derive for each channel realization an upper bound on the error probability of the joint ML decoder of \(K\) ensembles of IC’s transmitted on the unconstrained multiple-access channel, assuming each IC is \(D_l \cdot T_i\)-complex dimensional. Then, we show that the transmission scheme proposed in IV-A attains the optimal DMT for \(N \geq (K + 1) \cdot M - 1\) and \(\max(r_1, \ldots, r_K)\).

In accordance with the definitions in IV-B we denote the effective channel of any set of users pulled together by \(H^{(1),(s)}\), where \(s \subseteq \{1, \ldots, K\}\). We define \(H^{(1),(s)}\) by

\[
H^{(1),(s)} = \rho^{-1} \sum_{i=1}^{s} \eta_i D_l \cdot T_i \cdot \eta_i \cdot \tilde{h}_i \quad i \leq s \leq |s| \cdot D_l \cdot T_i
\]

where \(\rho = \frac{1}{s}\) is the \(i\)th singular value of \(H^{(1),(s)}\), \(1 \leq i \leq |s| \cdot D_l \cdot T_i\). We also define \(\eta_i = (\eta_1^{(s)}, \ldots, \eta_{|s|}^{(s)} D_l \cdot T_i)^T\). Note that in our setting \(N T_i \geq K \cdot D_l \cdot T_i\).

Theorem 4. Consider \(K\) ensembles of \(D_l \cdot T_i\)-complex dimensional IC’s transmitted on the unconstrained multiple-access channel with effective channel \(H^{(1),(s)}\) and densities

\[
\eta_1 = \ldots = \eta_{|s|} = \ldots = \eta_{|s|}^{(s)} D_l \cdot T_i
\]

Note that in IV-B we considered the case of the first \(k\) users where \(k = 1, \ldots, K\). The extension to any \(s \subseteq \{1, \ldots, K\}\) is straightforward.
\( \gamma_i^{(r)} = \rho^{T_i r_i}, \ i = 1, \ldots, K. \) The average decoding error probability of the joint ML decoder is upper bounded by
\[
\overline{P_e}(H_{\text{eff}}^{(i)}), \rho) \leq \sum_{s \subseteq \{1, \ldots, K\}} \overline{P_e}(\eta_i^{(s)}), \rho) = \sum_{s \subseteq \{1, \ldots, K\}} D(\|s\| \cdot D_l \cdot T) \rho^{-T_i((\|s\| D_l - \sum_{\epsilon \in s} r_i)) + \sum_{i=1}^{\|s\|} T_i \eta_i^{(s)}},
\]
where \( D(\|s\| \cdot D_l \cdot T) \) is a constant independent of \( \rho \), and \( \eta_i^{(s)} \geq 0 \) for any \( s \subseteq \{1, \ldots, K\} \) and any \( 1 \leq i \leq \|s\| \cdot D_l \cdot T \).

**Sketch of the proof:** The proof is based on dividing the error event into events of error for different sets of users (disjoint events). Then, the upper bound on the error probability of the point-to-point channel derived in [8] is used to upper bound the probability for each of these events.

We wish to emphasize that the constraint of \( \eta_i^{(s)} \geq 0 \) for \( i = 1, \ldots, \|s\| \cdot D_l \cdot T \) and for any \( s \subseteq \{1, \ldots, K\} \) results from the fact that we upper bound the same ensemble for any channel realization. In cases where it is possible to fit an ensemble to each channel realization, i.e. the case where the transmitter knows the channel, the upper bound applies also without this restriction.

Now we are ready to lower bound the transmission scheme DMT. Let us denote the maximal multiplexing gain by \( r_{\text{max}} = \max(1, \ldots, K) \), and also assume \( l = \lfloor r_{\text{max}} \rfloor \).

