Information spread in networks: Control, games, and equilibria

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***Abstract*—We design intervention schemes to control information spread in multi-agent systems. We consider two information spread models: linear distributed averaging and virus spread dynamics. Using the framework of differential games, we design a dynamical optimization framework that produces strategies that are robust to adversarial intervention. For linear dynamics, we show that optimal strategies make connection to potential-theory. In the virus spread case, we show that optimal controllers exhibit multiple switches. Moreover, we establish a connection between game theory and dynamical descriptions of network epidemics, which provides insights into decision making in infected networks. Finally, we present initial building blocks for network controllability using a limited number of controls.**

I. Introduction

Various global patterns in computer, social, and biological networks stem from local interactions among nodes. Examples include birds flying in formation, propagation of rumors and computer viruses, and epidemics. While a large body of literature is dedicated to modelling information diffusion in networks, controlling the diffusion subject to the network dynamics received limited attention. A common practice has been to design static controllers or to assume that all the nodes in the network can be controlled.

Interesting problems in social and biological networks as well as multi-agent systems have been studied in the literature. In [1], Kempe, Kleinberg, and Tardos have studied the problem of finding the optimal set of nodes to maximize the spread of influence in a social network. They have proposed a polynomial-time algorithm based on submodular functions that finds a near-optimal solution. A rumor source estimator based on the infected nodes and the underlying network structure was obtained in [2]. Limiting behavior of the voter model and opinion dynamics in the presence of stubborn agents was investigated in [3], [4]. In [5], the epidemic threshold in the models where the curing rate is proportional to the degree of the node was analyzed. A competition between two opposing campaigns to influence the largest set of nodes was studied in [6], where a greedy algorithm was proposed to find the best set of nodes for one campaign to limit the influence of the other.

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The controlled parameters in all the above problems are chosen at the initial time and are left static thereafter. These designs, therefore, cannot handle dynamically changing networks or the presence of other strategic players in the network. In fact, the proper framework to construct controllers suitable for competitive dynamic environments is differential game theory. Moreover, a common theme in current research is to assume that the network designer can control all the nodes in the network in order to limit the infection's spread. In reality, such freedom in placing controllers may not be possible. As networks grow in size to include millions of nodes, reducing the number of controllers required to counter the infection's spread will result in vast cost reductions.

Motivated by the problems of controlling the spread of influence and epidemics in networks, the main focus of this paper is to *construct dynamic control strategies capable of controlling the information spread under practical constraints*. In our development, we will consider optimal control design as well as state-feedback stabilization. In particular, we ask the following questions:

- What are the optimal strategies for limiting or amplifying the information diffusion in networks?
- Using a limited number of bounded controllers, when is it possible to steer the state of the network to a desired value?

To study these questions, we consider two models of information spread. The first model is the linear continuous-time distributed averaging algorithm, which is a popular information spread model where an agent updates its value as a linear combination of the values of its neighbors. Averaging dynamics is the basic building block in many multi-agent systems, and it is widely used whenever an application requires multiple agents, who are graphically constrained, to synchronize their measurements. Examples include formation control, coverage, distributed estimation and optimization, and flocking [7], [8], [9]. The second model is a nonlinear one, and it was recently proposed to describe virus spread in networks [10].

In practice, communication among agents performing averaging is prone to different types of non-idealities which can affect the convergence properties of the associated distributed algorithms. Transmission delays [11], noisy links [12], [13], and quantization [14] are some examples of non-idealities that are due to the physical nature of the application. In addition to physical restrictions, researchers have also studied averaging dynamics in the presence of malicious nodes in the network [15], [16].

In our work, we study the interaction between an adversary and a network designer over a network of nodes performing distributed averaging. The adversary is capable of disconnecting certain links in the network, while the designer can change the weights of certain links. Both the adversary and the designer are constrained by their physical capabilities, e.g., battery life and communication range. To capture such constraints, we allow the adversary and the designer to affect only a fixed number of links. In [17], we derived the worst-case attack on the network in the absence of a network designer.

Such an interaction between a network designer and an adversary can occur in various practical applications. For example, in a wireless network, the adversary can be a jammer who is capable of breaking links by injecting high noise signals that disrupt the communication among nodes. The link weights in such a network represent the capacities of the corresponding links. The designer can modify the rate of a certain link using various communication techniques such as introducing parallel channels between two nodes as in orthogonal frequency division multiple access (OFDMA) networks [18].

Our model is different from the models in the current literature in two ways: (i) the adversary and the designer compete over a dynamical network. This is different from the problems studied in the computer science and economics communities where the network is usually static [19]; (ii) the players in our model are constrained and do not have an infinite budget. This enables us to model practical scenarios more closely rather than allowing the malicious behaviour to be unrestricted as in [15], [20], [21].

The second model we study is a nonlinear one that has been recently proposed to model virus spread in networks. Viruses, misinformation, and rumors can diffuse rapidly through a network via local interactions. Modeling the spread of misinformation in networks as well as the control of such phenomena have received wide interest in the literature [22], [23], [24], [10], [25], [26]. A typical approach for modeling the spread of infection has been via describing the local interactions among individuals within the network. An example of such models is the so-called n-intertwined Markov model [10], which belongs to the susceptible-infected-susceptible (SIS) class where each node can either be healthy or infected.

The flow that prescribes the evolution of the n-intertwined model is nonlinear. When the curing rate is high, the states of the individual nodes are provably convergent to the healthy state. When the curing rates are low, however, a strictly positive equilibrium point arises in the n-intertwined dynamics, which is referred to as the "metastable" state in the literature. At this stage, a residual infection will persist in the network. One main focus of our work is characterizing the stability properties of this equilibrium, which has not been addressed in the literature.

By formulating a decentralized control system, we also study the case where the curing rates at a limited number of nodes can be controlled. We identify conditions under which the network can be stabilized to the origin. In particular, for path, star, and tree graphs, we characterize sufficient conditions for stabilization of the infection dynamics. Moreover, we propose a dynamic optimization framework that allows for designing controllers that minimize the total infection in the network at minimum cost. Several simulations illustrate our results.

Organization

The rest of this paper is organized as follows. In Section II, we formulate and solve two Stackelberg games for information spread control over distributed averaging networks. Section III discusses the stability properties of the n-intertwined model. We also present stabilizing and optimal controllers for infected networks. Future research directions and concluding remarks are presented in Section IV.