**Theorem 5.** Consider \( K \) sequences of ensembles of \( D_l \cdot T_i \)-complex dimensional IC’s transmitted over the unconstrained multiple access channel, where each user transmits multiplexing-gain \( r_i \) using \( G_i^{(r)} \), \( i = 1, \ldots, K \). The DMT of this transmission scheme is lower bounded by
\[
d_{\text{DMT}}^{(\text{FC})}(r_{\text{max}}).
\]

**Sketch of the proof:** We base the proof on the upper bound on the error probability derived in Theorem 4. This upper bound consists of the sum of several terms, one for each \( s \subseteq \{1, \ldots, K\} \). Each term depends on the determinant corresponding to its effective channel \( |H_{\text{eff}}^{(i)}|, H_{\text{eff}}^{(i)}|^{-1} \). We upper bound each term determinant (for each \( s \)) to get a new upper bound on the error probability. The upper bound is based on the fact that a determinant equals to the multiplication of the orthogonal elements of its columns (when the number of rows is larger than the number of columns). We average this multiplication over all channel realizations and show that at large \( \rho \) the diversity order of the most dominant error event is lower bounded by \( d_{\text{DMT}}^{(\text{FC})}(r_{\text{max}}) \).

In Theorem 3 we have shown that for \( N \geq (K + 1) M - 1 \) the DMT of any IC is upper bounded by \( d_{\text{DMT}}^{(\text{FC})}(r_{\text{max}}) \). On the other hand in Theorem 5 we have shown that there exists a sequence of IC’s that attain DMT which is lower bounded by \( d_{\text{DMT}}^{(\text{FC})}(r_{\text{max}}) \). Hence, the transmission scheme must attain the optimal DMT.

In the next theorem we prove the existence of a sequence of lattices that attains the optimal DMT as in Theorem 5.

**Theorem 6.** For each tuple of multiplexing gains \((r_1, \ldots, r_K)\) there exist \( K \) sequences of 2D-lattices transmitted over the unconstrained multiple access channel that attain diversity order of \( d_{\text{DMT}}^{(\text{FC})}(r_{\text{max}}) \), where regular lattice decoder is employed, where \( l = \lfloor r_{\text{max}} \rfloor \).

**Sketch of the proof:** An upper bound on the error probability of an ensemble of lattices is obtained based on the Minkowski-Hlawka theorem [14], followed by the same averaging technique we used to prove Theorem 5.

**V. Discussion**

In this section we discuss the results presented in the paper. As an illustrative example we consider the case \( K = M = 2 \). We consider the symmetric case where \( r_1 = r_2 = r \), and explain based on Theorem 2 why for \( N = 2 \) IC’s are suboptimal, and on the other hand based on Theorem 4 and Theorem 5 why IC’s attain the optimal DMT when \( N \geq 5 \). The analysis in this section is somewhat loosed and we refer the reader to [12] for the full analysis.

We begin by giving a short reminder to the behavior of lattices in a point-to-point channel when \( M = N = 2 \), as presented in [8]. We consider in this discussion lattices although the results apply to IC’s in general. In this case, the optimal DMT equals \( d_{\text{DMT}}^{(\text{FC})}(r) = 4 - 3r \) in the range \( 0 \leq r \leq 1 \), and in order to attain it the average number of dimensions per channel use, \( D \), must be equal to \( \frac{4}{3} \). We wish to explain why when \( D \neq \frac{4}{3} \) the optimal DMT is not attained in the range \( 0 \leq r \leq 1 \). For lattices, obtaining multiplexing gain \( r > 0 \) requires scaling each dimension of the lattice by \( \rho^{-\frac{4}{3}} \). However, the scaling is too strong and does not enable to attain the optimal DMT for any \( r > 0 \) (there are not enough degrees of freedom to attain the straight line \( 4 - 3r \)). On the other hand when \( D > \frac{4}{3} \), the lattice “fills” too much of the space and the channel induces error probability that does not enable to attain diversity order of \( 4 \) for \( r = 0 \), and therefore does
not allow attaining the optimal DMT in the range \(0 \leq r \leq 1\). Hence, choosing \(D = \frac{4}{3}\) balances the effect of the scaling and the channel on the lattice and allows to attain the optimal DMT in the range \(0 \leq r \leq 1\). We now follow this intuition to discuss the case of multiple-access channel.