Notation and Terminology

We denote the set of edges in a graph $\mathcal{G}$ by $\mathcal{E}(\mathcal{G})$. When clear from the context, we will drop the argument of any set defined on a graph. To emphasize the effect of link removal by the adversary, we will sometimes write $\mathcal{G}(u(t))$ to denote the graph resulting after the adversary acts at time $t \in \mathbb{R}_{\geq 0}$ with action $u(t)$. We will use $\sum_{j>i}(.)$ to mean $\sum_{j=2}^{n}\sum_{i=1}^{j-1}(.)$, $[.]^T$ to denote the transpose of a vector or a matrix $[.]$, and $\mathbf{1}$ to denote the n-dimensional column vector of 1's. We denote the set cardinality operator by card$\{.\}$. The Euclidean norm of a vector is denoted by $||.||_2$, the ℓ_1-norm of a vector is denoted by $||.||_1$, and the absolute value of a real number is denoted by $|.|$. The (i,j)-th element of a matrix X is denoted by X_{ij}. We will often use x to refer to a function or its value at a given time instant; the context should make the distinction clear.

For a real vector function $f : \mathbb{R}^n \to \mathbb{R}$, the k-th derivative with respect to x_i, i.e., $\frac{\partial^k}{\partial x_i^k} f(x)$, is denoted by $\nabla_i^k f(x)$. We denote the i-th eigenvalue of a matrix X by $\lambda_i(X)$. The identity matrix is denoted by I. We will use the words "strategy" and "action" interchangeably; since we are seeking optimal open-loop strategies in this paper, both terms are equivalent. We use the game theoretic notation x_{-i} to refer to the vector comprised of the optimization variables of all players except that of player i.

We call a directed graph weakly connected, if it contains a path between every two nodes when all the edges are viewed as undirected ones. A directed acyclic graph (DAG) is a directed graph with no directed cycles.

II. Robust Distributed Averaging

The main goal of this section is to introduce the distributed averaging problem in the presence of an adversary, and to derive optimal strategies for the designer and the adversary who have conflicting objectives. Because the order in which the players act affects the resulting utilities, we formulate two problems based on the order of play, allowing each player to have the *first-move-advantage* in the two problems. The proofs of the results presented in this section can be found in [17], [27].

Consider a connected network of n nodes and m links described by a weighted undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with vertex set $\mathcal{N}$, $|\mathcal{N}| = n$, and edge set $\mathcal{E}$, $|\mathcal{E}| = m$. The value, or state, of the nodes at time instant $t \in \mathbb{R}_{\geq 0}$ is given by $x(t) = [x_1(t), ..., x_n(t)]^T$. The nodes start with an initial value $x(0) = x_0$, and they are interested in computing the average of their initial measurements, $x_{avg} = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$, via local averaging. We consider the continuous-time averaging dynamics given by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{1}$$

where the matrix A, $A_{ij} = a_{ij}$, has the following properties:

$$\begin{aligned} &A = A^T, && A\mathbf{1} = 0, \\ &A_{ij} \geq 0, && A_{ij} = 0 \iff \{i,j\} \notin \mathcal{E}, \quad i \neq j. \end{aligned}$$

Define $\bar{x} = \mathbf{1}x_{avg}$ and let $M = \frac{\mathbf{1}\mathbf{1}^T}{n}$. A well-known result states that, under the above assumptions, the nodes will reach consensus as $t \to \infty$, i.e., $\lim_{t\to\infty} x(t) = \bar{x}$ [7]. To achieve their respective objectives, the designer and the adversary control the elements of A as we describe next. This will render the matrix A to be time-varying.

The adversary attempts to slow down convergence by breaking at most $\ell \leq m$ links at each time t. Let $u_{ij}(t) \in \{0,1\}$ be the weight the adversary assigns to link $\{i,j\}$ at time t. He breaks link $\{i,j\}$ when $u_{ij}(t) = 1$. Define $r := \binom{n}{2}$. The action set of the adversary can then be written as

$$\begin{aligned} U \;=\; &\{w \in \mathbb{R}^r : w = [w_{12}, ..., w_{1n}, w_{23}, ..., w_{(n-1)n}]^T, \\ &w_{ij} \in \{0,1\}, w_{ij} = 0 \text{ if } \{i,j\} \notin \mathcal{G}, ||w||_1 \leq \ell\}. \end{aligned}$$

The set of admissible controls, $\mathcal{U}$, consists of all functions that are piecewise continuous in time and whose range is U. Given a time interval $[0,T]$, we can formally write

$$\begin{aligned} \mathcal{U} \;=\; &\{u : [0,T] \to U \mid u \text{ is a piecewise continuous} \\ &\text{function of } t\}. \end{aligned}$$

We introduce a network designer who attempts to accelerate convergence by controlling the weights of the edges. The designer can change the weight of a given link by adding $v_{ij}(t)$ to its weight a_{ij}. We assume that $v_{ij}(t) \in \{0, b\}$ and that the number of links the designer modifies is at most $\ell \leq m$. Given the above specifications, we can write down the $\{i,j\}$-th element, $i \neq j$, of the matrix $A(u(t), v(t))$ as

$$A_{ij}(u(t), v(t)) = (a_{ij} + v_{ij}(t))(1 - u_{ij}(t)).$$

We require that the resulting matrix is a negative Laplacian of the graph; hence, we must have $A_{ii}(u(t), v(t)) = -\sum_{j\neq i} A_{ij}(u(t), v(t))$, for all i. With this definition, we can view the actions of the players as switches among the possible Laplacian matrices resulting from modifying the links. Moreover, the capability of the designer and the adversary to change the system matrix renders it as "switched" one. The optimal controllers for such systems can exhibit Zeno effect, i.e., they may switch infinitely many times over a finite interval. In order to explicitly eliminate the possibility of infinite switching, we make the following assumption in the remainder of this section.

Assumption 1. *Let $r_1 < \ldots < r_{K_u}$ be the switching times of u and $s_1 < \ldots < s_{K_v}$ be those of v, where (u, v) is an arbitrary pair of controllers. Assume that K_u, K_v are finite, and that there exists a globally minimum dwell time $\tau > 0$ such that*

$$\begin{aligned} \tau \;\leq\; &\min\{r_{i+1} - r_i, s_{i+1} - s_i, |r_i - s_j| : 1 \leq i \leq K_u, \\ &1 \leq j < K_v\}, \end{aligned} \tag{2}$$

over which the system matrix $A(u, v)$ is time-invariant.

Given a time interval $[0, T]$, introduce the following functional:

$$J(u, v) = \frac{1}{2}\int_0^T k(t)\, ||x(t) - \bar{x}||_2^2\, dt,$$

where the kernel $k(t)$ is positive and integrable over $[0, T]$. This constitutes the utility function of the adversary, and that of the designer is $-J(u, v)$. We will study two problems. In the first one, the adversary acts first by selecting the links he is interested in breaking. Then, the network designer optimizes his choices over the resulting graph $\mathcal{G}(u(t))$. In this case, the action set of the designer can be written as

$$\begin{aligned} V(u(t)) = \big\{&w \in \mathbb{R}^r : w = [w_{12}, ..., w_{1n}, w_{23}, ..., w_{(n-1)n}]^T, \\ &w_{ij} \in \{0, b\}, w_{ij} = 0 \text{ if } \{i,j\} \notin \mathcal{G}(u(t)), ||w||_1 \leq b\ell\big\}. \end{aligned}$$

The set of admissible controls for the designer, $\mathcal{V}(u)$, consists of all piecewise continuous functions whose range is $V(u)$. Formally, we define

$$\begin{aligned} \mathcal{V}(u) \;=\; &\{v : [0,T] \to V(u(t)) \mid v \text{ is a piecewise} \\ &\text{continuous function of } t\}. \end{aligned}$$

The max–min problem can now be formally written as[1]

$$\begin{aligned} &\sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{V}(u)} && J(u,v) \\ &\text{subject to} && \dot{x}(t) = A(u(t), v(t))x(t), \quad x(0) = x_0. \end{aligned}$$

In the second problem, the order is reversed. Because the designer acts first in this problem, he can optimize over the entire graph $\mathcal{G}$. Thus, the action set of the designer in this problem is $V := V(0)$ and the set of its admissible controls is $\mathcal{V} := \mathcal{V}(0)$; the sets of actions and admissible controls of the adversary remain the same. We can then write

$$\begin{aligned} &\inf_{v\in\mathcal{V}} \sup_{u\in\mathcal{U}} && J(u,v) \\ &\text{subject to} && \dot{x}(t) = A(u(t), v(t))x(t), \quad x(0) = x_0. \end{aligned}$$

[1]Even though existence of a maximum and a minimum has not yet been shown at this stage, we will still call this the "max–min" problem in anticipation of such an existence result later in the paper. The formal definition below is still in terms of sup and inf. The same argument applies to the min–max problem to be introduced shortly.

In a computer network, the max–min problem allows the network designer (who is the maximizer here) to architect networks that are robust against strategic virus diffusion. The min–max problem finds applications in army combat situations where the designer (the minimizer) attempts to counter the attacks of the enemy intending to disrupt the communication among agents. For both problems, we make the following assumption:

Assumption 2. *The initial matrix $A(0,0)$, the time interval $[0,T]$, the values ℓ and b, and the initial state x_0 are common information to both players.*

The following remark is now in order.

Remark 1. *(Problem Complexity) Let us consider the problem of the adversary for a given strategy of the designer. Assume that the adversary can act at K given time instances over the interval $[0,T]$. Then, for $\ell \leq m$, assuming that $||u(t)||_1 = \ell$ for all t, the total number of links that need to be tested in a brute-force approach is*

$$\binom{m}{\ell}^{K_u} \geq \left(\frac{m}{\ell}\right)^{\ell K_u}. \tag{3}$$

Clearly, the brute-force approach leads to an exponential number of computations as a function of K. The same argument applies to the problem faced by the network designer.

A. Optimal Strategies

We will now present the solutions to the two problems introduced above by working directly with the objective functional. In what follows, we will often drop the time index and other arguments for notational simplicity. We will be using the term "connected component" to refer to a set of connected nodes which have the same values.

The following quantities will be central to the derivation of the optimal strategies:

$$\nu_{ij} := -(x_i - x_j)^2, \quad w_{ij} := (a_{ij} + v_{ij})\nu_{ij}. \tag{4}$$

Define the set operator $\Phi : S(\mathcal{G}') \subset \mathbb{R} \to \mathcal{E}(\mathcal{G}') \subset \mathcal{E}(\mathcal{G})$ that returns the links in $\mathcal{E}(\mathcal{G}')$ that correspond to the elements of $\mathcal{S}(\mathcal{G}')$. Also, define $\Phi_i : S(\mathcal{G}') \to \mathcal{E}(\mathcal{G}')$ that returns the links in $\mathcal{E}(\mathcal{G}')$ corresponding to the *smallest* i elements of the set $S(\mathcal{G}')$. When card$\{S(\mathcal{G}')\} < i$, we set $\Phi_i(S(\mathcal{G}')) = \Phi(S(\mathcal{G}'))$. We also adopt the convention $\Phi_0(.) = \{\emptyset\}$.

The Min–Max Problem

Let $\mathcal{L}(v) = \{(a_{ij} + v_{ij})\nu_{ij} : \{i,j\} \in \mathcal{E}(\mathcal{G})\}$ and define the set $\mathcal{L}_\ell(v) = \Phi_\ell(\mathcal{L}(v))$. [2] By a possible abuse of notation, we let the k-th element of $\mathcal{L}_\ell(v)$, $\mathcal{L}_{\ell,k}(v)$, correspond to the link $\{i,j\} \in \mathcal{L}_\ell(v)$ and to the *value* $(a_{ij} + v_{ij})\nu_{ij}$ associated with it; the context should make it clear as to which attribute of $\mathcal{L}_{\ell,k}(v)$ we are referring to. We assume that $\mathcal{L}_{\ell,1}(v) \geq \ldots \geq \mathcal{L}_{\ell,\ell}(v)$. Further, define the sets $\mathcal{P}(v) = \{a_{ij}\nu_{ij} : \{i,j\} \notin \mathcal{L}_\ell(v)\}$ and $\overline{\mathcal{P}}(v) = \{\nu_{ij} : \{i,j\} \notin \mathcal{L}_\ell(v)\}$. We also define

$$[v_{\mathcal{S}}(b)]_{ij} = \begin{cases} b, & \{i,j\} \in \mathcal{S} \\ 0, & \{i,j\} \notin \mathcal{S} \end{cases}$$

The following theorem presents the optimal strategy of the adversary in the min–max problem.

Theorem 1. *Under Assumptions 1 and 2, and for a fixed strategy v of the designer, the optimal strategy of the adversary in the min–max problem is*

$$u^\star_{ij}(v) = \begin{cases} 1, & \{i,j\} \in \mathcal{L}_\ell(v) \\ 0, & \{i,j\} \notin \mathcal{L}_\ell(v) \end{cases}$$

If the adversary has an optimal strategy of breaking fewer than ℓ links, then either $\mathcal{G}$ has a cut of size less than ℓ or the nodes have reached consensus by time t. In either of these cases, breaking ℓ links is also optimal.