### A. Why IC’s are Suboptimal for \(K = M = N = 2\)

The optimal DMT of finite constellations in the symmetric case equals

\[
\min_{D_{1,2}} \max_{r} \left( d_{2,2}^{(FC)}(r), d_{1,2}^{(FC)}(2r) \right)
\]

where \(D = \{0 \leq D_1 \leq 2, 0 \leq D_2 \leq 2, D_1 + D_2 \leq 2\}\), i.e. the bounds derived in [8] for the point-to-point channel. In case the dimensions of any subset of the users do not “align”, i.e. in case a certain subset of the users has average number of dimensions per channel use that is too large or too small to attain the optimal DMT, we get sub-optimality. The solution to the optimization problem in (16) is \(d_{2,2}^{*(IC)}(r) = 4(1-r)\), which is smaller than the optimal DMT for any \(0 < r < 1\). Let us explain the reason for the sub-optimality. First, note that in the symmetric case we must choose \(D_1 = D_2\) to maximize the IC’s DMT, i.e. the users have the same average number of dimensions per channel use. Since \(N = 2\) each user can not transmit more than one average number of dimensions per channel use, where in [8] it was shown that each user needs to transmit \(\frac{3}{4}\) average number of dimensions per channel use in order to attain \(d_{2,2}^{(FC)}(r)\) in the range \(0 \leq r \leq \frac{3}{2}\). In addition, the maximal diversity order each user may attain is 4 since \(M = N = 2\), and also \(d_{2,2}^{*(IC)}(r)\) is a straight line. Hence, even when transmitting one dimension per channel use the DMT must be smaller than \(6 - 6r\). Therefore, in this case the dimension mismatch manifest itself in the fact that \(N\) is too small even to attain the first line of \(d_{2,2}^{*(FC)}(r)\). This sub-optimality is presented in Figure 2.

### B. Why IC’s Attain the Optimal DMT for \(N \geq (K + 1)M - 1\)

For the case where \(N \geq (K + 1)M - 1\) there is a sufficient amount of dimensions per channel use for each user, to avoid the dimension mismatch which does not allow to attain the optimal DMT for the case \(N < (K + 1)M - 1\). However, the condition that there is no dimension mismatch is merely a necessary condition in order to attain the optimal DMT. In this subsection we will explain why the optimal DMT is attained based on the transmission scheme presented in Subsection IV-A and on the effective channel presented in IV-B.

We consider as an example the case \(M = K = 2\) and \(N = 5\), and show why in this case the single user performance \(d_{2,5}^{*(FC)}(r)\) is attained. For simplicity let us focus on the symmetric case. Essentially, we show in this example that IC’s attain the first line of the optimal DMT, \(10 - 6r\), which coincides with the optimal DMT \(d_{2,5}^{*(FC)}(r)\) in the range \(0 \leq r \leq 1\). The transmission scheme is \(G_0^{(1,2)}\) presented in (11). Note that each user transmits an optimal transmission scheme of a point-to-point channel with 2 transmit and 5 receive antennas. Therefore, the DMT of the error events of each of the users, is upper bounded by \(10 - 6r\) the optimal DMT in the range \(0 \leq r \leq 1\). What is left to show is that the DMT of the error event of the two users is also upper bounded by \(10 - 6r\). In this case we consider the effective lattice of the two users pulled together, i.e. an error event of a lattice transmitted over a point-to-point channel with 4 transmit and 5 receive antennas. For this lattice the average number of dimensions per channel use equals \(D_1 + D_2 = \frac{10}{3}\). We will show that for \(r = 0\) this lattice attains diversity order 10. This will lead to DMT \(10 - 6r\) since the lattice DMT is a straight line and \(D_1 + D_2 = \frac{10}{3}\).

At the receiver, the effective radius of the lattice of the two users pulled together at \(r = 0\) is

\[
r_{e,fj}^2 = |V|^{-1} = \gamma_{rec}^{-1} = |H_{eff}(0,K)|^2 = \frac{1}{|\rho_{min}(D_{2,2})|}
\]