Consider the following numerical example for the worst-case attack in the absence of the network designer. We study a complete graph with $n = 4$. The matrix $A(0,0)$ is generated at random and is equal to

$$A(0,0) = \begin{pmatrix} -2.1293 & 0.0326 & 0.5525 & 1.5442 \\ 0.0326 & -1.2191 & 1.1006 & 0.0859 \\ 0.5525 & 1.1006 & -3.1447 & 1.4916 \\ 1.5442 & 0.0859 & 1.4916 & -3.1217 \end{pmatrix}$$

We fix $\ell = 2$, $T = 2$, and $x_0 = [1,2,3,4]^T$ – hence, $x_{avg} = 2.5$. We computed the optimal control using Theorem 1, which was found to be $u^\star(t) = [0,1,0,1,0,0]^T$ for $t \in [0,2]$. Indeed, at $t = 0$, the highest w_{ij} values are $w_{13}(0) = 2.2101$ and $w_{14}(0) = 13.8979$ which confirms the conclusion of Theorem 1. In this particular example, $w_{13}, w_{1,4}$ remain dominant throughout the problem's horizon, and hence the control is stationary. Fig. 1 simulates the network at hand with and without the presence of the adversary. Note that the adversary was successful in delaying convergence.

The following theorem presents the optimal strategy of the designer in the min–max problem.

Theorem 2. *In the min-max problem, and under Assumptions 1 and 2, the optimal strategy of the designer is to run Algorithm 1 in Table I, and to set $v^\star_{ij} \in \{0, b\}$ if $\nu_{ij} = 0$. Further, it is optimal for the designer to modify ℓ links.*

The Max–Min Problem

Let $\mathcal{F}(\mathcal{G}') = \{\nu_{ij} : \{i,j\} \in \mathcal{E}(\mathcal{G}')\}$ for some graph $\mathcal{G}'$ and let $\mathcal{F}_\ell(\mathcal{G}') = \Phi_\ell(\mathcal{F}(\mathcal{G}'))$. Then, $\mathcal{F}_\ell(\mathcal{G}')$ is the set containing the smallest ℓ values in $\mathcal{F}(\mathcal{G}')$. Also, define the set $\mathcal{D}(\mathcal{G}') = \{a_{ij}\nu_{ij} : \{i,j\} \in \mathcal{E}(\mathcal{G}')\} \cup \{(a_{ij} + b)\nu_{ij} : \{i,j\} \in \mathcal{E}(\mathcal{G}')\}$ and let $\mathcal{D}_\ell(\mathcal{G}') = \Phi_\ell(\mathcal{D}(\mathcal{G}'))$. The following theorems specify the optimal strategies of the adversary and the designer.

Theorem 3. *Under Assumptions 1 and 2, and for a fixed strategy u of the adversary, the optimal strategy of the network*

[2] It should be noted that although $\{i,j\}$ and $\{j,i\}$ belong to $\mathcal{E}(\mathcal{G})$, we include $(a_{ij} + v_{ij})\nu_{ij}$ only once in $\mathcal{L}(v)$. This applies to all the definitions to follow.

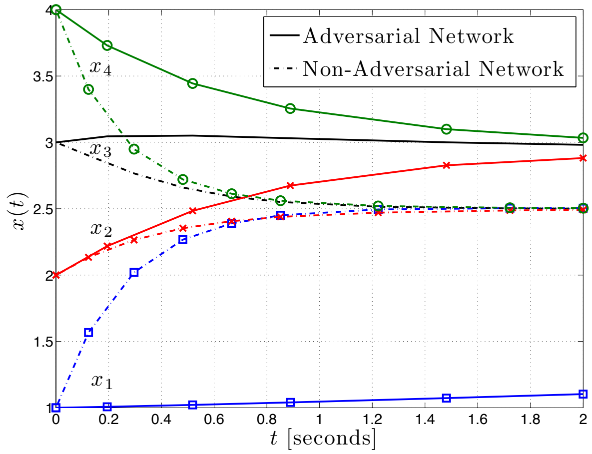

Fig. 1: Effect of adversary on the convergence to consensus. $T = 2$, $n = 4$, $\ell = 2$, and $x_0 = [1, 2, 3, 4]$.

TABLE I: Algorithm I: Computing the optimal strategy for the minimizer in the min–max problem.

0: **input:** a strategy v with $||v||_1 = 0$
1: **for** $i = \ell \downarrow 1$
2:     **if** $\exists \mathcal{S} \subset \Phi(\mathcal{P}(0)), |\mathcal{S}| = i,\ \mathcal{L}_{\ell,i}(0) \notin \mathcal{L}_\ell(v_\mathcal{S}(b))$
3:         Set $v^\star_{ij} = b, \forall \{i,j\} \in \mathcal{S} \cup \Phi_{\ell-i}\left(\overline{\mathcal{P}}(v_\mathcal{S}(b))\right)$.
4:         **Exit** for loop.
5:     **end**
6: **end**
7: **if** $||v||_1 = 0$
8:     Set $v^\star_{ij} = b$ for all $\{i,j\} \in \Phi_\ell\left(\overline{\mathcal{P}}(0)\right)$.
9: **end**

designer in the max–min problem is given by

$$v^\star_{ij}(u) = \begin{cases} b, & \{i,j\} \in \mathcal{F}_\ell(\mathcal{G}(u)) \\ 0, & \{i,j\} \notin \mathcal{F}_\ell(\mathcal{G}(u)) \end{cases}$$

If the designer has an optimal strategy of modifying fewer than ℓ links, then either $\mathcal{G}$ has a cut of size less than ℓ or the nodes have reached consensus by time t. In either of these cases, breaking ℓ links is also optimal.

Theorem 4. *In the max–min problem, and under Assumptions 1 and 2, the optimal strategy of the adversary is given by*

$$u^\star_{ij}(t) = \begin{cases} 1, & \{i,j\} \in \mathcal{D}_\ell(\mathcal{G}) \\ 0, & \{i,j\} \notin \mathcal{D}_\ell(\mathcal{G}) \end{cases}$$

Further, it is optimal for the adversary to break ℓ links.

Remark 2. *(Potential-Theoretic Analogy) When the graph is viewed as an electrical network, $a_{ij} + v_{ij}$ can be viewed as the conductance of link $\{i,j\}$, and $x_i - x_j$ as the potential difference across the link. Therefore, according to Theorems 2 and 3, the optimal strategy of the designer in both problems involves finding the links with the highest potential difference (or the lowest ν_{ij}'s) and increasing the conductance of those links by setting $v_{ij} = b$. This leads to increasing the power dissipation across those links, which translates to increasing the information flow across the network and results in faster convergence. The optimal strategy of the adversary should therefore involve breaking the links with the highest power dissipation. But power dissipation is given by $(a_{ij} + v_{ij})(x_i - x_j)^2$, and this is exactly what the adversary targets according to Theorems 1 and 4.*

B. Complexity of the Optimal Strategies

We next study the complexity of the optimal strategies. We first start with the max–min problem. Assuming, as in Remark 1, that the players switch their strategies a total of K times over $[0, T]$, we conclude that the worst-case complexity of the strategy of either player is $\mathcal{O}(K \cdot m \log m)$ as their strategies involve merely the ranking of sets of size at most $2m$. As for the min–max problem, the complexity of the adversary's strategy is $\mathcal{O}(K \cdot m \log m)$. The main bottleneck in the strategy of the designer is step 2 in Algorithm I. The size of the set $\mathcal{P}(0)$ is at most $m - \ell$; thus, the worst-case complexity for the designer is $K \cdot \sum_{i=1}^{m-\ell} \binom{m-\ell}{i} \approx K \cdot \sum_{i=1}^{\ell} (m-\ell)^i$. By comparison with (3), we conclude that the derived optimal strategies achieve vast complexity reductions.

C. A Sufficient Condition for the Existence of an Saddle-Point Equilibrium

Recall the definition of a saddle-point equilibrium (SPE).

Definition 1 ([28]). *The pair $(u^\star, v^\star)$ constitutes an SPE if it satisfies the following pair of inequalities*

$$J(u, v^\star) \leq J(u^\star, v^\star) \leq J(u^\star, v), \tag{5}$$

for $u \in \mathcal{U}$, $v \in \mathcal{V}$.

Thus far, we have solved the min–max and max–min problems separately. To prove the existence of an SPE, it remains to verify whether the pair of inequalities (5) can be satisfied under some assumptions. Define $\gamma := \frac{4||x_0||^2_\infty}{\epsilon^2}$, $\epsilon > 0$. We assume that ϵ is chosen to guarantee $\gamma > 1$.

Theorem 5. *Given $\epsilon > 0$, under Assumptions 1 and 2, a sufficient condition for the existence of an SPE for the underlying zero-sum game between the network designer and the adversary is to select b such that*

$$0 \leq b \leq \min_{\{i,j\},\{k,l\} \in \mathcal{E}} |\gamma a_{ij} - a_{kl}|, \tag{6}$$

given that $a_{ij} \neq a_{kl}$ and $a_{ij} > \gamma a_{kl}$ whenever $a_{ij} > a_{kl}$, for all $\{i,j\}, \{k,l\} \in \mathcal{E}$.

Remark 3. *The condition derived in the above theorem requires the network to be "sufficiently diverse" in the sense that the weights of the links have to be not only different from each other, but also apart by a factor of γ. This is because the proof (as given in [27]) requires obtaining uniform bounds on the ν_{ij}'s defined in (4). If we allow b to vary with time, then one can find less restrictive conditions to ensure the existence of an SPE.*

III. Virus Spread Control

In this section, we study the problem of control design in infected networks. Toward this goal, we start by introducing a generic propagation model that describes the interaction among nodes in an infected network as a noncooperative game. This model provides a set of dynamical systems which describe the propagation of infection over networks. We derive a condition for the existence and uniqueness of Nash equilibrium, which can be checked distributedly. Interestingly, we show that the n-intertwined model, a recently proposed model that describes virus spread in networks, can be obtained as a special case of our generic model.

Focusing on the n-intertwined model, we first start by studying the local and global stability properties of its equilibrium points. Then, we present preliminary results on the stabilizing control design when the curing rates in the network are low. In particular, we identify graph classes that can be stabilized using a limited number of controllers. Finally, we present a dynamic optimization framework that enables regulating the infection levels over networks while minimizing the cost of control. The proofs of the results we present in this section can be found in [29].

A. Generic Dynamical Model for Infected Networks

Consider a network of n nodes that is described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices, and $\mathcal{E}$ is the set of edges. Let A be the adjacency matrix of the graph with entries $a_{ij} \in \mathbb{R}_{\geq 0}$, where $a_{ij} = 0$ if and only if $\{i, j\} \notin \mathcal{E}$. Let $0 \leq x_i \leq 1$ be the rate with which node i sends messages. The objective function of each node i, denoted $f_i : \mathbb{R}^n \to \mathbb{R}$, is comprised of a local utility function $U_i : [0, 1] \to \mathbb{R}$, and a component that is influenced by the neighboring agents of the node. The influence of node j on node i is described via the function $\tilde{g}_{ij} : [0, 1] \times [0, 1] \to \mathbb{R}$. We can then write the objective function of node i as

$$f_i(x_i, x_{-i}) = U_i(x_i) + \sum_{j \neq i} \tilde{g}_{ij}(x_i, x_j). \tag{7}$$

An interesting form of the influence function $\tilde{g}_{ij}$ is the one used in the following:

$$f_i(x_i, x_{-i}) = U_i(x_i) + x_i \sum_{j \neq i} a_{ij} g_{ij}(x_j). \tag{8}$$

The benefit of working with the particular structure in (8) is twofold: (i) it highlights the fact that x_i is a rate as it multiplies the total influence of the neighboring nodes $\sum_{j \neq i} a_{ij} g_{ij}(x_j)$; (ii) the second derivative of f_i with respect to x_i is independent of x_{-i}, which allows us to design concave games when U_i is selected to be concave in x_i.

Each node is interested in maximizing its own objective function f_i. Formally, we can write the problem of the i-th agent as

$$\max_{0 \leq x_i \leq 1} f_i(x_i, x_{-i}), \quad \text{for each fixed } x_{-i}. \tag{9}$$

When f_i is concave in x_i, and because the objective function of each player depends on the actions of other players, problem (9) describes a concave game [30]. The solution concept we are interested in studying here is the pure-strategy Nash equilibrium (PSNE).

Definition 2 ([28]). *The vector $x^\star$ constitutes a PSNE if*

$$f(x_i^\star, x_{-i}^\star) \geq f(x_i, x_{-i}^\star), \quad \forall i \in \{1, \ldots, n\}.$$

According to this definition, no agent has an incentive to unilaterally deviate from the person-by-person optimal solution $x^\star$. The next proposition establishes the existence and uniqueness of the PSNE for the game in (9), when it is concave.

Proposition 1 ([30]). *Under the following diagonal dominance condition:*

$$2\left|\nabla_i^2 U_i(x_i)\right| > \sum_{j \neq i} \left|\nabla_j \nabla_i (a_{ij}\tilde{g}_{ij}(x_i, x_j) + a_{ji}\tilde{g}_{ji}(x_j, x_i))\right|, \tag{10}$$

the concave game in (9) admits a unique PSNE.

B. Stability of the n-intertwined Markov Model

In the remaining parts of this section, we take epidemic networks as a specific example of infected networks, and we assume that the graph $\mathcal{G}$ is connected (or weakly connected if $\mathcal{G}$ is directed). In particular, we work with a recently proposed virus spread model called the n-intertwined Markov model [10], which we will first briefly review.