where \(V = \gamma_{rec}^{-1}\) is the volume of the Voronoi region of the effective lattice at the receiver. Recall that for lattices \(r_{packing} \geq \rho_{min} = \frac{d_{min}^{(lattice)}}{2}\), where \(r_{packing}\) and \(d_{min}^{(lattice)}\) are the packing radius and the minimal distance of the lattice respectively. We are interested in the event where \(r_{eff}^2\) is in the order of the additive noise variance \(\rho^{-1}\). In this case \((\rho_{min}^{-1})^2\) is in the order of the noise variance or worse, and so the error probability does not reduce with \(\rho\). In subsection IV-C it is shown that this event is the dominant error event in determining the DMT of the transmission scheme. From (17) we get that \(H_{eff}(0,K)\) determines the effective radius in the receiver. From (11) and the description of the effective channel in subsection IV-B we get that \(H_{eff}(0,K)\) is a block diagonal matrix, where 4 of its blocks equal \(H = \mathbb{C}^{5 \times 4}\). For large \(\rho\), the most probable error event \(r_{eff}^2 = \rho^{-1}\) occurs when the determinant of \(H\) reduces with \(\rho\), and the determinants of the rest of the blocks in \(H_{eff}(0,K)\) remain constant with \(\rho\). Note that if \(|H^H| = \rho^{\alpha}\) then most likely that the smallest singular value of \(H\) equals \(\rho^{\alpha}\) and the rest of the singular values remain constant [3]. In this case we get \(|H^H| = \rho^{\alpha}\) with a PDF proportional to \(\rho^{-2\alpha}\). By assigning \((D_1 + D_2)T = 20\) and \(|H_{eff}(0,K)| = \rho^{\alpha}\) in (17) we get that

\[
r_{eff}^2 = |H^H| - \gamma_{rec}^{-1} = \rho^{-2\alpha}
\]

with a PDF proportional to \(\rho^{-2\alpha}\). Hence, \(r_{eff}^2 = \rho^{-1}\) when \(\alpha = -5\). Based on subsection IV-C we get for large \(\rho\) that this is the most dominant error event, and by assigning \(\alpha = 5\)
in the PDF we get that this event happens with probability $\rho^{-10}$. Therefore, in this case diversity order of 10 is attained.

For general $N = (K + 1) M - 1$ each user transmits an optimal transmission scheme of a point-to-point channel with $M$ transmit and $N$ receive antennas. Since the users do not cooperate, in the worst case we get that $H_{\text{eff}}^{(i=0),K}$ has $N - M + 1$ blocks that equal $H \in \mathbb{C}^{N \times K \cdot M}$. For large $\rho$ we get that $|H^{'H}| = \rho^{-\alpha}$ with a PDF proportional to $\rho^{-(N - K \cdot M + 1)\alpha}$. In this case $\left(\sum_{i=1}^{K} D_i\right) = K \cdot M \cdot M$ and so we get

$$r_{\text{eff}}^2 \geq |H^{'H}| = \frac{\rho^{-(N - M + 1)\alpha}}{K \cdot M \cdot M}.$$  

(19)

Since $N = (K + 1) M - 1$ we have enough equations at the receiver to get $N - M + 1 = K \cdot M$ and $N - K \cdot M + 1 = M$. Hence, by substituting in (19) we get

$$r_{\text{eff}}^2 \geq \rho^{-\frac{\alpha}{N}}.$$  

(20)

with a PDF proportional to $\rho^{-(N - K \cdot M + 1)\alpha} = \rho^{-M \cdot \alpha}$. Therefore, we get for $\alpha = N$ that $r_{\text{eff}}^2 = \rho^{-1}$ with probability $\rho^{-MN}$, which leads to a diversity order $MN$ at $r = 0$. In addition $\sum_{i=1}^{K} D_i = \frac{K \cdot M \cdot N}{N - M + 1}$ and so the first line of the optimal DMT is attained. Note that we considered the case of the $K$ users pulled together. For any error event of the users in $s \subseteq \{1, \ldots, K\}$, the diversity order will be larger or equal to $MN$ at $r = 0$.

In summary, since the users do not cooperate we get in the worst case $N - M + 1$ occurrences of $H$ in the blocks of $H_{\text{eff}}^{(i=0),K}$. However, when $N \geq (K + 1) M - 1$ there is a sufficient amount of receive antennas to compensate for the impact of $H$ on $r_{\text{eff}}$, by decreasing the probability that $H$ has small determinant.

VI. CONCLUSIONS

This work studies the DMT of the unconstrained multiple-access channel. For the case $N \geq (K + 1) M - 1$ an explicit upper bound on the optimal DMT of IC’s for any multiplexing-gain tuple is presented. The upper bound coincides with the optimal DMT of the multiple-access channel for finite constellations. A transmission scheme that attains this upper bound is also introduced and analyzed. On the other hand for the case $N < (K + 1) M - 1$ (assuming $K > 1$) an analytical expression of an upper bound on the DMT of any sequence of IC’s is presented for the symmetric case. By comparing this upper bound to the optimal DMT in the symmetric case, we show that IC’s are sub-optimal in this case.

REFERENCES