The proposed model is based on viewing each node in the network as a Markov chain with two states: infected or cured. The curing and infection of each node in the network are described by two independent Poisson processes with rates $\delta_i > 0$ and $\beta_i > 0$, respectively. The transition rates between the two states depend on the infection probabilities of the neighboring nodes as well as their curing and infection rates. A mean-field approximation is made in [10] to capture the effect of neighbors on a given node via the total expected infection. This facilitates the derivation of an ODE that described the evolution of the probability of infection of node i. Let $p_i(t) \in [0, 1]$ be the infection probability of node i at time $t \in \mathbb{R}_{\geq 0}$ and define $p = [p_1, \ldots, p_n]^T$. Let $D = \text{diag}(\delta_1, \ldots, \delta_n)$, $P = \text{diag}(p_1, \ldots, p_n)$, and $B = \text{diag}(\beta_1, \ldots, \beta_n)$. The n-intertwined Markov model is then given by

$$\dot{p}(t) = (AB - D)p(t) - P(t)ABp(t). \tag{11}$$

The n-intertwined Markov Model as a Concave Game

The following lemma establishes a relationship between virus spread in networks and concave games.

Lemma 1. *The dynamics of the n-intertwined Markov model are best-response dynamics of a concave game whose players are the nodes, and their objective functions are given by*

$$f_i(p_i, p_{-i}) = -\frac{\delta}{2}p_i^2 + p_i(1 - \frac{p_i}{2})\beta \sum_{j \neq i} a_{ij} p_j. \tag{12}$$

In the homogeneous case, i.e., when the curing and infection rates do not vary per node, and when the graph is undirected, it was shown in [10] that, starting from an arbitrary initial infection profile, the state converges to zero exponentially fast if

$$\lambda_1(A) < \frac{\delta}{\beta}, \tag{13}$$

where $\lambda_1(A)$ is the largest eigenvalue of the adjacency matrix A of the graph. By applying the diagonal dominance condition in (10) to (12), we obtain

$$2\delta > \beta \sum_{j \neq i} a_{ij}(1 - p_i - p_j).$$

Define $R_i := \sum_{j \neq i} a_{ij}$. A sufficient condition for the above inequality to hold is

$$\frac{1}{2} \max_i R_i < \frac{\delta}{\beta}. \tag{14}$$

Note the similarities between (13) and (14). The two conditions are related by the Gershgorin Circle Theorem which states that every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i) = \{x \in \mathbb{R} : |x - a_{ii}| \leq R_i\}$.

While (14) is more restrictive than (13), it is easier to compute and can be converted to a linear condition by requiring $\frac{1}{2} R_i < \delta/\beta$ for all i. More importantly, when converted to the linear version, condition (14) can be checked in a distributed fashion.

In [31], the condition for exponential stability of the origin was extended to the heterogeneous setting. In principle, the following inequality provides a sufficient condition for stability of the origin

$$\lambda_1(AB - D) < 0. \tag{15}$$

This condition is transformed into a centralized eigenvalue equation in [31]. The extension of our condition to the heterogeneous case is straightforward:

$$\frac{1}{2} \sum_{j \neq i} a_{ij} \beta_j < \delta_i, \quad i = 1, \ldots, n.$$

Note that this general condition still maintains the same attractive features of (14).

Stability

Under the assumption that the steady-state exists, the equilibrium points of the dynamics (11) were derived in [10], [31]. Solving the equation

$$0 = (AB - D)p - PABp \tag{16}$$

for the steady-state leads to a quadratic equation in p_i which can have multiple solutions. However, it was shown in [31] that the origin $p = 0$ is the only nonnegative vector that solves (16) when condition (15) is satisfied. Using the comparison lemma, it was shown in [32] that the origin is globally exponentially stable (GES) when (15) is satisfied, and we provided a Lyapunov based proof for this result in [29]. The following lemma provides a condition for the instability of the origin.

Lemma 2. *The origin is unstable when* $\lambda_1(AB - D) > 0$.

For directed graphs, we provide the following result for weakly connected DAGs.

Lemma 3. *In a weakly connected directed acyclic graph, the origin is the unique equilibrium, and it can be stabilized by assigning an arbitrarily small but positive curing rate* δ_i *to every node.*

Interestingly, when condition (15) is violated, another valid probability vector arises in addition to the origin. This other equilibrium point is called the "metastable" state; we denote it by $p^\star$. The metastable state has the property that $p_i^\star > 0$ for all i. This clearly shows that there will be a residual epidemic when the ratio δ_i/β_i is below a certain threshold. Henceforth in this section, we assume that $\delta_i > 0$ and $\beta_i > 0$ for all i. The existence of this equilibrium is established in [10], [31]. The next theorem provides the stability properties of this equilibrium point.

Theorem 6. *Assume that* $p(0) \neq 0$, *the graph* $\mathcal{G}$ *is connected, and* $\lambda_1(AB - D) > 0$. *Then, the metastable state* $p^\star$ *is GAS. Further, the metastable state* $p^\star$ *is locally exponentially stable.*

The following numerical experiments demonstrate the global stability of $p^\star$. The infection rates, the edge weights, and the initial infection profile were generated randomly. The curing rates were selected in a way that violates (15).

Fig. 2 shows the state of a ring graph with 20 nodes. The figure also plots the Lyapunov function $V(\tilde{p}) = \frac{1}{2}\tilde{p}^T\tilde{p}$, where $\tilde{p} = p - p^\star$. As claimed, the system converges to the strictly positive state $p^\star$, and the Lyapunov function decays monotonically to zero.

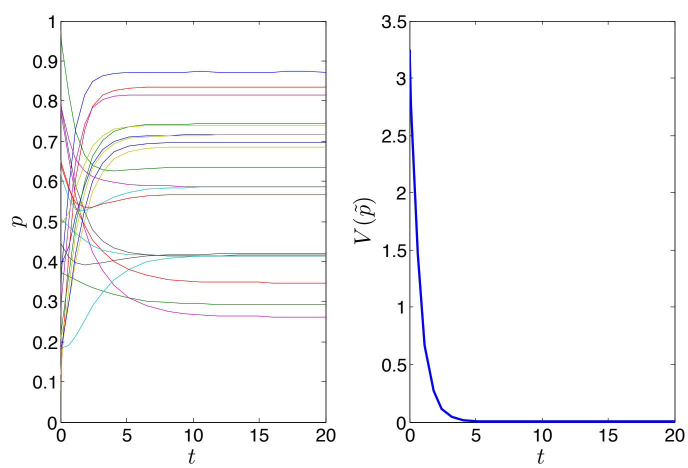

Fig. 2: Stabilization of a ring graph with 20 nodes.

Fig. 3 shows the same simulation for a random graph with 100 nodes. The probability that an edge occurs in the graph was selected to be $3/10$. The specific graph realization used in this experiment contained 1704 edges. Again, we observe that the state converges to $p^\star$. It is interesting to note that convergence here is faster than the case of the ring graph.

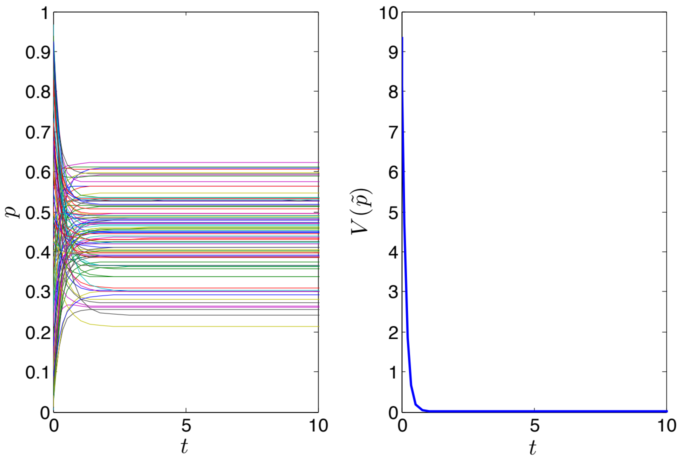

Fig. 3: Stabilization of a random graph with 100 nodes and 1704 edges.

C. Control Design

We now investigate the possibility of reducing the infection across the network by altering the curing rates at a limited number of nodes belonging to a set $S_{\text{control}} \subset \mathcal{V}$. Toward this end, throughout this section, we replace δ_i with $u_i(t)$, where $i \in S_{\text{control}}$. Given the necessary conditions presented in the recent paper [33], we will use the assumption that there exists a small curing rate of α_i at any node in $\mathcal{F} = \mathcal{V} \backslash S_{\text{control}}$. This amount of self healing may, however, not be enough to stabilize the system to the origin. By construction, we have $S_{\text{control}} \cap \mathcal{F} = \emptyset$ and $S_{\text{control}} \cup \mathcal{F} = \{1, \ldots, n\}$. Let $U(t)$ be a diagonal matrix such that $U_{ii}(t) = u_i(t)$ if and only if $i \in S_{\text{control}}$ and zero otherwise. Similarly, let Γ be a diagonal matrix such that $\Gamma_{ii} = \alpha_i$ if and only if $i \in \mathcal{F}$ and zero otherwise. The dynamics can then be written as:

$$\dot{p}(t) = (AB - \Gamma - U(t))p(t) - P(t)ABp(t), \tag{17}$$

We will now provide stabilization and dynamic optimization frameworks for infected networks.

Stabilization

We are interested in answering the following question: *When condition (15) is initially violated, can we stabilize the system to the origin by controlling the nodes in S_{control} only?* Note that the system (17) is affine in controls. To see this, define $h(p) = (AB - \Gamma)p - PABp$ and $g_i(p) := -p_i e_i$, where $\{e_1, \ldots, e_n\}$ is the fundamental basis. We can then write

$$\dot{p} = h(p) + \sum_{i \in S_{\text{control}}} g_i(p) u_i.$$

When zero is unstable for the drift vector field $\dot{p} = h(p)$, the only feasible design problem, when the controllers must be bounded, is to find a control u that would drive $p^\star$ as close as possible to zero. We are currently investigating this question.

In what follows we consider two special cases for which a limited number of controllers can stabilize the system.

Lemma 4. *The star graph can be stabilized by placing an appropriate controller at the root node and arbitrarily small α-self-loops everywhere else.*

Lemma 5. *In an odd (or even) length path graph, a maximum of $(n-1)/2$ (or $n/2$) controllers are required to stabilize the network, provided that all other nodes implement arbitrarily small α-self-loops.*

Similar results can be obtained for other classes of graphs. The key idea behind the above results is to place the controllers in such a way that no path can be drawn between two nodes in $\mathcal{F}$ without passing through a node in S_{control}. For example, in a tree with an even number of levels, stabilization can be achieved by controlling the nodes in every other level, and placing arbitrarily small α-self-loops everywhere else.

Next, we will compare the performance of Sontag's universal controller to a constant controller based on the cost of control as given by $\int_0^T u_i(t)dt$. The horizon of the simulation, T, is chosen to be 100. Consider a star graph with 10 nodes. By Lemma 4, we know that it suffices to control the root node to stabilize the network. Let node 1 be at the root. We assume that the remaining nodes implement a self loop $\alpha = 0.1$. Fig. 4 illustrates the performance of a constant controller $u_1 = 8$, while the performance of Sontag's universal controller is shown in Fig. 5. We observe that the stabilization properties of both controllers are similar. However, Sontag's universal controller incurs a lower cost compared to the constant controller; the total cost incurred under the constant controller is 800, while that incurred under Sontag's controller is 738.6.

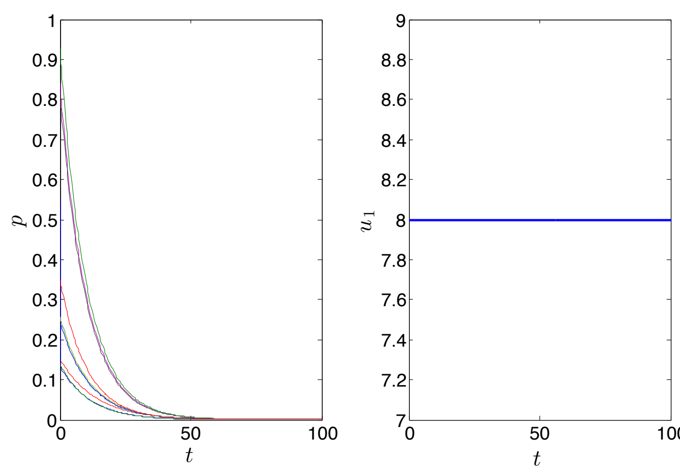

Fig. 4: A star graph with a constant controller implemented at the root. $n = 10$.

Optimal Control

We now focus on designing optimal controllers for infected networks. We assume that the designer can control the curing rates of all nodes, i.e., $\mathcal{F} = \emptyset$; however, there is a cost associated with increasing the curing rate of any node. We assume that there are minimum and maximum curing rates

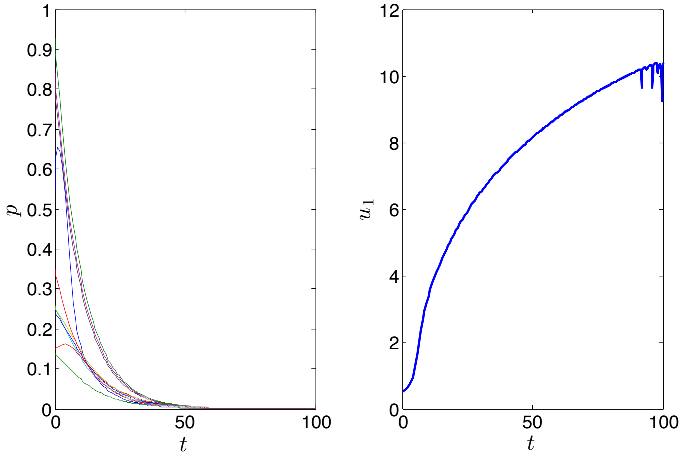

Fig. 5: A star graph with Sontag's universal controller implemented at the root. $n = 10$.

such that $\underline{u} \le u_i(t) \le \overline{u}$, for all i. The set of admissible controls, $\mathcal{U}$, consists of all functions that are piecewise continuous over $[0, T]$, where T is given. The designer aims to reduce the infection probabilities across the network, while minimizing the cost associated with modifying the curing rates. Let $c \in \mathbb{R}^{n\times 1}_{\ge 0}$ be the cost associated with the state, and let $d \in \mathbb{R}^{n\times 1}_{\ge 0}$ be the cost associated with the control.

In order to minimize the cost associated with the state, the designer must attempt to stabilize the state to the origin. To this end, we will linearize the dynamics around the origin to obtain $\dot{p} = (AB - U)p$. Consider the following optimal control problem:

$$\begin{array}{ll} \inf\limits_{u\in\mathcal{U}} & J(u) = \displaystyle\int_0^T [c^T p + d^T u] dt \\ \text{subject to} & \dot{p} = (AB - U)p, \quad p(0) = p_0. \end{array}$$

The Hamiltonian associated with this problem is

$$H(p, q, u) = c^T p + d^T u + q^T (AB - U)p,$$

where q is the costate vector. Assuming an optimal controller exists, the Pontryagin's Minimum Principle (PMP) [34] states that there exists a costate vector q satisfying the following conical equations along the optimal trajectory:

$$\begin{aligned} \dot{p}^\star &= (AB - U^\star)p^\star, \quad p^\star(0) = p_0, \\ \dot{q}^\star &= -\frac{\partial}{\partial p} H = -(AB - U^\star)^T q^\star - c, \quad q^\star(T) = 0. \end{aligned}$$

Further, by PMP the optimal control minimizes the Hamiltonian:

$$u^\star = \arg\min_{\underline{u} \le u_i \le \overline{u}} H(p^\star, q^\star, u),$$

which yields, for $i = 1, \dots, n$,

$$u_i^\star = \begin{cases} \overline{u}, & d_i - p_i^\star q_i^\star < 0 \\ \underline{u}, & d_i - p_i^\star q_i^\star > 0 \\ \{\underline{u}, \overline{u}\}, & \text{otherwise} \end{cases} \tag{18}$$

Using the continuity of $q^\star$ and the terminal condition imposed on it, we conclude that $u^\star = \underline{u}\mathbf{1}$ over $[T - \epsilon, T]$, where $\epsilon > 0$ is small.

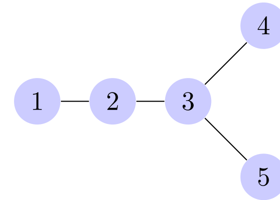

Fig. 6: An infected graph with node 3 having high probability of infection and high cost on control.

Consider the network shown in Fig. 6, and let $d = [1, 1, 10, 1, 1]^T$ such that node 3 has a high control cost. Also, let $p(0) = [0.1, 0.01, 0.9, 0.01, 0.01]^T$, where we assigned a high probability of infection to node 3. Let $\underline{u} = 0.1$, $\overline{u} = 1$, $T = 100$, and $c = \mathbf{1}$. Unity infection rates were assigned to all the nodes, and the edge weights were generated randomly.

Fig. 7 shows the state of the network above after implementing the controller given in (18). Note that $u_3 = \underline{u}$ throughout $[0, T]$ because controlling this node is expensive. Nevertheless, although the neighboring nodes have low initial probability of infections, the optimal controllers intelligently increases the curing rates of these nodes, who enjoy low control cost, in order to help cure node 3. It is interesting to note that all the controllers, except u_3, exhibit multiple switches between $\underline{u}$ and $\overline{u}$.

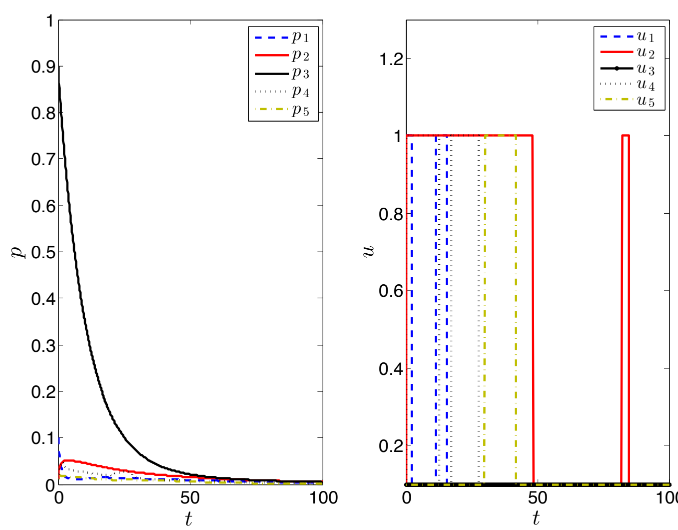

Fig. 7: State and optimal control of a network with a highly infected node whose control cost is high.

IV. Conclusion

In this paper, we focused on designing optimal and stabilizing controllers for the purpose of controlling information spread in networks. As representative of our recent work in this area, we considered two models to describe information spread: linear distributed averaging and the n-intertwined model. Designing controllers with practical constraints was the main feature of our designs for both dynamical models.

For distributed averaging networks, we considered an adversarial attack whose objective is to slow down the convergence of the computation at the nodes to the global average. We introduced a network designer whose objective is to assist the nodes reach consensus by countering the attacks of the adversary. The adversary and the network designer are capable of targeting links. We have formulated and solved two Stackelberg games that capture the competition between the players in this attack. The derived strategies were shown to exhibit a low worst-case complexity and admit a potential-theoretic analogy. Also, we provided a sufficient condition for the existence of a pure strategy saddle-point equilibrium.

For infected networks, we considered a dynamical model that describes the interaction among nodes as a concave game and demonstrated that the n-intertwined model is a special case of it. This alternative description provides a new condition, which can be checked collectively by agents, for the stability of the origin. When the curing rates in the network are low, we showed that the metastable state $p^\star$ is GAS and locally exponentially stable. We proposed a method that allows for stabilizing the state to the origin using a limited number of controllers. We further provided a dynamical optimization approach to regulate infection probabilities across the network while minimizing the cost of control and demonstrated that the optimal controllers may exhibit multiple switches.

Future work will focus on studying decentralized information spread controllers. Another potential research direction is studying the fundamental limits of network controllability using a limited number of controllers.

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